Introduction to Algebraic Topology (Part-I)
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Lecture No. 48
Classification

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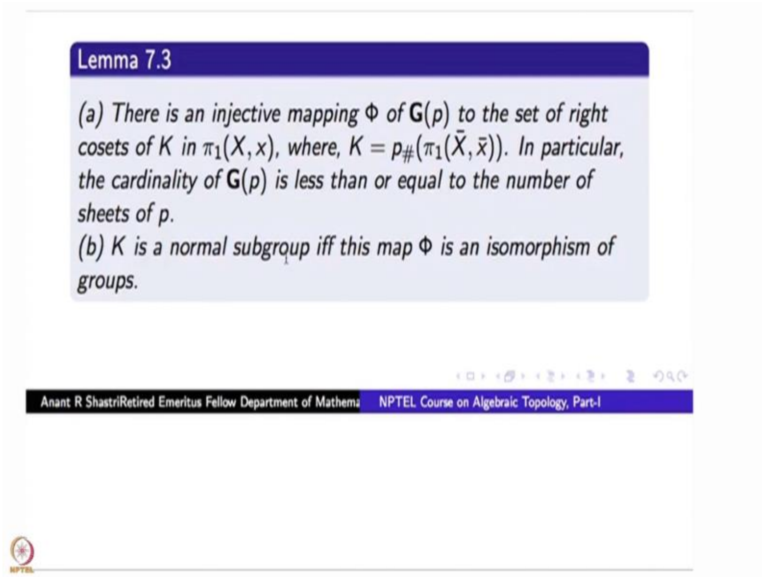

We have started classification of covering projection; let us continue that work now. The first lemma we had last time we already stated it, let me recall it. There is an injective mapping Φ from the group of covering transformations of p, i.e., Galois group of the covering transformation p into the set of right cosets of K in $\pi_1(X, x)$, where $K = p_\#(\pi_1(\bar{X}, \bar{x}))$. In particular, the cardinality of the group of covering transformation is less than or equal to the number of sheets of p. Why this part follows?--- the latter part? Because we have already established a bijection from the set of right cosets of K with the fibre of $p : \bar{X} \to X$.

The number of points in the fibre is the cardinality of the sheet; number equal to number of sheets. Since, there is an injection, so cardinality will be less than or equal to this number. So, this part is OK, once we define an injective map. The second part says that K is a normal subgroup, if and only if the map Φ which is only a set-theoretic function becomes an isomorphism of groups. So, this is possible because, if K is normal, then the right cosets form a group, first of all.

So, the map Φ is from one group into another group, and the claim is that it is actually an isomorphism. Not only it is injective, it will be an isomorphism automatically. That means it is onto also. So, several things you have to verify, so let us do them one by one; first, the definition of the map Φ.

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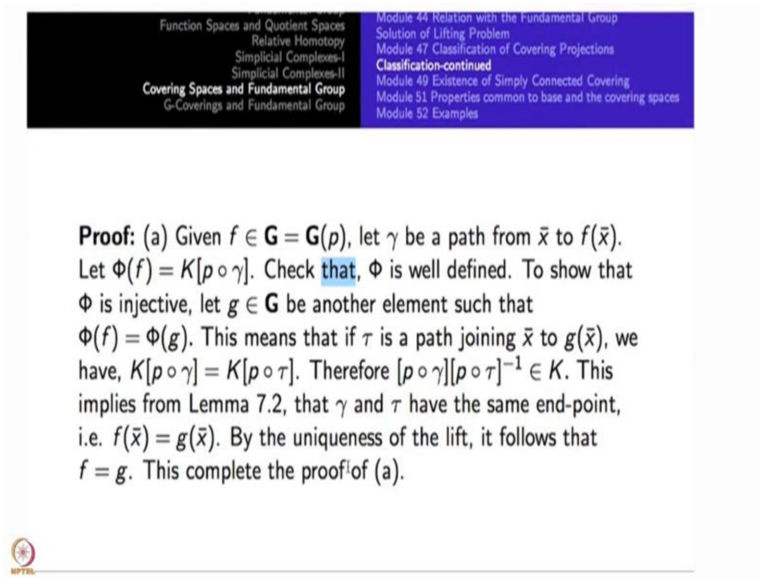

Given a covering transformation f of p, look at the image of $\bar{x}$ under this covering transformation. Since, p of f is f which follows that; sorry $p \circ f = p$, it follows that $p(\bar{x}) = p(f(\bar{x}))$. That means $f(\bar{x})$ and $\bar{x}$ are in the same fibre of p. So, I am going to take $\Phi(f) = K[p \circ \gamma]$, γ is a path from $\bar{x}$ to $f(\bar{x})$. When you take a path from $\bar{x}$ to $f(\bar{x})$ its image under p will be a loop in X, because both $\bar{x}$ and $f(\bar{x})$ are mapped onto the same point, namely, here in this particular case, onto $x \in X$. Therefore $p \circ \gamma$ is a loop at x.

Take its class and take its right coset, K of that; the right coset of K with that element, this is the definition of Φ. Now, the only ambiguity at all is in the choice of that the path joining $\bar{x}$ to $f(\bar{x})$ Take another path, then what happens? Suppose ω is another path. Then $\gamma * \omega^{-1}$ is a loop at $\bar{x}$ and and hence $k = [p \circ (\gamma * \omega^{-1})] \in K$. It follows that $K[p \circ \omega] = Kk[p \circ \omega] = K[p \circ (\gamma * \omega^{-1} * \omega)] = K[p \circ \gamma]$. Thus the coset class is independent of the different path that we are taking.

To show that Φ is injective, proof is similar now. Take another $g \in G(p)$. That is g is a covering transformation of p such that $\Phi(g) = \Phi(f)$. Suppose, the path joining $\bar{x}$ to $g(\bar{x})$ is τ. Then it means that $K[p \circ \gamma] = K[p \circ \tau]$. But, then $[p \circ \gamma][p \circ \tau]^{-1} \in K$. This means that the loop

$(p \circ \gamma) * (p \circ \tau^{-1})$ lifts to loop at $\bar{x}$. As seen before, this in turn implies $f(\bar{x}) = \gamma(1) = \tau(1) = g(\bar{x})$. Remember what are f and g? They are covering transformations.

That means $p \circ f = p, p \circ g = p;$ which in turn means that they are lifts of p through p. At one point they agree, then they must agree everywhere, by the unique lifting property. So, the first part (a) is complete, where definition and injectivity of Φ are established. As an immediate consequence, cardinality of $G(p)$ is less than the number of sheets. Now, let us come to (b): K is normal if and only if Φ is an isomorphism. I will just assume that this Φ is surjective; instead of isomorphism, let us just assume surjectivity and see what happens. Suppose Φ is surjective.

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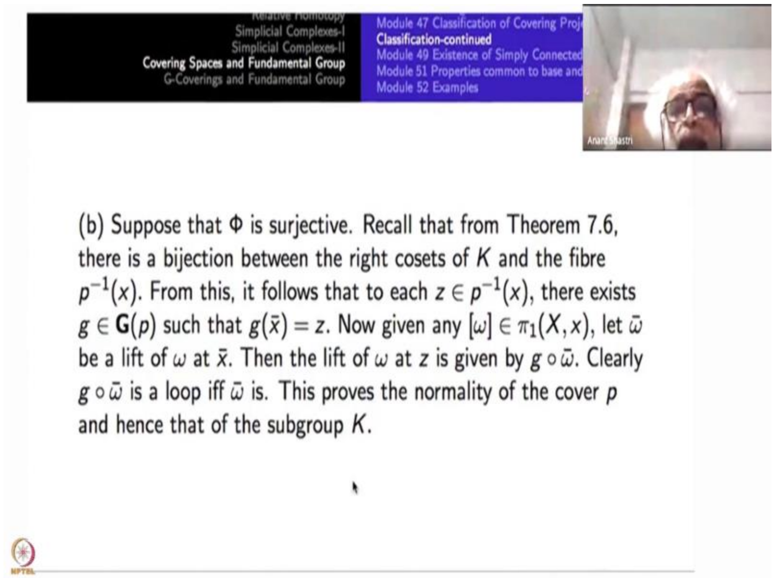

First of all, we have already another bijection, which I am going to use again; the bijection between the right cosets of K and the fibre $p^{-1}(x)$. From this, it follows that to each $z \in p^{-1}(x)$ there exists a $g \in G(p)$ such that $g(\bar{x}) = z$. Because $\Phi(g) = K[p \circ \tau]$ where τ is path from $\bar{x}$ to $g(\bar{x})$ and then this right coset is mapped to the end point of τ. Thus we have a bijection from $G(p)$ onto $p^{-1}(x)$.

Now, take any point any element of $[\omega] \in \pi_1(X, x)$ and lift ω at $\bar{x}$ to a path $\bar{\omega}$. Now for any point $z \in p^{-1}(x)$, how to get the lift of ω? Take the element $g \in G(p)$ which corresponds to z viz., $g(\bar{x}) = z$. It follows that $g \circ \bar{\omega}$ is the answer. (The lift of omega at z; z is also a point over x; I can lift it at any point in p inverse of x, it is a starting point. Look at g operating a point omega bar, g of omega bar; g is a transformation from x to x, X bar to x bar. You can compose omega bar with g; g omega bar operating a point 0 is g of x bar that z by definition. That means the starting point of g composite omega bar is nothing but z. But, g composite omega bar is p of that is nothing but the same omega bar; because p composite G is nothing but p itself.)

Now it is clear that $\bar{\omega}$ is a loop iff $g \circ \bar{\omega}$ is a loop, because g is a homeomorphism. (You take a path omega lifted at x bar; it maybe a loop or it may not be a loop.

If it is a loop, then g of that is a loop for all g; and that will give you for all elements, because of a surjectivity start with z. The loop the lifts of omega at all the points z in the fibre, they are all loops. Or, if it is one of them is not a loop, then none of them is a loop; so that if and only if omega bar is a loop; this is a loop if and only if omega bar is a loop.) This proves that the covering is normal; but then we have already established covering is normal iff the subgroup K is normal. So, surjectivity of Φ gives you normality of K.

So, in particular phi is an isomorphism; it is it is surjective also k must be normal. Now, we can do the converse. Suppose, K is normal; then I have to prove that Φ is surjective, and is a homomorphism. While proving normality, I did not have to worry about Φ being a homomorphism or not; just the bijection was enough, surjection was enough. Now, I have to do the other way round also; namely, once $K$is normal the converse part.

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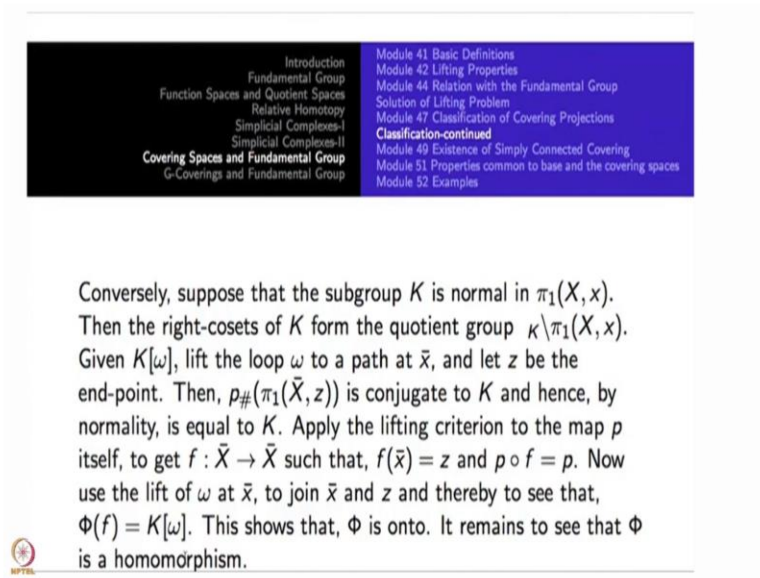

Suppose K is normal subgroup of $\pi_1(X, x)$. Then I have to show that p is a homomorphism and it is surjective; injectivity we have already done. So, what is the meaning of K is normal? The right cosets of K form the quotient group, which we denoted by $K\backslash\pi_1(X, x)$. Otherwise it is only a cosets space. Now, it is a group. Even an element in this group how do you represent? It is by a coset $K[\omega]$. Lift this ω to a path at $\bar{x}$; look at its end-point z. $p_{\#}(\pi_1(\bar{X}, z) \subset \pi_1(X, x)$ i.e., look at the fundamental group $\pi_1(\bar{X}, z)$ and take its image under $p_{\#}$. These things we have seen. It is equal to the conjugate to K by same element $[\omega]$. From $\bar{\omega}$ you comeback. By the normality a conjugate of K is equal to K.

So, what you have is you have two copies of $p : \bar{X} \to X$ but, the base point you have changed. One is $\bar{x}$ and another one is z. We apply lifting criteria to the map $p : (\bar{X}, \bar{x}) \to X$, through the map $p:(\bar{X},z)\to X$. Since $p_{\#}(\pi_1(\bar{X},\bar{x}))\subset p_{\#}(\pi_1(\bar{X},z))$, the map can be lifted. So, applying the lifting criteria to one of the maps the map p itself; so both of them are p, but the base points are different. So, you can demand that $\bar{x}$ goes to z and get a function f so that $p\circ f=p$ and $f(\bar{x})=z$. This is means that it is a lift of p through p itself. So, p composite f is p and p of $\bar{x}$ bar is z. Now, reverse the role, lift the map $p:(\bar{X},z)\to X$ through $p:(\bar{X},\bar{x})\to X$. What you get?

You will get a $g:\bar{X}\to\bar{X}$ such that $g(z)=\bar{x}$ and $p\circ g=p$. But then, $p(g\circ f)=p, p(f\circ g)=p$. Also $gf(\bar{x})=\bar{x}; f\circ g(z)=z$. g By the uniqueness of the lifting, we get $g\circ f=Id_{\bar{X}}=f\circ g$. That means f and g are homeomorphisms, which means that f is covering transformation. So, what I have done is I have done something more than this one. So here what I have? I have first shown that the normality implies that the map $\Theta\circ\Phi:G(p)\to p^{-1}x$ is surjective first; so, it is a bijection.

So, how do you do that? By the lifting criterion get $f\in G(p)$ such that $f(\bar{x})=z$. Since Θ is already known to a bijection, this shows that Φ is onto.

Now, you have to show that this Φ is homomorphism, surjectivity is done; it is a homomorphism, so you have to show that. So, here is the workout.

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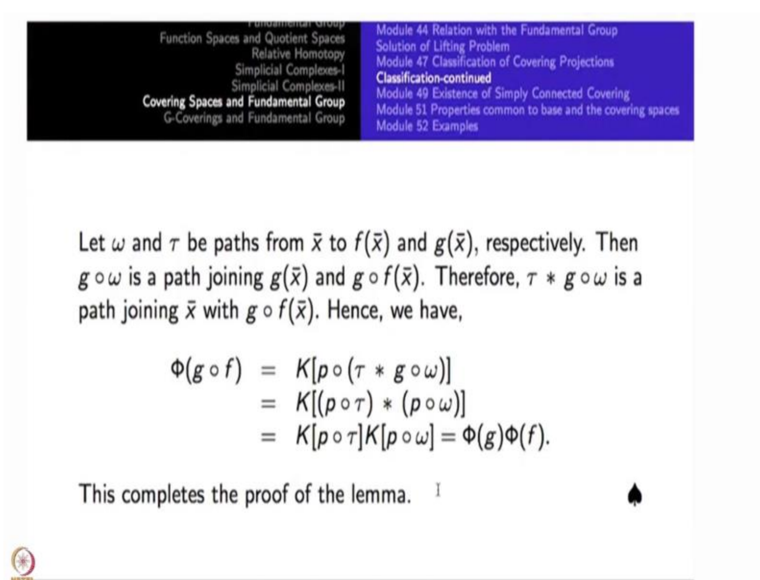

Given $f, g \in G(p)$, take paths ω, τ so that $\omega(0) = \bar{x} = \tau(0)$, $\omega(1) = f(\bar{x})$ and $\tau(1) = g(\bar{x})$. Then, if you take $g \circ \omega$, it is a path in $\bar{X}$ from $g(\omega(0))$ to $g(\omega(1))$, i.e., from $g(\bar{x}) = \tau(1)$ to $g \circ f(\bar{x})$. All these paths are inside $\bar{X}$ itself; $g \circ \omega$ is a path from $g(\bar{x}) = \tau(1)$ to $g \circ f(\bar{x})$. because I am applying g to the path ω; initial point is $\bar{x}$ for ω and end-point is $f(\bar{x})$. Therefore g of that will be initial point for $g \circ \omega$ and the end-point will be $g \circ f(\bar{x})$. Therefore, if you take $\tau * (g \circ \omega)$, this is another path in $\bar{X}$. τ starts at $\bar{x}$ and ends at $\tau(1) = g(\bar{x})$; and this one starts at $g(\bar{x})$ and ends at $g \circ f(\bar{x})$. Therefore, this path is from $\bar{x}$ to $g \circ f(\bar{x})$.

Therefore, by the definition of Φ, we have $\Phi(f) = K[p \circ \omega]$; $\Phi(g) = K[p \circ \tau]$ and $\Phi(g \circ f) = K[p \circ (\tau * (g \circ \omega))]$. it will be the end-point of phi composite f will be the end-point But, what is p of this path? It is $p \circ (\tau * g \circ \omega) = (p \circ \tau) * (p \circ \omega)$, because $p \circ g = p$. By the group law on the quotient group of right cosets, we get $\Phi(g \circ f) = (K[p \circ \tau])(K(p \circ \omega]) = \Phi(g) \circ \Phi(f)$. (How to multiply right cosets? $(Ka_1)K(a_2) = K(a_1 a_2)$.)

So, you see you had taken left cosets, then you will not get a group homomorphism; but you would have got an anti-group homomorphism. The way we write the composite of functions is the reason for that. So, here you get the correct law provided you take right cosets.

So this completes the proof of the lemma. So, let us go ahead; this is only a beginning as it is, as the name indicates. So, some immediate corollaries of it are stunning; this is something which you can remember very easily.

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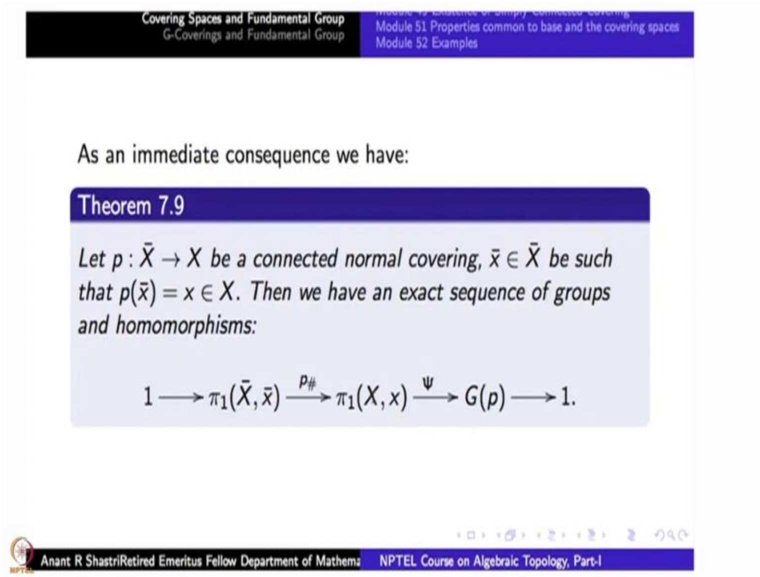

So, what it says? Start with connected normal covering blah blah blah; all those basic assumptions are there. We have to assume X is locally path connected, path connected beyond that. Take a point $\bar{x}$ in the covering space, such that it comes to x in the bottom. Then we are in the exact sequence of groups and homomorphisms. What is it? The trivial group to $\pi_1(\bar{X}, \bar{x})$ followed by $p_\# : \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, x)$. This part is exact just means that the kernel of $p_\#$ is equal to image of this, which is equal to (1). That means $p_\#$ is an injective. This part we have already we have already established.

Now, we have another group here; another group of homomorphisms $\Psi : \pi_1(X, x) \to G(p)$ the group of covering transformation. And the last one is trivial means that Ψ is a surjective homomorphism. In the middle we have Kernel of Ψ is precisely equal to $p_\#$ of this, which is the normal subgroup K. So this follows easily by the first isomorphism theorem of group theory, so here it is. All I have to know is that this notation $K = p_\#(\pi_1(\bar{X}, \bar{x}))$.

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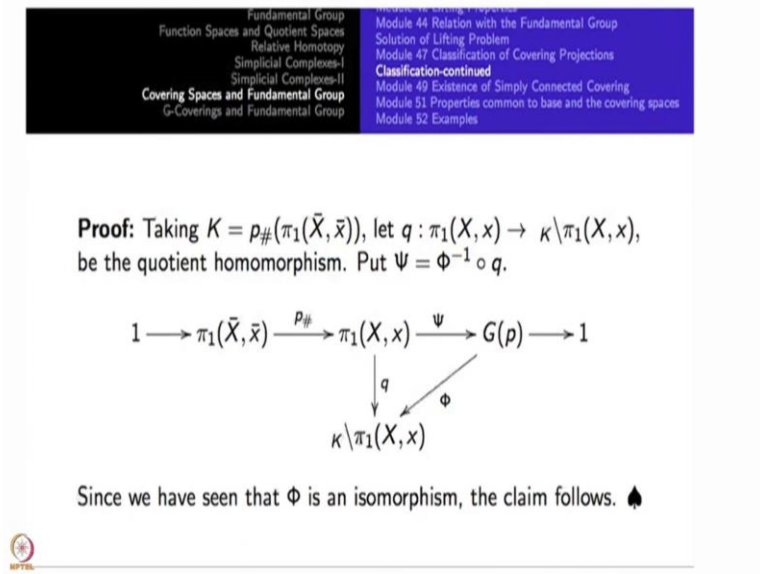

Then, look at q which is the quotient map, quotient homomorphism from $\pi_1(X, x)$ to the right cosets, this is a quotient map. Put $\Psi = \Phi^{-1} \circ q.$ Since q surjective, it follows that Ψ is surjective. Since kernel of q is equal to K, so is kernel of Ψ.

(Ignore the rest of the explanation here: We have established that Φ is an isomorphism from what? From where to where? This lemma are where lemma what it is say? Phi is an isomorphism from; of course I should not expect you to tell this, so I am. So, what we have seen in this lemma I am just recalling; this group of covering transformation so to the right cosets. So, what you have is a first (try to) morphism this picture; given isomorphism this way is when isomorphic here map here is a right cosets.

The right cosets to G p this is an isomorphism; but this map is this, this is a this is surjective mapping. The kernel of q is same thing as p check, by the very definition of right cosets and the quotient group. K is nothing but image of this and this is injective. So, by the image if you go go to the quotient map, this is an exact sequence. To change this one by this map and go here, and what is the definition of psi? You just try as, you just take this phi inverse, phi composite phi inverse. Go back here q composite phi inverse; so that is chi.

So, it is actually first isomorphic then; if they are homeomorphism such that, right cosets have gone to the same point. That means this kernel of this one is K. Therefore, modulo K it is inside the morphism, whichever way you like. So, I have fairly here write down, if you if you do not know the first isomorphism; then also this one will fall. All that you have to know is the quotient quotient map as kernel is precisely K; and that is all and quotient map it is surjective map.

So, this quotient is get identified with G p. So, instead of writing this group here; I am writing G p now; which is more descriptive in terms of our covering transformation. You have fundamental group here, you have sub subgroup K and then you are going up here; and so fundamental group of this one is up stairs. The covering transformation of that is a quotient group, so this is the picture; so, this is now theorem which we keep referring to.)

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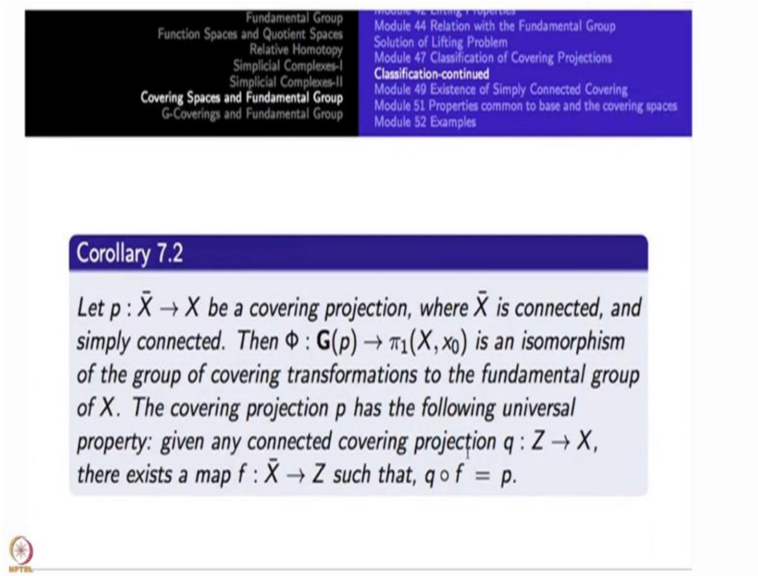

Corollary. This is again an interesting corollary, namely, you start with a connected covering space etcetera. Suppose the top space is simply connected, which means as in the earlier lemma, K is trivial. Then, quotient is same thing as the fundamental group itself; Therefore $G(p)$ will be isomorphic to $\pi_1(X, x)$ itself. So, this $\Phi : G(p) \to \pi_1(X, x)$ itself, because now the right cosets are nothing but elements of $\pi_1(X, x)$. Therefore, this Ψ itself is an isomorphism oform the group of covering transformation to $\pi_1(X, x)$. The covering projection p has the following universal property; this is the extra thing that we have that we have to digest, which is not at all difficult.

What it says, given any covering projection q from Z to X, there exists a map $f : \bar{X} \to Z$ such that $q \circ f = p$. That is, p which is a covering projection from a simply connected space factors into f composite q, where q is any given connected covering projection onto X. Thus you see that a simply connected covering space $\bar{X}$ as a big space which `sits' over all coverings of X. Only thing we have to assume that Z is connected. That is why $\bar{X}$ is called as universal covering. We are going to say what is universal property first; we are going to make a definition because of this property. So, here is what I have drawn a picture.

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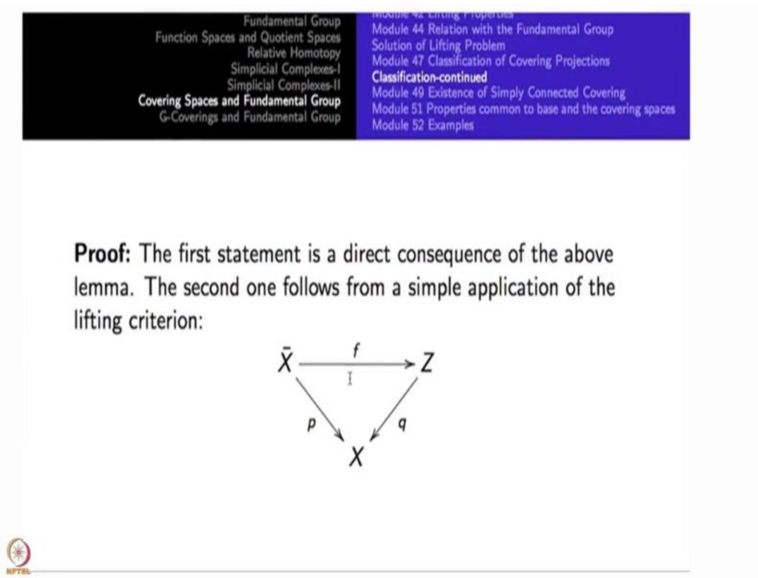

X is here, p is the covering projection; and this is connected and simply connected, π_1 is tirvial and so its image is also trival. By the lifting criterion, says that this can be lifted. Once trivial group it is inside the image of q check; that was the lifting criterion, so it can be lifted up. In fact, there will be several lifts depending upon the first base point one, base point where you want to go. We have to fix the base point. Then it is unique. So, this p is factored into two maps. Here q is given; you have some choice in f , but not much.

In fact, this f itself becomes a covering projection; right now we are not proving that presently. So in conclusion, this we begin with a simply connected covering $\bar{X}$ of X, then it will sit over every covering of X. What we will prove that this f itself is a covering; so universal covering is a covering for everybody. So, that is why we are making a definition here, namely universal covering; so, let us look at with definition now.

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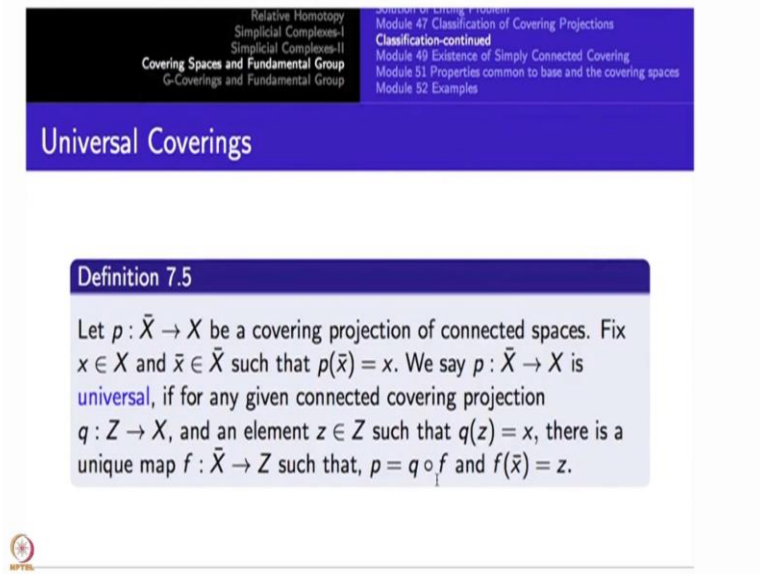

Start with a covering projection $p : \bar{X} \to X$. Let us assume that all spaces are connected. Fix a base point $x \in X$ and a base point $\bar{x} \in \bar{X}$ above x; $p(\bar{x}) = x$. We say p is universal covering projection-- you can just say `universal', if for any given connected covering projection $q : Z \to X$, and an element $z \in Z$, such that, $q(z) = x$, (see, $\bar{x}$ and z are both sitting over x; that is necessary, as soon as this much is mentioned), there is a unique map $f : \bar{X} \to Z$, such that $q \circ f = p$ and $f(\bar{x}) = z$.

Uniqueness follows by unique lifting property; existence follows by lifting criteria, in the case $\bar{X}$ is simply connected.

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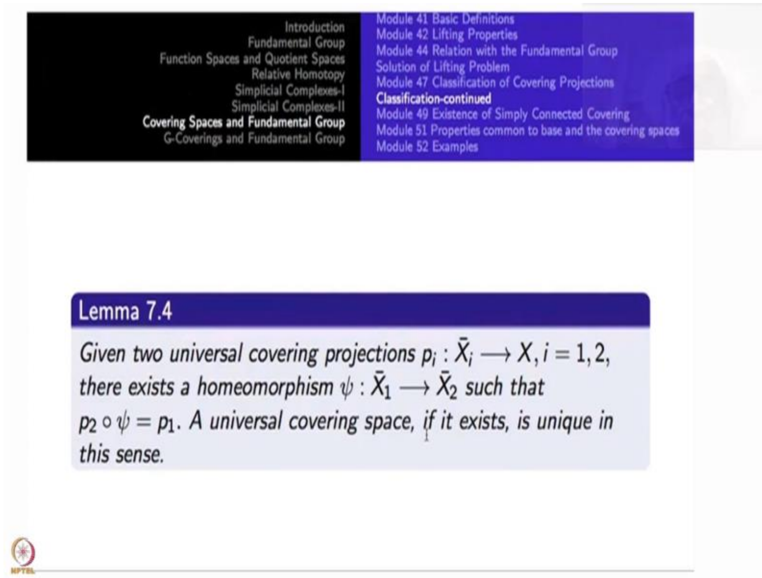

So, we have just made a definition, we have just given the definition of universal covering. Let us now see one important property of universal coverings. Given two universal coverings, $p_i : (\bar{X}_i, \bar{x}_i) \to (X, x), i = 1, 2$ of the same space X, you do not have much choice, namely there are homeomorphic to each other. Actually, the equivalence of coverings they are the same name; namely, there a homeomorphism $\psi : (\bar{X}_1, \bar{x}_1) \to (\bar{X}_2, \bar{x}_2)$ such that $p_2 \circ \psi = p_1$. Remember, such a thing is you have defined equivalence of covering projection.

Therefore, a universal covering space if it exists is unique up to the equivalence. There is only one equivalence class of universal covering space. Maybe there is none, we do not know that yet; It happens that universal covering space is may not exist.

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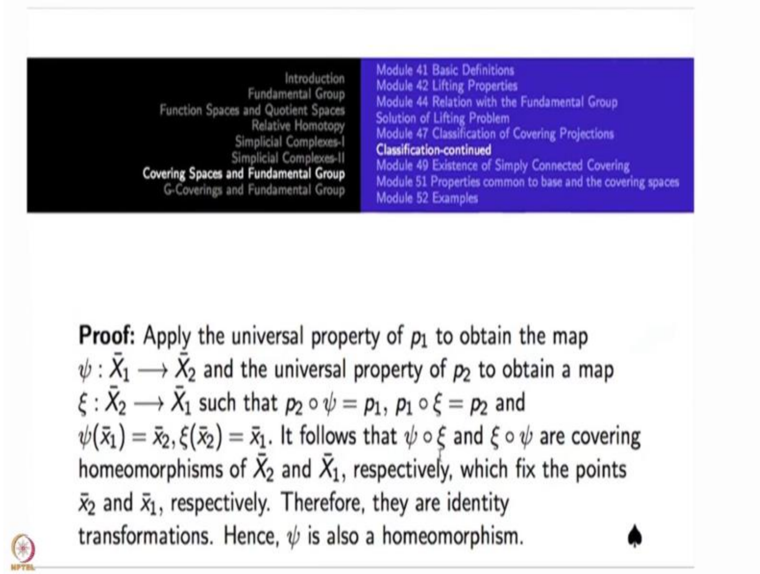

So, how to show that this uniqueness? Very easy; you use the universal covering both ways. You will get maps from the other one way, other way and so on; just now I have shown that f is a homeomorphism and so on; just here same thing you have to do. Apply universal property of p1 p_1 to obtain the map $\psi : \bar{X}_1 \to \bar{X}_2$ and the universal property p_2 to obtain a map $\xi : \bar{X}_2 \to \bar{X}_1$. These maps satisfy $p_2 \circ \psi = p_1, p_1 \circ \xi = p_2, \psi(\bar{x}_1) = \bar{x}_2;\ \xi(\bar{x})_2) = \bar{x}_1$.

Now, you look at $\psi \circ \xi(\bar{x}_2) = \bar{x}_2$ and $\xi \circ \psi(\bar{x}_1) = \bar{x}_2$. Therefore, by the uniqueness part, they must be a identity maps; which means one is the inverse of the other. So, this is why universal covering space if it exist, it is unique.

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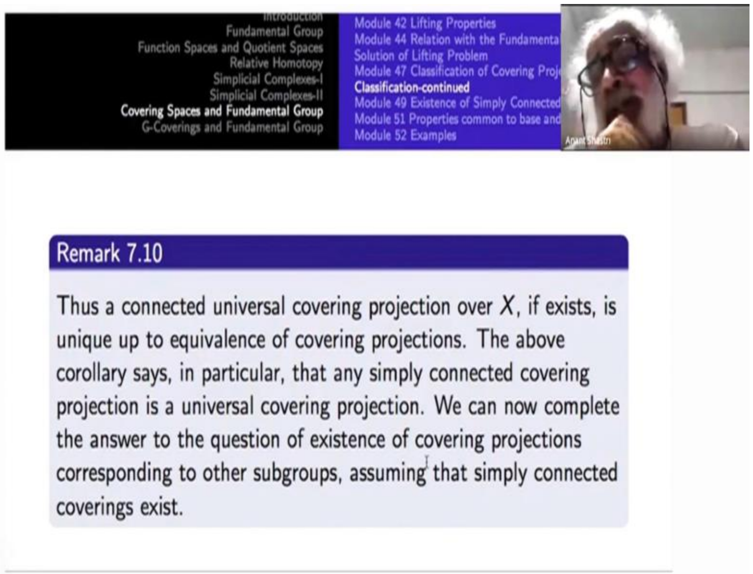

So, this is the remark. Connected universal covering space over X, if it exists, is unique up to equivalence of covering projections. In particular, previous corollary says that simply connected covering are universal coverings. Simply connected coverings are universal coverings; but we still do not know whether they exist. We can now complete the answer from the question of existence of covering projection, corresponding to other subgroups, assuming that simply connected covering exist. So, that is the next project; we will stop here.