Introduction to Algebraic Topology (Part I)
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Lecture 47
Classification of Coverings

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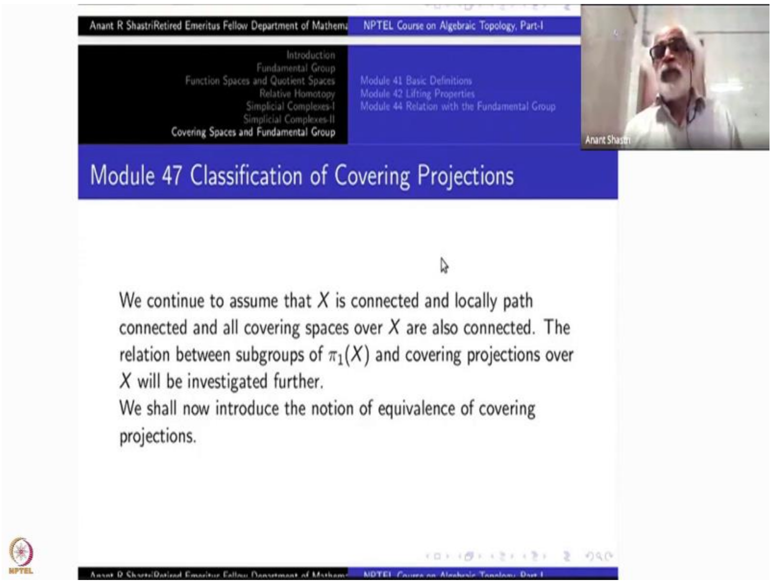

So, today we shall take up another theoretical topic, classification of covering projections, like classification of spaces, now we are classifying the functions which are all covering projections. Obviously, what we are going to do is we are going to fix the base space, base space is fixed, up to homeomorphism. The covering projections will be classified up to homeomorphism of the top space, which take one projection to another projection.

It is not just classification of the top spaces, up to homeomorphism. The homeomorphisms should preserve the covering projection. So, this leads to what are known as covering transformations and so on. So, we shall go on introducing some of these. And finally, we will see that they again use group theory namely they are closely related to subgroups of $\pi_1(X)$. So, let us go through this topic.

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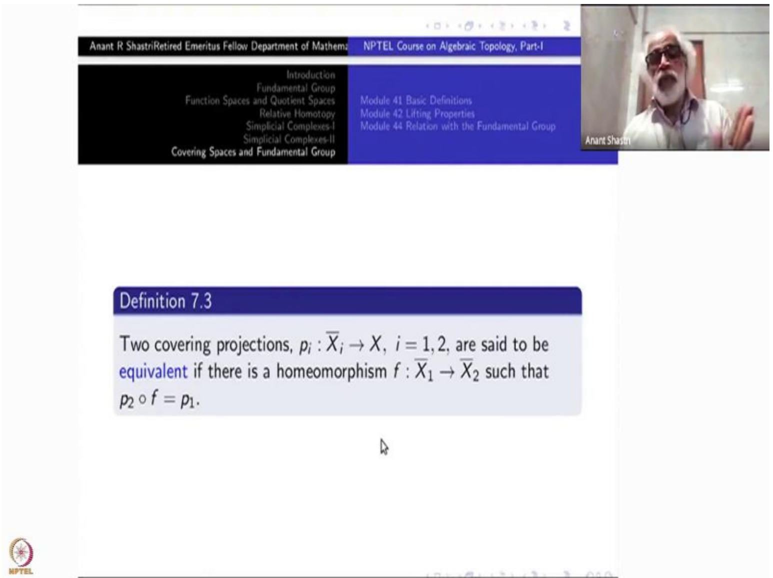

So, here is the definition. Take two covering projections $p_1 : \bar{X}_1 \to X$ and $p_2 : \bar{X}_2 \to X$ over X. We say they are equivalent if there is a homeomorphism $f : \bar{X}_1 \to \bar{X}_2$ from X1 bar X2 that when you compose it with p_2 it gives you p_1. So, the the projection map p_1 is taken to p_2 by f. So that is the meaning of this one. Is the definition clear?

Now, this is usually seen that it is an equivalence relation first of all. If there is a map f like this you take f^{-1} that will give you $p_1 \circ f^{-1} = p_2$. So, this means that p_1 is equivalent to p_2 implies p_2 is equivalent to p_1; every p_1 is equivalent to itself --- no problem. Again, if there is a $p_3 : \bar{X}_3 \to X$ and a homeomorphism $g : \bar{X}_2 \to \bar{X}_3$ such that $p_3 \circ g = p_2$, composite G equal to then you take $p_3 \circ g \circ f$, that will be equal to p_1. So, that will show that $g \circ f$ is a covering transformation and gives an equivalence between p_1 and p_3. So, this is a transitive relation. So, equivalence under this one is actually equivalent, you are calling it equivalent, so it is the equivalence relation. So, it is justified, so equivalence classes of covering projections are what we want to study.

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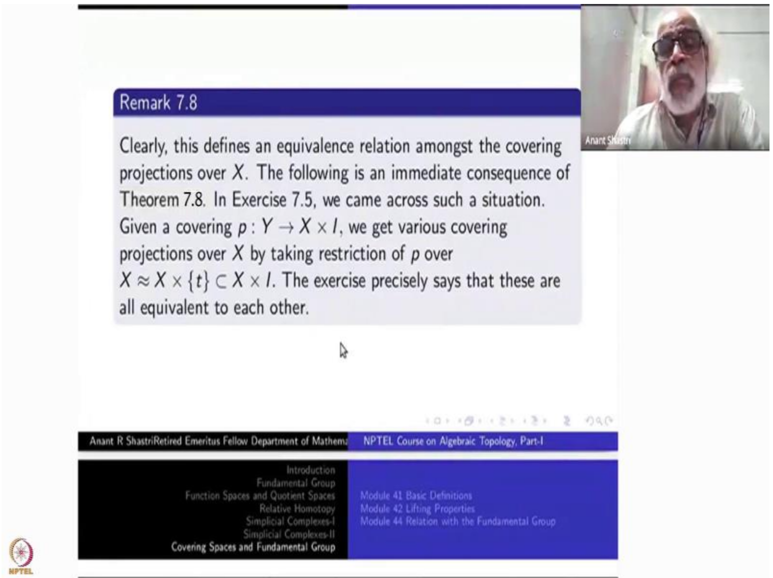

Given a covering projection $p : Y \to X \times \mathbb{I}$, remember there is an exercise given to you earlier, restricted over $X \times \{t\}$, you get various covering projections. And the exercise was that these are all equivalent without giving the definition of equivalence. Now in terms of this definition what you have to show is that all these covering projections, when you think of $X \times \{t\}$ as a copy of X, just by forgetting t, i.e., under that homeomorphism $x \mapsto (x, t)$ all these covering projections restricted $X \times \{t\}$, are all equivalent to each other, that was the exercise.

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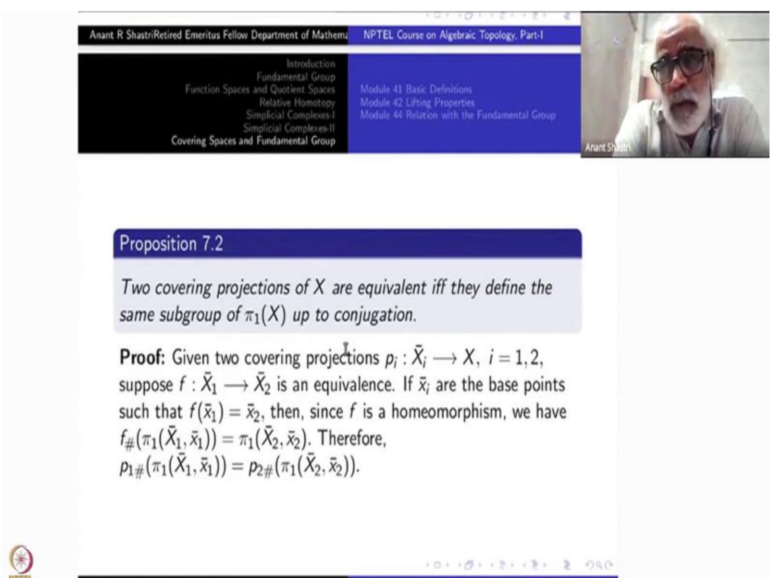

So, that is an example provided you have solved that, we will see some more things like that here. Two covering projections over X are equivalent if and only if they define the same subgroup of $\pi_1(X)$ up to conjugation. Therefore, if two corresponding sub groups are not conjugate to each other then you will know that the covering projections are not equivalent. So, this is the starting point of relating the equivalence classes of covering projections, to subgroups of $\pi_1(X)$.

So, let us see how this one comes, if you want a straight one-line proof: p_1 and p_2 are covering projections $f : \bar{X}_1 \to \bar{X}_2$ is an equivalence. Let us, pick up base points $\bar{x}_1, \bar{x}_2$ sitting over the same base point $x \in X$ such that $f(\bar{x}_1) = \bar{x}_2$. (This can be ensured by first choosing $\bar{x}_1 \in p_1^{-1}(x)$ and then putting $\bar{x}_2 = f(\bar{x}_1)$.

Since f is a homeomorphism, we have $f_{\#}(\pi_1(\bar{X}_1, \bar{x}_1)) = \pi_1(\bar{X}_2, \bar{x}_2)$. This we have seen because f is homeomorphism. Therefore, when you apply $(p_1)_{\#}(\pi_1(\bar{X}_1, \bar{x}_1)) = (p_2)_{\#}(f_{\#}(\pi_1(\bar{X}_1, \bar{x}_1))) = (p_2)_{\#}(\pi_1(\bar{X}_2, \bar{x}_2))$.

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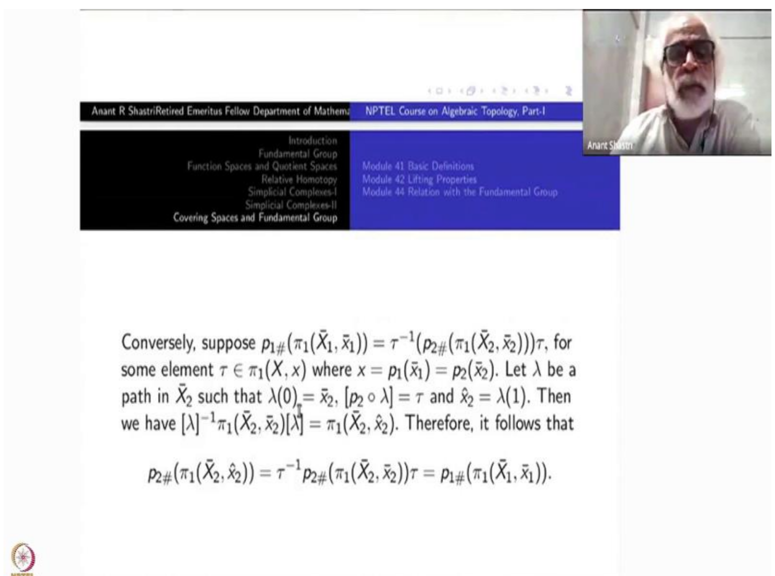

So, conversely, now suppose these two are conjugate to each other. This time I do not know what point I have to take. I have taken two subsgroups $(p_1)_{\#}(\pi_1(\bar{X}_1, \bar{x}_1))$ and $(p_2)_{\#}(\pi_1(\bar{X}_2, \bar{x}_2))$ at some other points $\bar{x}_1 \in \bar{X}_1$ and $\bar{x}_2 \in \bar{X}_2$. (Of course the only thing I know is that $p_1(\bar{x}_1) = p_2(\bar{x}_2) = x$.) Suppose these subgroups are conjugate to each other, by some element $\tau \in \pi_1(X, x)$. There is no f here. I have chosen $\bar{x}_1, \bar{x}_2$ sitting over the same point $x \in X$. Then only I get two subgroups of $\pi_1(X, x)$ and I am assuming that these two subgroups are conjugate to each other, that means this one is conjugate of that one.

Now, I will produce a map $f : \bar{X}_1 \to \bar{X}_2$ which takes $\bar{x_1}$ to $\bar{x}_2$ and this map f will be a homeomorphism such that when you compose f with p_2, it will be equal to $p_1.$ So that is what we want. So, let λ be a path in $\bar{X}_2$ such that its starting point $\bar{x}_2$, $[p_2 \circ \lambda] = \tau \in \pi_1(X, x).$ So, I have lifted a representative loop of the class τ that is what λ is. Let the endpoint of this λ be denoted by $\hat{x}_2$.

Then we have $[\lambda]^{-1}(\pi_1(\bar{X}_2, x_2))[\lambda] = \pi_1(\bar{X}_2, \hat{x}_2).$ This is, we know, λ is a path $\bar{x}_2$ to $\hat{x}_2$. So, a loop at $\bar{x}_2$ will get converted into loops at $\hat{x}_2$, and we know $h_{\lambda^{-1}}$ is the isomorphism. Therefore, it follows that if you apply $(p_2)_\#$, on one side we get $(p_2)_\#(\pi_1(\bar{X}_2, \hat{x}_2).$ And on the other side, $(p_2)_\#([\lambda^{-1}] = \tau^{-1}$, then we have $(p_2)_\#(\pi_1(\bar{X}_2, \bar{x}_2))$ followed by τ. WE have assumed that this latter group is equal to $(p_1)_\#(\pi_1(\bar{X}_1, \bar{x_1})).$

So, what I got now is, for some other point $\hat{x}_2 \in \bar{X}_2$, the image of the fundamental group is the same as the image of $\pi_1(X_{\bar{x}_1}).$ So, equality was given in the in the earlier case, in the forward case, in the backward case, we had get this one after starting with a conjugate, the conjugate is that when you fix your points yourself.

Now, the conjugate will tell you what point you should take so that two groups become equal that is the idea. So, now $(p_2)_\#$ and $(p_1)_\#$ have the same image, because I have changed the base point in $\bar{X}_2$. Equivalently I could have changed the base point in $\bar{X}_1$, by reversing the arguments, these are symmetric argument. So, now what?

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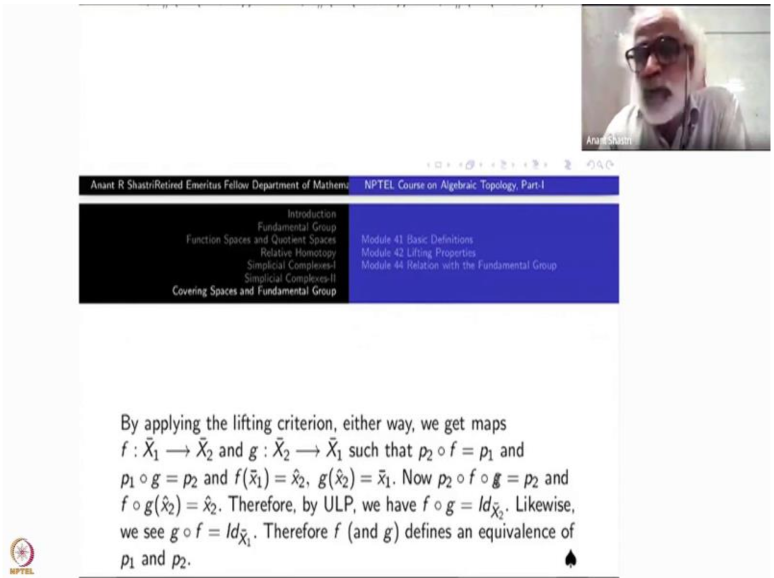

Now, you apply the lifting criterion. You see I have told you that we keep on applying the lifting criterion to get map $f : \bar{X}_1 \to \bar{X}_2$ or $g : \bar{X}_2 \to \bar{X}_1$, taking p_1 to p_2 or the other way. So, you have $p_2 : (\bar{X}_2, x_2) \to (X, x)$ and $p_1 : (\bar{X}_1, \bar{x}_1) \to (X, x)$, put $Y = \bar{X}_1$ apply the lifign criterion to get $f : Y \to \bar{X}_2$, but Y is nothing but $\bar{X}_1$. So, you can do the other way around also. So, both ways you will get maps f and g such that $p_2 \circ f = p_1$, and $p_1 \circ g = p_2$. Also, we have $f(\bar{x}_1) = \hat{x}_2, g(\hat{x}_2) = \bar{x}_1$. So, this is the starting point condition. Whenever you are lifting you are specifying the image of one point, the lift of one single point you have to specify. Now, if you take $g \circ f$ or $f \circ g$ wkat happens? $p_2 \circ (f \circ g) = p_1 \circ g = p_2$, and $p_1 \circ (g \circ f) = p_2 \circ f = p_1$.

So, $g \circ f$ (respectively, $f \circ g$) is a lift of p_1 through p_1 (respectively, is a lift of p_2 through p_2). But identity map is also a lift and $g \circ f(\bar{x}_1) = \bar{x}_1$. So, identity map and $g \circ f$ agree at one point and $\bar{X}_1$ is connected (of course, we have to assume this, we have assume that all spaces involves are connected.) Therefore, we conclude that $g \circ f = Id_{\bar{X}_1}$, by the uniqueness of the lifts. Similarly, $f \circ g = Id_{\bar{X}_2}$.

So, f, g are inverses of each other automatically. So, that gives you that lifting problem which you have solved is used here to show that if the fundamental group level p1 check and p2 check give you conjugate subgroups, then the coverings are equivalent, what are the assumption? Assumptions that they are connected coverings.

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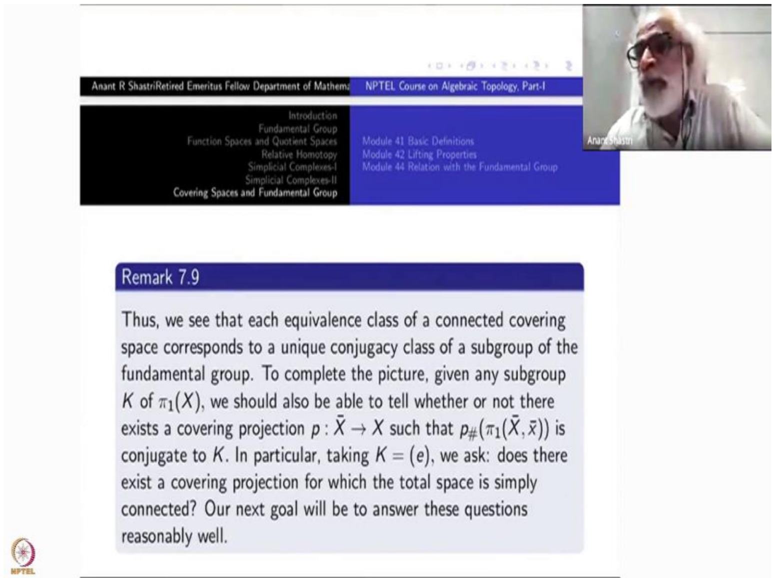

So, what we have seen is that an equivalence class of a connected covering space correspond to a unique conjugacy class of a subgroup of the fundamental group of the base space. In fact, take a covering projection $p : \bar{X} \to X$. You have freedom to choose the base point above inside the fibre $p^{-1}(x)$. As you keep changing the base point, the subgroups will become different but they are all conjugate, to within a space if you change the space if you change a base point, what happens is, you are getting conjugation loops.

Nothing else happens if you change the whole space but keep the group as conjugate to itself, then it has to be a homoeomorphic to it, the base points may get shuffled, that is all. If you fix the base points also one going to other even the same conjugacy subgroup will get no other conjugate subgroup. The number of conjugacy classes will also depend upon these number of, number of these things up to something, of course several of them may give you same conjugacy class.

So, to complete the picture given any subgroup K of $\pi_1(X, x)$, we should also be able to tell whether there exist any covering correspond to K So, this question is not discussed yet. So, what we have got so far is: each equivalence class of a covering goes to a single conjugacy class of a subgroup. If you have subgroup, will there be a covering which comes to that, that is the fundamental problem here, namely, ontoness of this assignment .

If two coverings go to the same conjugacy class of a subgroup, then they are equivalent. So that is, the assigment taking an equivalence class to a conjgacy class of a subgroup is injective mapping. To complete the picture we ask whether it is surjective, surjective means what? Given any subgroup $K \subset \pi_1(X, x)$, can you construct a covering $p : \bar{X} \to X$ such that $p_{\#}(\pi_1(\bar{X}, \bar{x}))$ is a conjugate of K?. Of course, once it comes to one of the conjugates you can actually make it a subgroup also by changing the base point in $\bar{X}$. That much you know already.

In particular suppose I take the trivial group $K = (1)$, the trivial group. That is is also allowed, you have to answer this question to all the subgroups of $\pi_1(X, x)$. So, take the trivial group, what is the corresponding covering space? Does there exist a covering projection for which the total space is simply connected? Remember p check is injective mapping, if $p_{\#}(\pi_1(\bar{X}, \bar{x})) = (1)$ it follows that $\pi_1(\bar{X}, \bar{x}) = (1)$. So, given any space do we have a covering that has that total space simply connected? -- the top space is simply connected?

So, this is a special question, I mean a particular case of the general question. It turns out that the method of attacking this problem is first to solve this problem, the simplest problem, namely the case $K = (1)$. Once you solve this one, it will tell you how to get other groups. So, that is the way it is done, I mean in theory, how to solve this problem, you solve the simple problem first. Our next goal is to ask this now solve this problem.

So, I repeat, instead of going to next topic, I repeat this one, what we want is now starting with a space which is locally path connected and path connected, let us assume that, take any point, does not matter which one, you get the fundamental group of that space at that point. Look at a subgroup, will there be a covering projection corresponding to this subgroup means top space of the covering projection must have fundamental group isomorphic to the subgroup under $p_{\#}$? $p_{\#}$ is a monomorphism, so onto the image it will be an isomorphism. So, this is our goal. So, let us make a few definitions.

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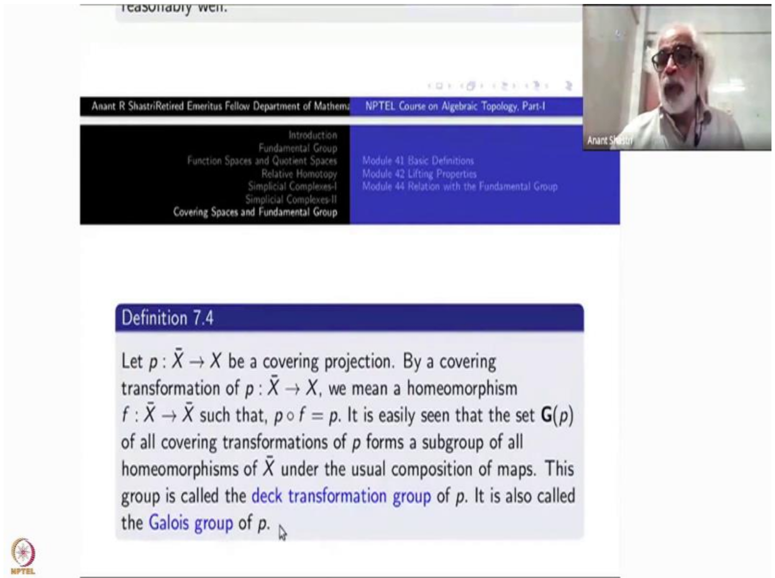

Starting with a covering projection $p : \bar{X} \to X$. What is the meaning of covering transformations? So, let us understand this one. $\bar{X}$ to $\bar{X}$ itself, f is a homeomorphism such that $p \circ f = p$. So, in other words it is an equivalence of X with itself. There may be several of them. Togehter they will form a group under composition, because if f and g are covering transformations of p, then $g \circ f$ is also a covering transformation of p. So is f^{-1}. Therefore, under composition of functions, this forms a group which we will denote by $G(p)$.

$G(p)$, the set of all covering transformations of p forms a subgroup of all the group of all self-homeomorphisms of $\bar{X}$, because each of them is a homeomorphism. And under the usual composition of maps, this becomes a subgroup and is called the deck transformation group. It is also called the Galois group of p. The deck transformation group of p or the Galois group of p, will of course depend upon p. it is not the group of all homeomorphisms of $\bar{X}$ in general. This is the definition of the group of deck transformations, or the Galois group of p.

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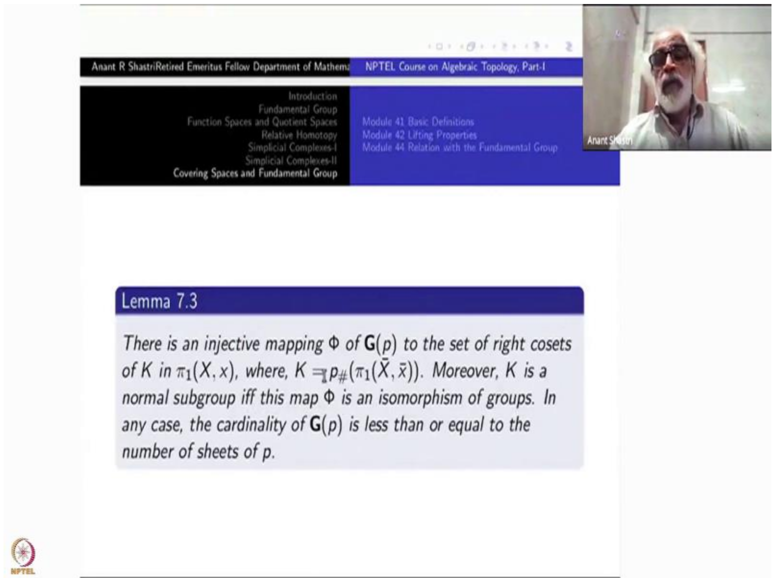

The basic lemma, again a group theoretic one here is the following: There is an injective map, just set theoretic function Φ from $G(p)$ into the set of all right cosets of $K = p_{\#}(\pi_1(\bar{X}, \bar{x}))$ in $\pi_1(X, x)$. Remember the right cosets of K have played some good role here earlier. I will recall it again. K is the image of $\pi_1(\bar{X}, \bar{x})$ under $p_{\#}$.

So, a subset of the set of all right cosets of K corresponds $G(p)$ under the injective function Φ. Moreover, K is a normal subgroup if and only if this Φ is an isomorphism, that means not only this function Φ, which we are going to define, is a bijection, it will be homomorphism, and hence, an isomorphism. In general, it is just a set theoretic function. So, K is a normal subgroup if and only if Φ is an isomorphism of groups.

In any case the cardinality of $G(p)$ is always less than or equal to the number of sheets of p, which is precisely equal to the number of points in p^{-1} of any point. That is the number of sheets of p. Why suddenly number of sheets come? Look at this. We have a bijection of the right cosets with the fibre as seen earlier. So, this is nothing but the right cosets as a set. So, the cardinality of a fibre of p= number of sheets of p, and Φ is an injective map into this one.

So, the cardinality of $G(p)$ will be less then cardinality of the fibres. So, this is the lemma, so non trivial part is to define this mapping and what? And show that it is injective and then, if K is normal, then this becomes surjective and the map becomes a homomorphism. So, this thing we will do next time, not today that is the plan. Thank you.