Introduction to Algebraic Topology (Part-I)
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Lecture – 49
Existence of Simply Connected Coverings

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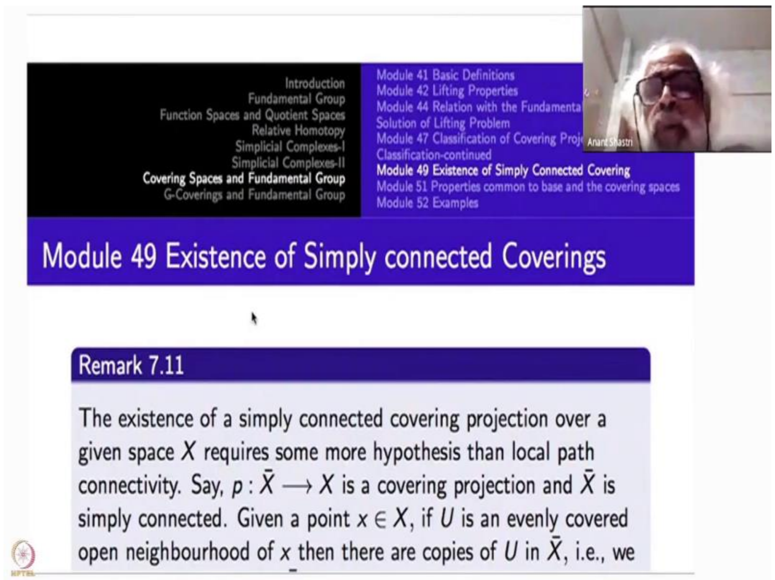

Today's topic is Existence of simply connected coverings. First we shall investigate how the existence of simply connected covering will actually solve the entire problem of existence of all the coverings. So, then we will come to the simply connected covering itself; so that is the plan. So, first let us see how once we have a simply connected covering over a given space; it will allow us to construct all other coverings.

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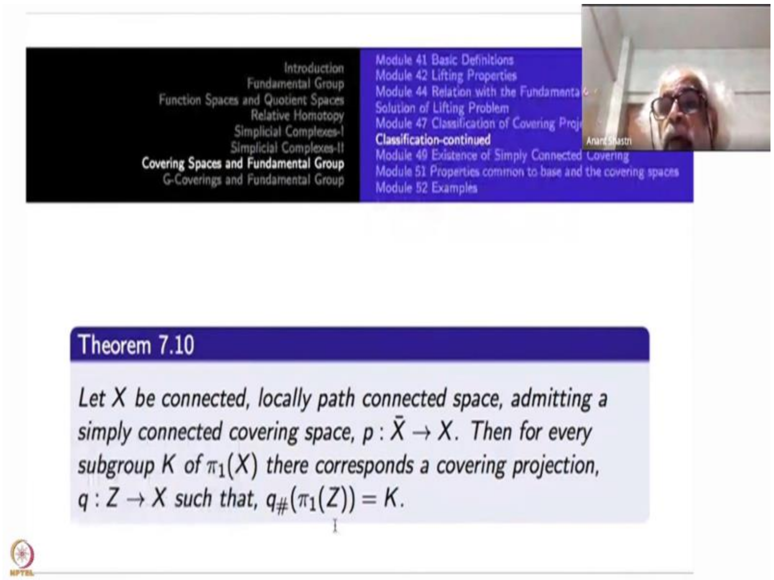

So, start with a connected, locally path connected space X, admitting a simply connected covering space; let us denote it by p from X bar to X. Then, to for every subgroup K of the fundamental group pi1(X), there corresponds a covering projection; which we will denote by q from Z to X such that if you take q#(pi1(Z)), it will be equal to K. Strictly speaking, there will be a base point z bar such that, q(z bar) = x etc; so, you have to mention these base points here, so that freedom is there. So, this will completely solve the problem now; for each subgroup you will have a corresponding covering.

And then we know how they are related and so on; namely, up to conjugation, there will be always equivalence and so on. So, the existence for arbitrary subgroup now.

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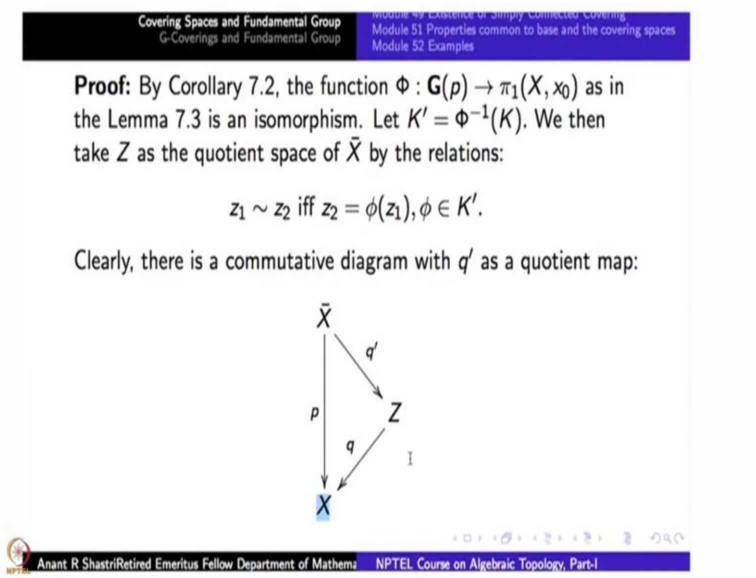

So, first we have this bijection Phi from G(p) to pi1(X, x0). Remember p is a universal covering, simply connected covering; therefore, this itself is an isomorphism. Put K' = phi inverse of K. That will give you a subgroup of the covering transformations G(p).

The group of covering transformation is identified with a fundamental group; so any subgroup K of the fundamental group can be thought of a subgroup of G(p), under this isomorphism; so K' = phi inverse of K. We take Z to be the quotient space of X bar by the relation z1 similar to z2 if and only if z2 = phi(z1) for some phi in K'.

This is nothing but under the orbit space of action of K' on X bar, we take the quotient space of X bar, the orbits of this action; so that is going to be our Z. So, claim is there is a commutative diagram as above, with q' from X bar to Z as a quotient map; q prime is from where to where? This from X bar to Z. So, what is that commutative diagram? it is this one.

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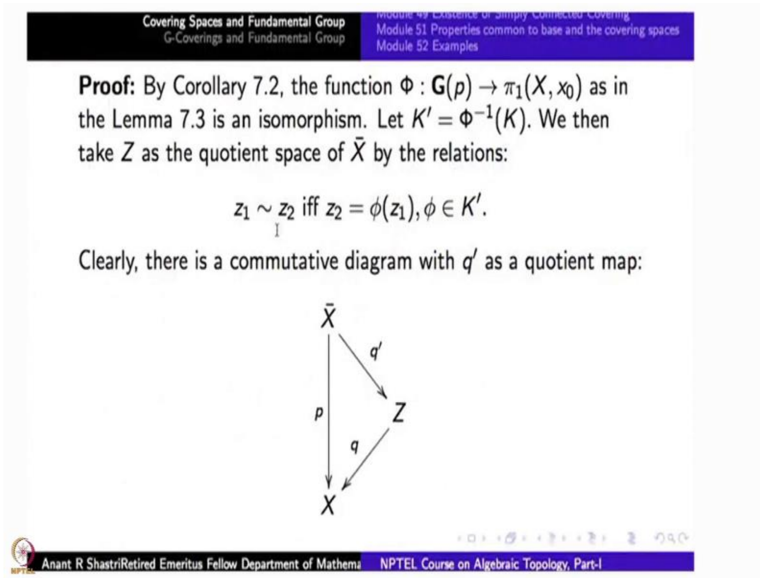

p from X bar to X respects the relation z2 = phi(z1). If you apply p to this equation, p(z2) = p compose phi(z1) = p(z1). Thus p(z1) = p(z2) is true whenever z1 similar to z2. Therefore, this p factors down to the quotient namely, on the equivalence classes of this relation.

So, let us denote this map by q from Z to X; this one by q'. Claim is this map which is defined by by p itself; q([z]) = p(z); it is a covering projection. This is connected, so this is connected no problem; you have to prove that q is a covering projection. We have to also prove that pi1(Z) (take a base point above this one that does not matter), is actually isomorphic to this K'.

That is when you take q# of pi1(Z), it will come to the subgroup K of the fundamental group pi1(X), whatever we have started with. Because, q is covering projection, then q# will be injective and maps onto the image whatever we have taken. So, there is work to do here; there is some work to do.

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If V is a connected open subset evenly covered by p, we claim that, it is also evenly covered by q. This follows from the fact that, if $p^{-1}(V) = \sqcup U_i$, then any covering transformation of p maps each U_i homeomorphically to another U_j. (Incidentally, it turns out that q' is also a covering projection but we shall not need this here.)

the Lemma 7.5 is an isomorphism. Let $K' = \Phi^{-1}(K)$. We then take Z as the quotient space of $\bar{X}$ by the relations:

$$z_1 \sim z_2 \text{ iff } z_2 = \phi(z_1), \phi \in K'.$$

Clearly, there is a commutative diagram with q' as a quotient map:

$$\begin{tikzcd}
\bar{X} \arrow[dd, "p"'] \arrow[dr, "q'"] & \\
& Z \arrow[dl, "q"] \\
X &
\end{tikzcd}$$

Suppose, V is a connected open subset of X, evenly covered by p. p is given to be a covering projection. So, the entire space X is covered by evenly covered open sets. Being locally connected, I can take smaller neighborhoods, which are connected at each point. So, they will cover and they will be even as well. So, because any subset of an even subset is even. Any subset of an evenly covered open set is automatically evenly covered. So, start with a connected evenly covered open set; we claim that it is evenly covered by q also. So, that will already settle that q is a covering projection. So, this follows from the fact that p inverse of V is disjoint union of Ui; I am just writing the meaning of evenly covered; write it in this way. Then, any covering transformation from f from X bar to X bar maps each Ui homeomorphically onto some other Uj. Because every covering transformation respects p; so f(p inverse of V) subset of p inverse of V. But, when you have written disjoint union, each Ui may not go to Ui itself; it may get shuffled that is all. So f from Ui to Uj is a homeomorphism, for each Ui is now connected also, because I assumed V is connected. So, it is like the connected components of this p inverse of V are getting shuffled; so that is the action of any covering transformation, all the covering transformations. Therefore, under the action of K, the image q inverse of V = q'(p inverse of V) = disjoint union of V_alpha, where the indexes alpha run over the right-cosets K' minus G(p) and q' $q' : U_i \to V_{\alpha_i}$ are homeomorphism. It follows that $q : V_\alpha \to V$ is a homemorphism. This proves that q, q' are both covering projections.

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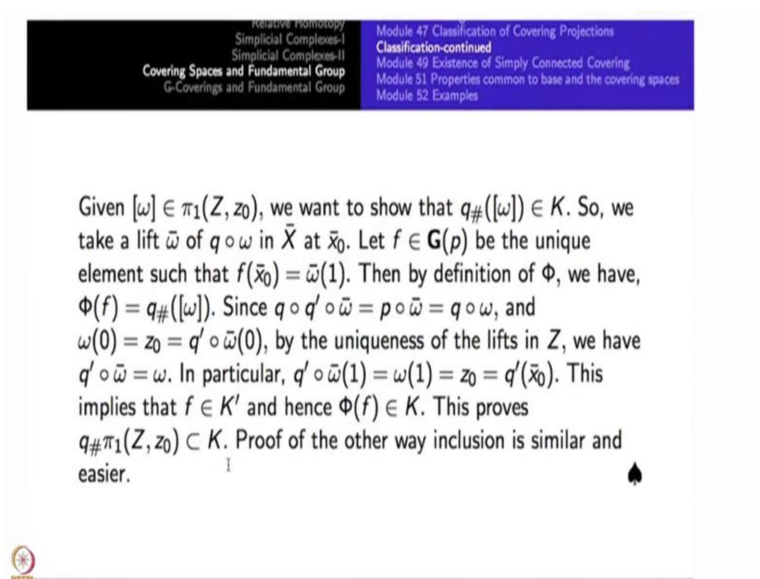

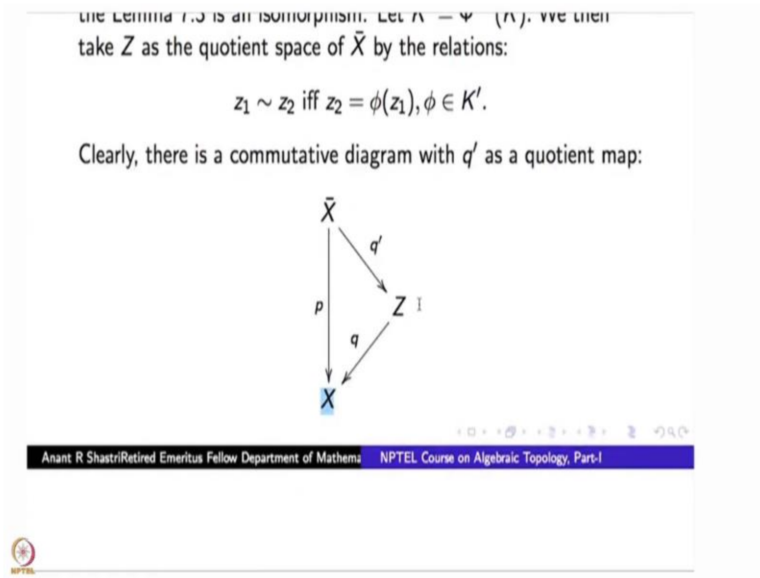

Given $[\omega] \in \pi_1(Z)$, we want to show that $q_{\#}([\omega]) \in K$. q check of omega is in K; $q_{\#}$ is anyway an injective homomorphism. ; so first we want to show that it is going inside K, and then we want to show that it is onto K. So, we take a lift $q \circ \omega$ in $\bar{X}$. Start with a loop in Z, take q of omega; that will be a path in X, at the point x_0. Take a lift $\bar{\omega}$ of it in $\bar{X}$. Everything loop can be lifted a path in $\bar{X}$ at the base point $\bar{x}_0$. There is a unique element $f \in G(p)$ such that $f(\bar{x}_0) = \bar{\omega}(1)$.

Then by definition of Φ, remember, $\Phi(f)$ is nothing but $q_{\#}[\omega] \in \pi_1(X, x_0)$. Since $q \circ q' = p$, we have $q \circ q' \circ \bar{\omega} = p \circ \bar{\omega} = q \circ \omega$. Therefore $q' \circ \bar{\omega}, \omega$ are two lifts of $q \circ \omega$ through q. We have to note their starting points are the same. Then it follows that $q' \circ \bar{\omega} = \omega$.

So, I took a path ω in Z, come down to X, lifted it to $\bar{X}$ to get a path, the image of that path is ω. This is picture. Start with a path here, come here, lift it here; then you come here, it will be the same path that we have started with. So, this is uniqueness follows by uniqueness of a path liftings.

So, it follows that starting point and end point of $\bar{\omega}$ are mapped onto the same point in Z by q'. That means $f \in K'$. Hence $\Phi(f) \in K$.

So, this proves q check of the entire pi 1 of Z is contained inside K. Now, we have to show that the K is contained inside this one; that proof is similar. So, therefore q check of pi 1 will be equal to K.

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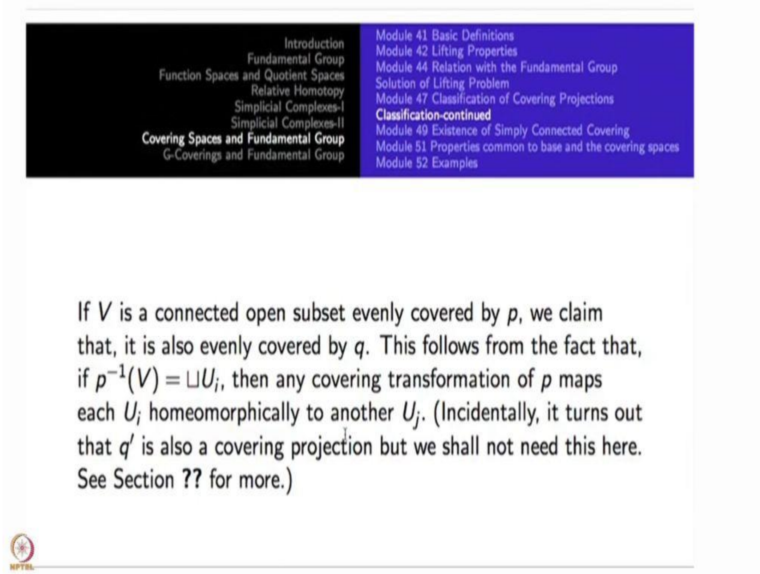

So, I repeat how the covering transformation is got, because the identification under subgroup is nothing but some Ui's have to be identified some Uj's. It is a full subgroup, they all of them will be identified to the single one component; because then it will transit it. So, remember all p inverse of V this Ui's is indexing say, is nothing but same thing as the group index inside; because there is a bijection between the fibre and the entire group. That is this you can I can be chosen as group (()) (14:29); so that is some shuffling here, and some of them are identified.

Rest of them go to homeomorphically onto p; so that is what is happening after identification. That is how q becomes a q becomes a covering transformation. And if you fix one single thing there that identification, all become its spread out again to some Ui. Now, this time the disjoint union will be much smaller, through the one single coset.

So, that was showing that q prime is also a covering projection. So, this is much easier than we thought, getting coverings for arbitrary subgroups is easier. Once you admit the covering for the that is trivial group, namely the simply connected covering. So, let us now proceed with existence of simply connected coverings.

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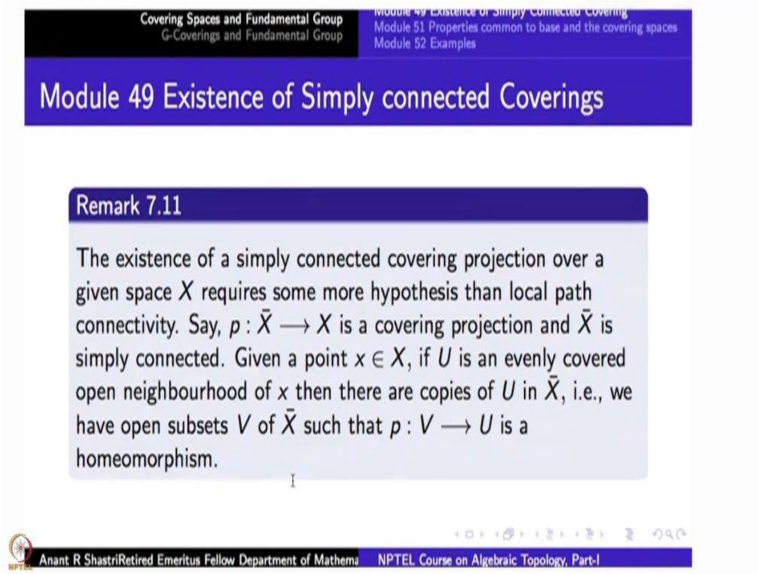

The existence of simply connected covering does not come freely. It requires some more conditions on this space X; let us see why. Suppose, $p : \bar{X} \to X$ is a covering projection and $\bar{X}$ is simply connected. Take any point $x \in X$ and U be an evenly covered neighborhood of x; then there are copies of U, copies of U means? which are homeomorphic to U, in $\bar{X}$.

That is, we have open subsets V of $\bar{X}$ such that $p : V \to U$ is a homeomorphism, because U is evenly covered. So this is the starting point, all the time we are doing it. Now, what happens? Let us see at the fundamental group level.

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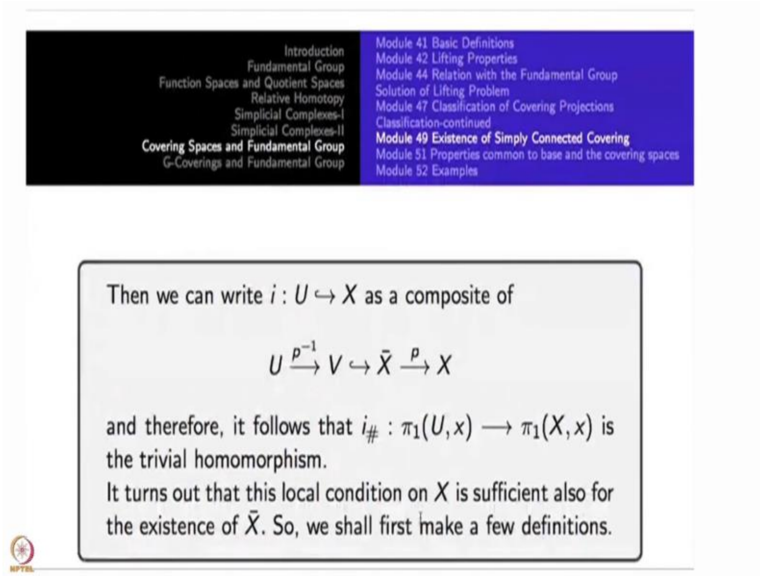

We can write the inclusion map $i : U \to X$ as a composite as follows: Since $p : V \to U$ is a homeomorphism, which the restriction of the covering map, I can take p^{-1}, which is like taking a branch of the logarithm function, for exponential function. Of course, this is not defined on the whole of X, but defined on U and there are many many such p inverses here.

One of them goes to V homemorphically. So, that p inverse followed by j, the inclusion of Vinto $\bar{X}$, following by p .This composite is nothing but the inclusion of U into X; because p, p inverse is identity. So, the inclusion map from U to X is broken up, or as a decomposed into $i = p \circ j \circ p^{-1}$.

Therefore, if you take a point here, start with any $x \in U$, then take $i_{\#} : \pi_1(U, x) \to \pi_1(X, x)$ this homomorphism factors through the group $\pi_1(\bar{X}, \bar{x})$ where $\bar{x} \in V$ is such that $p(\bar{x}) = x$. But $\pi_1(\bar{X}, \bar{x}) = (1)$. If a homomorphism is factored through a trivial group, the composite must be trivial.

So, what we have proved? U is to begin with a neighborhood of some arbitrary point of X with the property that it is evenly covered and connected. Then the inclusion map induces trivial homomorphism on the fundamental group.

So, this is a must and will happen, if there is a simply connected covering; so this is very important. So, we make a definition out of this observation.

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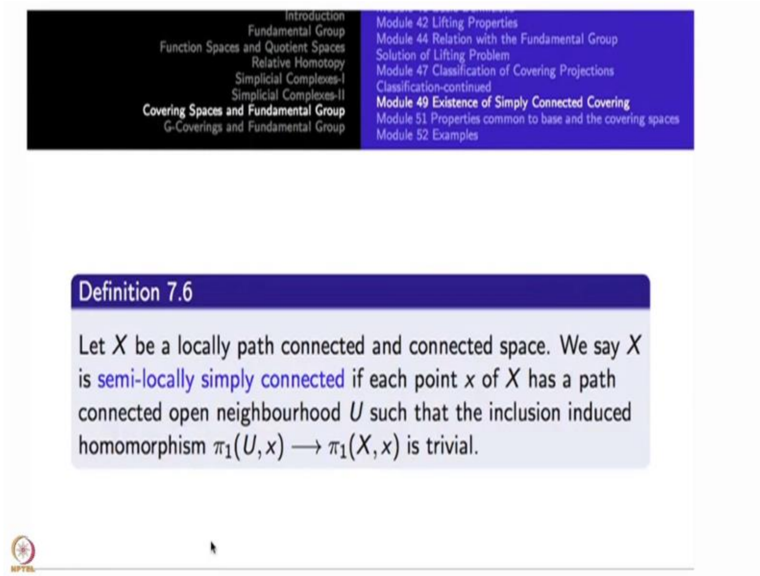

So, we start with a locally path connected space X. We say V is semi-locally simply connected (very strange kind of definition, but, wait a minute, we would explain it), if each point of $x \in X$ has a path connected open neighborhood U such that the inclusion induced homomorphism on the fundamental group is trivial. So, this was the condition that we have observed, holds there is a simply connected covering covering for X. So, we made this condition into a definition. then we put that this condition should be true; this now just the definition.

You could have required instead that it should be `locally simply connected; that is, $\pi_1(U,x) = (1)$. We do not need it; we need only the inclusion induced map of the homomorphism must be trivial. The inclusion map need not induce inclusion in the fundamental group. It that is also true, then we would have got $\pi_1(U,x) = (1)$. But that need not happen. For example, we take $\mathbb{S}^1 \subset \mathbb{D}^2$. Then $\pi_1(\mathbb{S}^1) \neq (1)$. But, the inclusion induced homomorphism is trivial because $\pi_1(\mathbb{D}^2) = (1)$. so that can happen.

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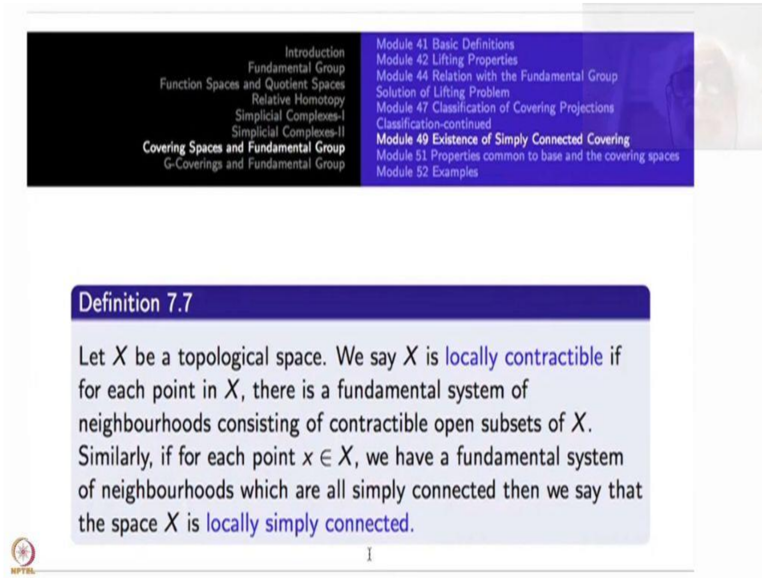

So, how to ensure that such things happen; there are stronger conditions like local contractibility. Locally contractible means what? At each point if we have a fundamental system of neighborhoods which are contractible. It is the same as saying that given any neighborhood of a point, there is an open neighghbourhood of that point contained inside that and that open set is contractible; the neighborhood should be open.

Similarly, we can define just locally simply connected which are defined just now, instead of semi-locally. What is this definition? Given any point and a neighborhood there is a smaller neighborhood of that point, which is simply connected; then this would be locally simply connected. What is easy here is the following.

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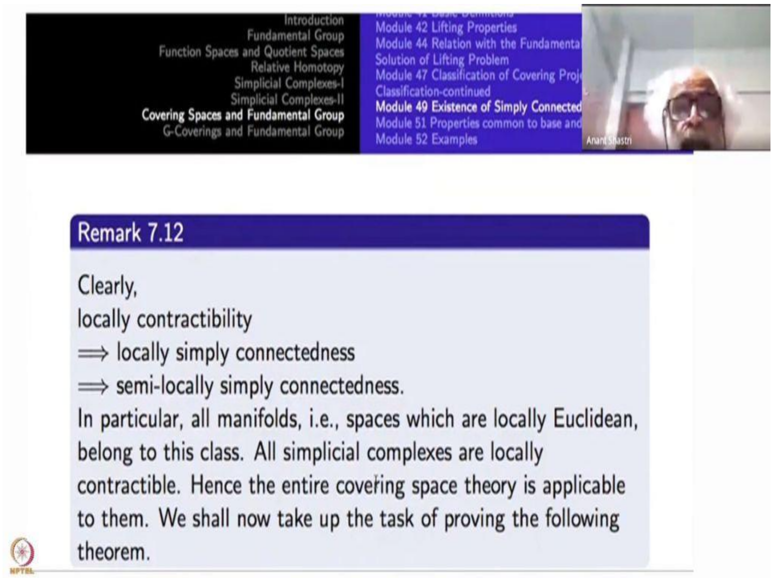

Namely, locally contractibility will imply local simply connectedness; that will imply semi-locally simply connectedness. So, what we are interested in is this just weakest condition, and it turns out to be good enough. This is always true, there is a simply connected covering for X.

These two this condition is too strong, this may not be true; this is also stronger; this may not be true. But, this will have to be true and if you put this condition, then we can guarantee the existence of simply connected covering for X. However, how to check this one? It is locally contractibility, you can go ahead; simply connectivity locally, you can go ahead.

So, what are the things that we guarantee this one? Manifolds of such things; so finishing a complex is such thing; we have not proved this yet. Maybe we will prove it in the next part, in a part-2 of this course. So, there are simplicial complexes which have this property and so on; there are lots of spaces which have this property, so it is not hopeless. So, entire covering space here is applicable to this manifolds, simplicial complexes CW complexes and so on. So, let us prove now the following theorem.

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Over a connected, locally path connected, semi-locally simply connected space; there exist a simply connected covering space. The connectivity path is optional; locally path connectivity and semi-locally path connectivity is not optional, you understand. If it is not connected what you can do is connected component; then prove for each connected component, and put them together. That will give you a universal the simply connected covering for all. So, that is why connectivity assumption is like a simplifying assumption, not mandatory.

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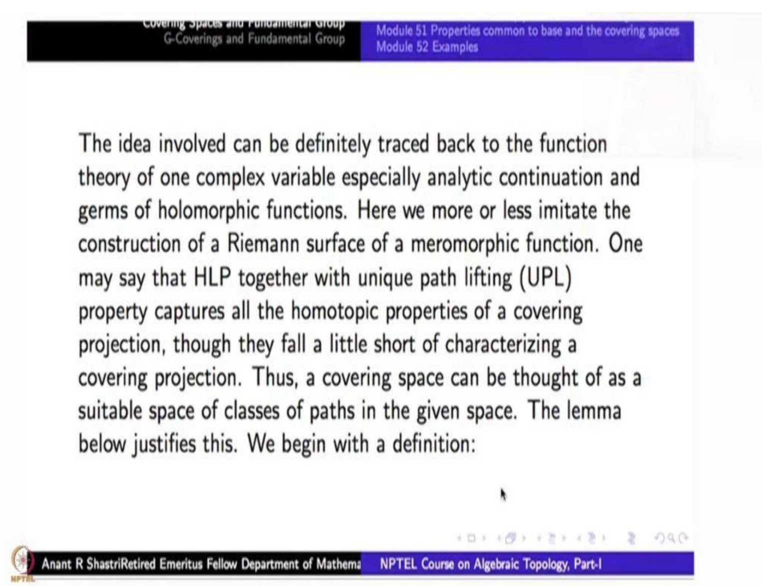

The idea involved can be definitely traced back to function theory of complex variables, one complex variable. That is especially what is called analytic continuation and germs of holomorphic functions. Here we more or less imitate the construction of a Riemann surface of a meromorphic function.

One may say that Homotopy Lifting Property together with unique path lifting property; there are two of them captures all the homotopic properties of a covering projection. Though they fall a little short of characterizing the covering projection, under some slightly stronger condition which is what it is; and that is what happens in function theory of one way variables.

Thus, a covering projection covering space can be thought of as a suitable space of classes of paths is in a given space. And that is why you needed existence of sufficiently many paths in the given space X; so that is given by locally path connected. So, there are plenty of paths in the space; but semi-locally simply connected will tell you that there are not too many of them. So, it brings some severity into the existence of path connected paths; it is really not too many of them, if you take many of them, when you go to a larger space, they will be all homotopic; that kind of result is semi-locally simply connected.

Take a loop low in a locally loop, locally you may not be able to make it homotopic. If you globally in the if you go into the whole space, it will in a homotopic. That will show that not too many funny kinds of loops are there; on the one hand there are plenty of them. On the other hand no no no no hold on; there are certain conditions on them, not too many. So, that is what is happening for existence of simply connected coverings. So, there is a quite a bit of work to do here; so let us do it one by one.

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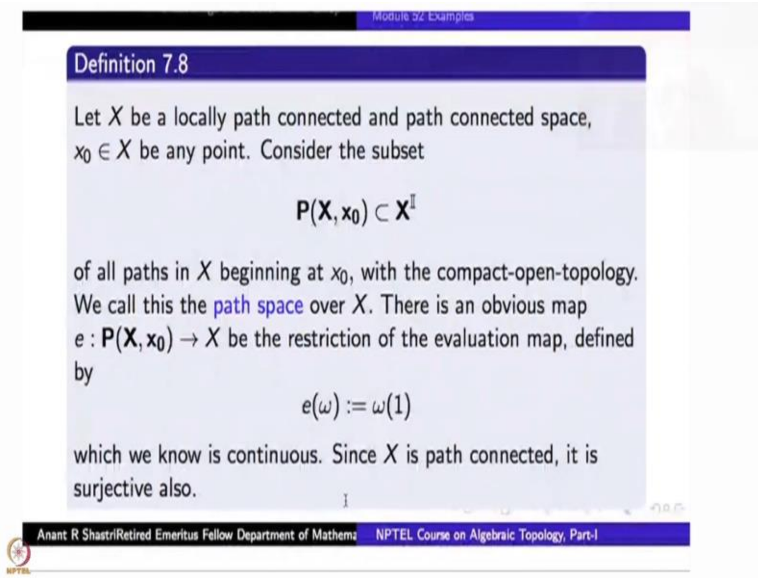

So, here is the definition. Path is have to be study, therefore we go to the functions space $X^{\mathbb{I}}$; what is it? All paths in X. Since, we want everything starting at a point, base point is fixed and so on. So, we start with a base point $x_0 \in X$. Then dont take the whole space $X^{\mathbb{I}}$. We would like to have only paths starting at that point. So, this is the some space $P(X, x_0) \subset X^{\mathbb{I}}$. Remember, on $X^{\mathbb{I}}$, we have to give the compact-open-topology; so same for the subspace namely compact-open-topology on $P(X, x_0)$ also.

So, this is called now the path space; actually this is path space is too big, we want this one. This is a path space, path is starting at x naught and all of them are inside X. So, remember there is the evaluation map $E : X^{\mathbb{I}} \times I \to X$; $E(\omega, t) = \omega(t)$. We can restrict this to get a map $e : P(X, x_0) \to X; e(\omega) = \omega(1)$.

Evaluation maps are nothing but coordinate projections restricted to P; that will be still continuous. So, what is this evaluation map I am taking here? e of the path omega take the end-point. Why I am taking end-point? The initial point is always x naught; so other one it is important the end-point.

Remember, in fundamental groups and while studying fundamental groups, you have to concentrate on. Initial points and end-points in the homotopy also, these two points have to be kept fixed throughout. So, that is why all these things are natural, they are not new and they are not strange; why they are coming again here, is obvious.

Because in the study of homotopy classes of paths or loops and so on, if you do not keep both the end-points fixed, everything will be null homotopic. There is nothing, no theory. So, we want x naught starting point is fixed already, now we are looking at the end-point.

So, that map is very important and that will tell you the entire story. So, this and e is evaluation map, its behavior is comparable with explanation map. So, it is just a chance that in mathematics using English language, that the same e will be denote both evaluation map as well as explanation map; and that is good for us anyway.

Now, X is path connected, therefore e is surjective. Take any point $x \in X$. There is a path ω starting from x_0 and ending at x. Then $e(\omega) = \omega(1) = x$ omega 1; so this means that e is surjective. e is continuous, e is a projection map; therefore, e is also an open map. Well, openness is not clear, because we are not taking the product topology; we are taking some stronger topology there. So, you have to wait for openness of this.

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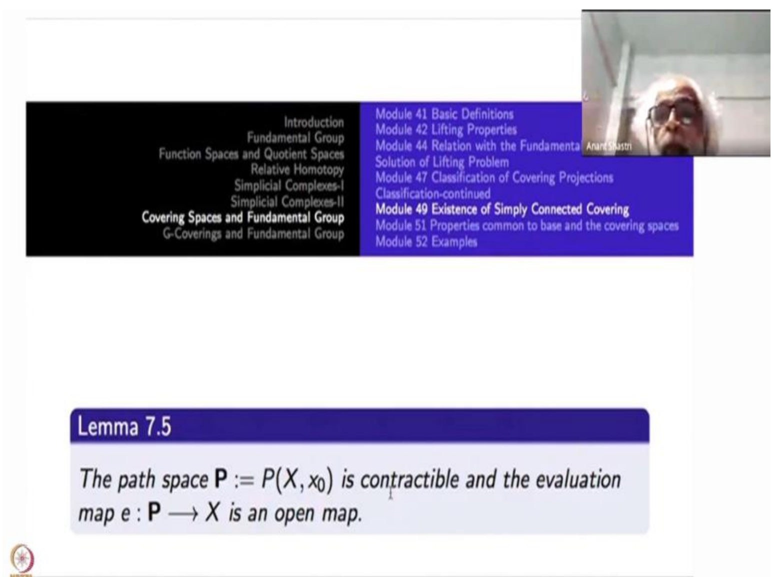

This space $P := P(X, x_0)$ is itself is contractible, and the evaluation map is an open map. Openness is not easy; not as easy as continuity. Continuity is fine, because we have this projection map and the the topology on X, x naught is finer than is finer than the topology, which is given by the

Cartesian-coordinate product topology. Evaluation map is continuity we have seen already in exponential correspondence. So, how to show that path space is contractible.

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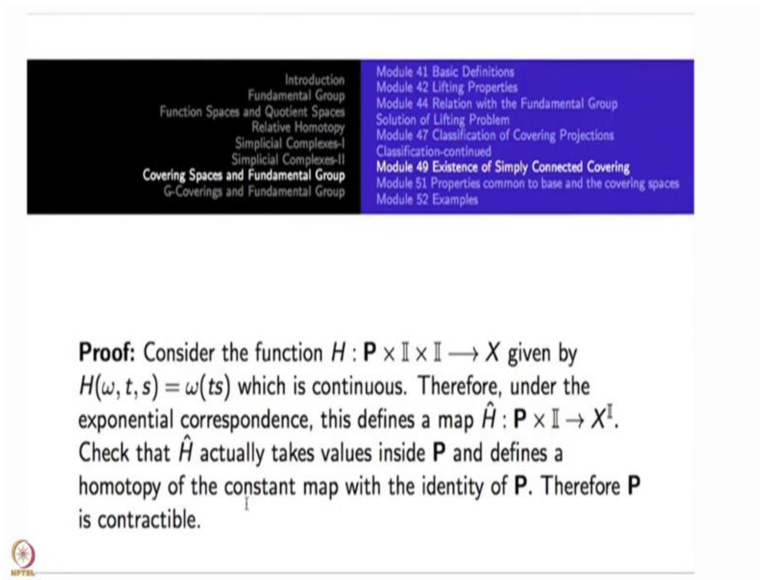

That means I have to construct $H : P \times \mathbb{I} \to P$ which is a homotopy of the identity map to a constant map; that is what I have to construct. So, I look at a very simple map here, which is again obtained by observing what happens inside $\mathbb{I}$. So, first I am taking this map on $X^{\mathbb{I}^1}$ instead of P cross I, but, late, I will restrict it to P cross I . So, then $H(\omega, t, s)$, where ω is a path in X, $t, s \in \mathbb{I}$, I am sending it to $\omega(ts)$. Look at what happens; if $s = 0$, irrespective what t is, it is $\omega(0) = x_0$. If $s = 1$, then it is $\omega(t)$, which is the identity map; so this is omega. Finally, if $t = 0$, irrespective of what s is, it is $\omega(0) = x_0$. So, all this is important. That will tell you that under the exponential correspondence, continuity it is obvious. Just because, $(t, s) \mapsto ts$ and ω is continuous. This is continuous function under the exponential correspondence, there is a map $\hat{H} : P \times \mathbb{I} \to X^{\mathbb{I}}$. But, where does starting point of each of the path is? For each path here starts at x_0 Therefore, this $\hat{H}$ takes values inside P; what is the definition of P? P$P = P(X, x_0)$ the set of all paths starting at x_0; that is all.

So, it is a map into P. And of course when s is 1, you see it is $\omega \mapsto \omega$. Therefore, this is a homotopy of the constant map to the identity map of P. So P is contractible. So, this is the proof of contractibility of P.

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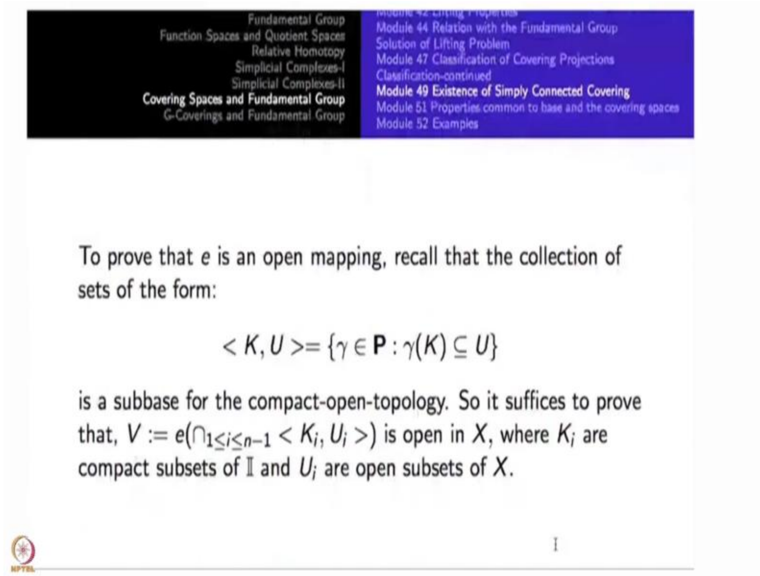

To prove that e is an open mapping, we have to show that for open subsets image is open. What are open subsets in P. This is the compact-open-topology; t So, what is compact-open-topology? Take a compact subset K of the interval $[0, 1]$. Take any open subset U of X. Then take all paths which take the compact subset inside U. So, that is the bracket $\langle K, U \rangle$ which is a subset of P. The set of of all such $\langle K, U \rangle$'s forms a subbase for the compact-open-topology; collection of all such things. Intersection of finitely many such elements forms a base; so it suffices to prove that under the evaluation map e, the image of every basic open set is open. Because then every other thing is union of such elements; e of the union is the union of e; e of the intersection is not the intersection of the e. For any function f, $f(A \cap B)$ may not be equal to $f(A) \cap f(B)$. So, you have to do this one, not just e of K, U is open; that is not enough. So, assume that K_i are compact subsets of $\mathbb{I}$ and U_i are open subsets of X, we have to prove $V := e(\cap_{i=1}^{n-1} \langle K_i, U_i \rangle)$ is open. So, let us complete this one.

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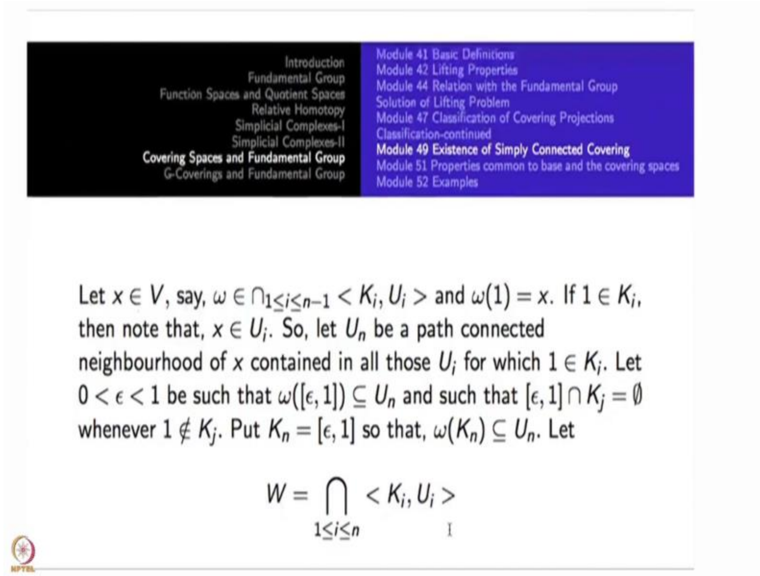

So, take $x \in V$, $e(\omega) = \omega(1) = x$, ω being in the intersection. This is the evaluation e of omega.

If 1 is in one of the K_i, $x \in U_i$, because $\omega(K_i) \subset U_i$. So, let us say, U_n is a path connected neighborhood of x contained in all the $U_i's$ for which $1 \in K_i$.

It should be intersection of finitely intersection of all those contained in all those Ui, for which 1 is inside Ki; Now use continuity of omega. Let us choose an epsilon between 0 and 1 such that, $\omega[\epsilon, 1] \subset U_n$. Because, $\omega(1) \in U_n$ and U_n is possible; moreover, ϵ can be chosen such that $[\epsilon, 1] \cap K_j = \emptyset$ whenever $1 \notin K_j$.

Put $K_n = [\epsilon, 1]$. so I have to go one more compact set along with K 1, K 2,... K n minus 1. Now, you take $W = \cap_{i=1}^{n} \langle K_i, U_i \rangle$, to be intersection of all these along with the bracket K n, U n also one more; so I am taking one more of its that is all. I have chosen this K_n and chosen U_n very carefully. So, once you do that you are home.

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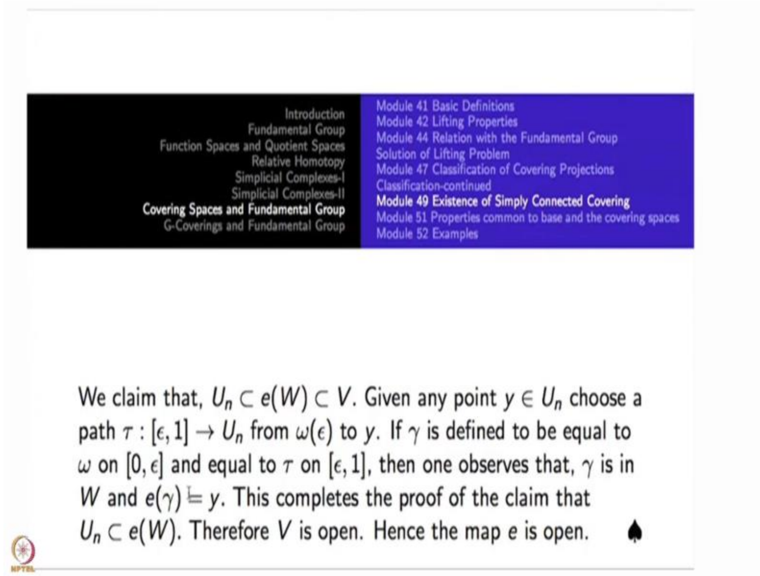

So now we claim that $U_n \subset e(W) \subset V$. So, I have found a neighborhood of x; U_n is neighborhood of x, that neighborhood is in the image of W; and this image is contained inside V. Every point is covered by an open subset in the image, which means the image itself is open; so this V itself will be open.

We want to show that $U_n \subset e(W)$. Take $y \in U_n$. Choose a path τ in U_n from $\omega(\epsilon)$ to y. $\omega(\epsilon) \in U_n$ and U_n is path connected; If γ is defined to be the restriction of ω on the interval , and equal to τ on $[\epsilon, 1]$, then $\gamma \in W$, because the first part is already inside $\cap_{i=1}^{n-1} \langle K_i, U_i \rangle$, this intersection.

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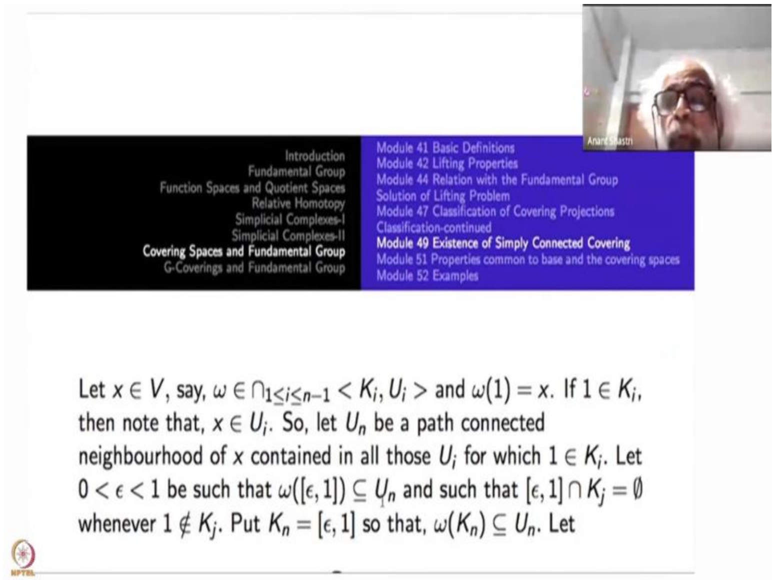

The latter part $\tau \in \langle K_n, U_n \rangle$ by choice. So, tau of this one is contained in this one; so it is an element of W now.

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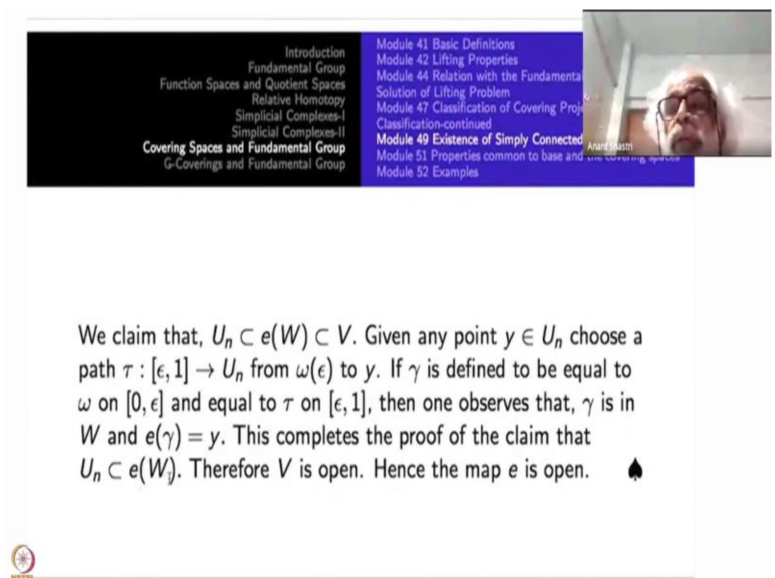

So, now $e(\gamma) = \gamma(1) = \tau(1) = y$. So, this completes the proof that $U_n \subset e(W)$.Un Therefore, V is open.

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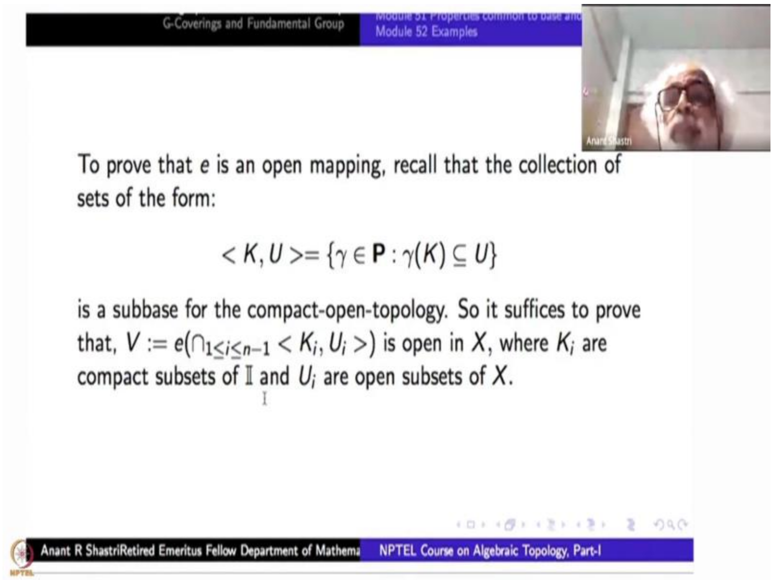

Therefore, the evaluation map is an open mapping. So, will stop here and continue the construction tomorrow. Thank you.