**Introduction to Algebraic Topology (Part I)**
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**Lecture 46**
**Lifting Problem**

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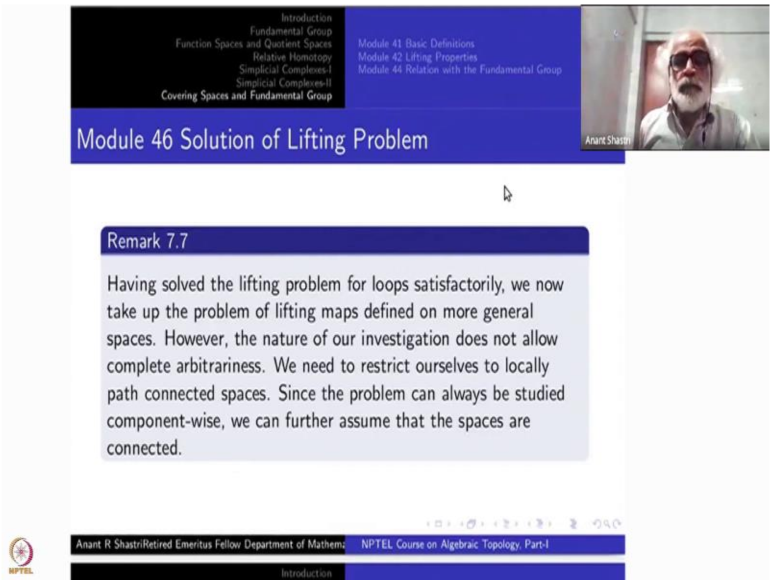


Hello, everybody. So, today's topic is solution of Lifting Problem. We have already solved this problem, in the simplest case namely for a loop, a loop based at a point $x_0$ can be lifted through the covering projection if and only if the loop represents an element of $\pi_1(X, x_0)$ should be in inside the image of $p_\#$, where $p$ is the covering projection, this is what we have seen.

So, now we want to expand this result to arbitrary spaces. The point here is, the space cannot be completely arbitrary, we need to assume some kind of hypthesis to ensure that enough paths must be there inside the space. So, we assume that it is locally path connected. And of course, after that we can assume that it is connected that is not a restriction, because we can always argue component wise.

So, let us assume that the space is locally path connected, take a map from this space to $X$, the base space, $p : \bar{X} \to X$ is the covering projection. So, then we ask the question whether this map can be lifted. So, it will depend upon the character of this map f, so what is the condition? The condition turns out to be completely in terms of algebra, so this is the gist of the thing here.

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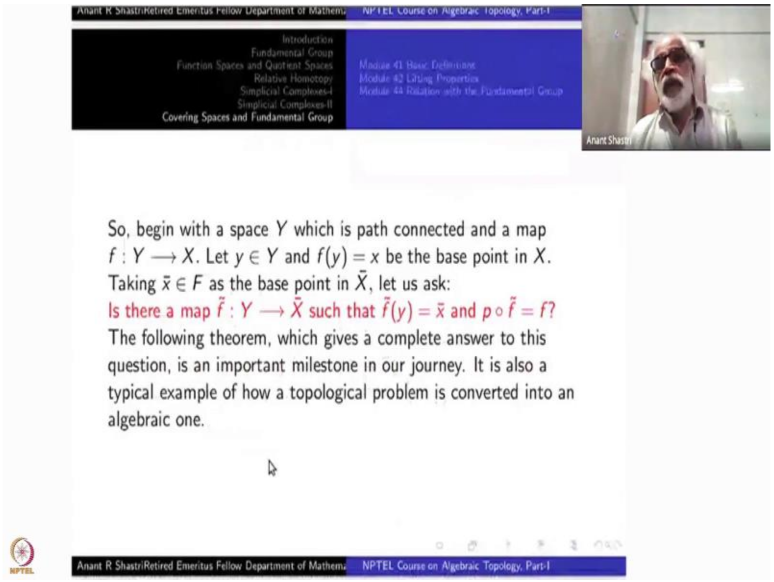


So, this is the question, we have $f : Y \to X$, where $p : \bar{X} \to X$ is the covering. Can you find $f$ a map $\hat{f} : Y \to \bar{X}$ such that $p \circ \hat{f} = f$? So finding a map $\hat{f}$. So, what we want to assume is that $Y$ is locally path connected and connected. So, because of our earlier criteria on loops, lifting of groups, this problem can also be converted into purely algebra. So, this is the theorem, the lifting criteria.

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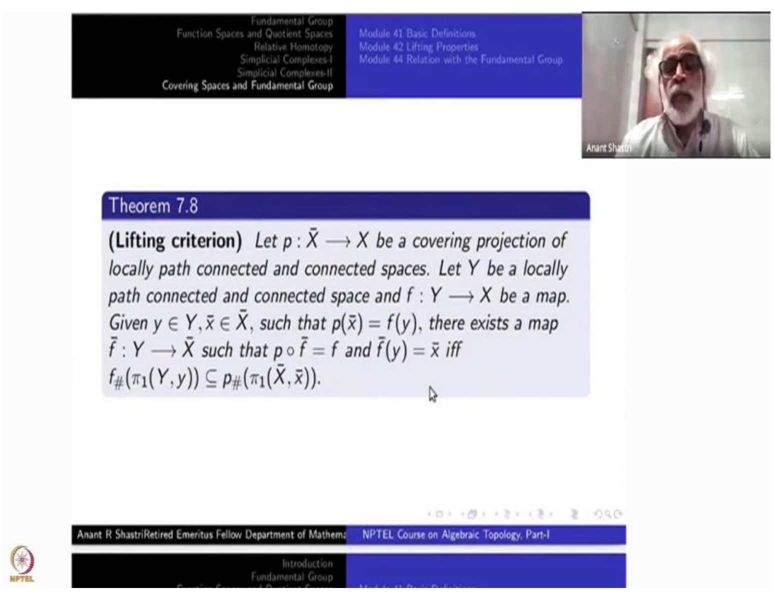


So, as usual $p$ is a covering projection, $Y$ is locally path connected and connected space, $f : Y \to X$ is a any continuous function, and we assume that $y \in Y$, $\bar{x} \in \bar{X}$ such that

$f(y) = x_0, p(\bar{x}) = x_0$. If this is the case, there exists a map $\hat{f} : Y \to \bar{X}$ as we want, namely, such that $p \circ \hat{f} = f$ and $\hat{f}(y) = \bar{x}$ if and only if $\pi_1(Y, y)$, under $f_{\#}$, goes inside $p_{\#}(\pi_1(\bar{X}, \bar{x})$, i.e., $f_{\#}(\pi_1(Y, y)) \subset p_{\#}(\pi_1(\bar{X}, \bar{x}))$.

Notice that left hand side and right-hand side, both are subgroups of $\pi_1(X, x_0)$, because, $f(y) = x_0$ and $p(\bar{x}) = x_0$. So, both of them, $f_{\#}$ and $p_{\#}$ are both homomorphisms into $\pi_1(X, x_0)$ pi 1 of X x. So, these two subgroups should must have the property that this one is entirely contained inside the other. If this happens, iff there is a map $\hat{f} : Y \to \bar{X}$ such that $p \circ \hat{f} = f$ and $\hat{f}(y) = \bar{x}$.

So that is `if and only if'. The `only f' part is very easy, because $p \circ \hat{f} = f$ you understand? So, when you pass on to the fundamental group level, namely, taking $\#$, you get $f_{\#}(\pi_1(Y, y)) = p_{\#}(\hat{f}_{\#}(\pi_1(Y, y))) \subset p_{\#}(\pi_1(\bar{X}, \bar{x}))$. So that is very easy by functoriality property.

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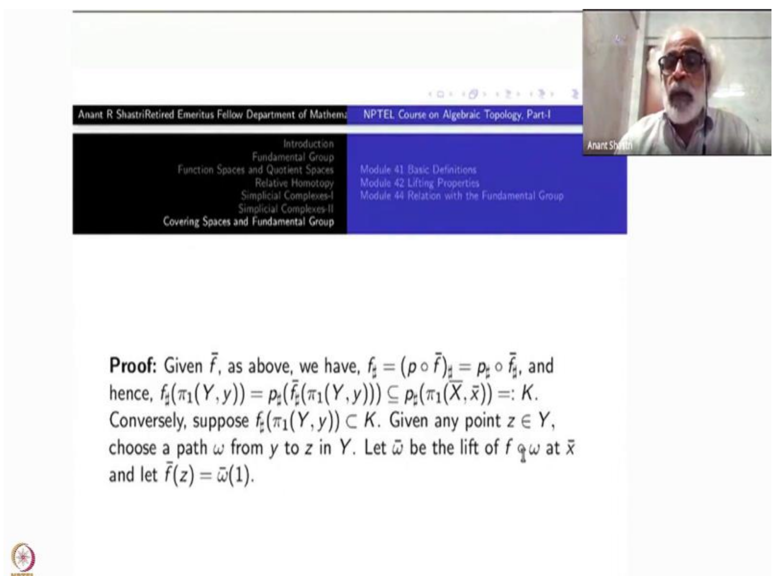


The induced map from the composite is the composite of the induce homomorphism, that is what we have to used. I repeat that we have $f_{\#} = p_{\#} \circ \hat{f}_{\#}$ because we have $f = p \circ \hat{f}$. And hence $f_{\#}(\pi_1(Y, y)) = p_{\#}(\hat{f}_{\#}(\pi_(Y, y)))$. But $\hat{f}_{\#}(\pi_1(Y, y))$ lands inside $\pi_1(\bar{X}, \bar{x})$. So, you have to follow it by $p_{\#}$. Recall that for $p_{\#}(\pi_1(\bar{X}, \bar{x}))$, the notation is $K$. $K$ is the subgroup of $\pi_1(X, x)$ and this subgroup must be contained inside the subgroup $K$.

Conversely, suppose this is contained inside $K$., Now we must construct $\hat{f}$. So here is the procedure which resembles more or less the primitive mapping theorem in two variable calculus that you must have learnt, If you remember you can see the analogy. Given any point $z$ belonging to $Y$, first we choose an arbitrary path from $y$ to $z$ inside $Y$. This is possible because $Y$ is path connected, connected locally path connected means its path connected, this is what I meant by having enough paths.

So, here all that I have used is the assumption that $Y$ path connected. We will use locally path connectedness also. So, start with any fixed $y \in Y$, then for any $z$, join it to $y$ by a path $\omega$. Take a lift of this path $f \circ \omega$ which is inside $X$ and is starting at $x_0$. So, take a lift $\bar{\omega}$, of this one, this will be a path inside $\bar{X}$ starting at $\bar{x}$. Look at its endpoint, the endpoint, I am going to define as $\hat{f}(z)$.

So, this kind of thing we have done right in the beginning, while computing the fundamental group of the circle. So, here for arbitrary function we are defining $\hat{f}(z) = \bar{\omega}(1)$. Now, you have to verify many things. First of all why $\hat{f}$ is well defined? Remember you have chosen some paths here, $y$ to $z$, there may be many paths, in fact, once $Y$ is path connected, there will be many paths. If I change the path, the end point of the lift may be different. So, I have to ensure that irrespective of what path you choose, the end point of the lift is the same.

So, this is the path independence absolutely. there is no homotopy or nothing. We started with $Y$ path connected, and locally path connected, we have joined the two points take the image of the path inside $X$ and then lift it. And we want the endpoint to be independent of what path $\omega$ we take, only thing is, it will be joining $y$ to $z$, that is all.

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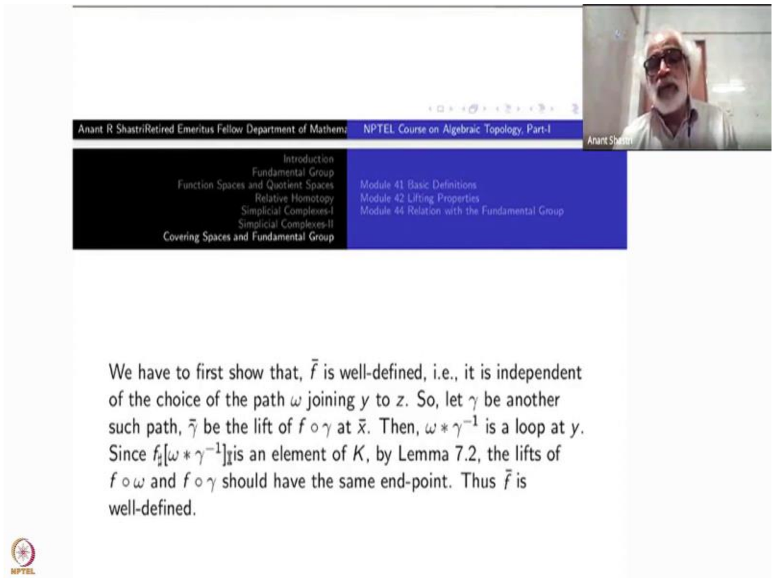


So, whatever we need, it turns out to be just there. We have to first show that $\hat{f}$ is well defined. So let $\gamma$ be another such path in $Y$, another such path means what? -- joining $y$ to $z$. Let $\bar{\gamma}$ be the lift of $f \circ \gamma$ at $\bar{x} \in \bar{X}$. Then we have to show that $\bar{\gamma}(1) = \bar{\omega}(1)$. That is what we have to show. Look at the loop $\omega * \gamma^{-1}$; that is a loop in $Y$, loop at $y \in Y$.

Now, if you take $f_\#([\omega * \gamma^{-1}]) \in K$ by hypothesis: the whole of $f_\#(\pi_1(Y, y)$ is inside $K$; so that is the hypothesis. Therefore, by our lemma for lifting the loop, this loop $f \circ (\omega * \gamma^{-1})$ lift to a loop $\bar{x}$ inside $\bar{X}$.

But that would mean that the two paths $\bar{\omega}, \bar{\gamma}$, which are lifts of $f \circ \omega, f \circ \gamma$ respectively, these two should have the same end points. So, this is our lemma 7.2 which was purely geometric, that we have seen. So, we are using it again here. So, these two have same endpoint, $\bar{\omega}(1) = \bar{\gamma}(1)$. So, set theoretically, we have already formed a function $\hat{f}$ such that $p \circ \hat{f} = f$ and $\hat{f}(y) = \bar{x}$.

Now, we have to show that it is continuous. What is immediate is that $p \circ \hat{f} = f$ automatically. What is $\hat{f}$? $\hat{f}(x) = \bar{\omega}(1)$. Therefore $p(\hat{f}(z)) = p(\bar{\omega}(1)) = f \circ \omega(1) = f(z)$. That is what we want. So, f bar followed by p is p composite f bar is f that is already there set theoretically. If $\hat{f}$ is continuous this will complete the task. But we are given $f$ is continuous, $p$ is continuous, you have to show $\hat{f}$ is continuous. So, you have to work harder there. So, let us see how the continuity comes.

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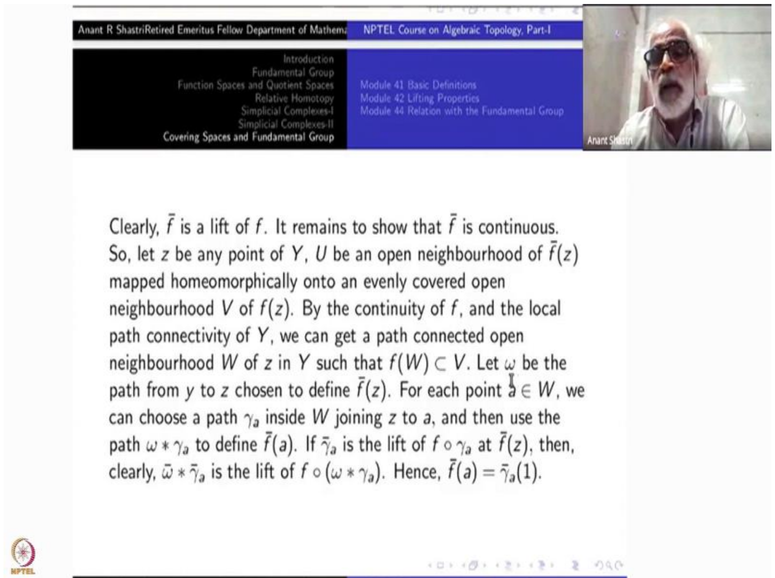


This is where we have use local path connectivity of $Y$. Take a point $z$ belonging to $Y$ at which I want to show that $\hat{f}$ is continuous. So, take a neighbourhood $U$ of $\hat{f}(z)$. Then I must produce a neighbourhood of $z$ which goes inside $U$ under $\hat{f}$; that is continuity. But inside inside every neighbourhood of $\hat{f}(z)$, I can choose a smaller one which is mapped homeomorphically onto an evenly covered neighbourhood of $V$ of $f(z) = p(\hat{f}(z)) \in X$; that means $p^{-1}(V)$ is a disjoint union of open sets each of which is mapped to $V$ homeomorphically, under $p$.

Now by continuity of $f$ and the local path connectivity of $Y$, we use both of them, we get a path connected open neighbourhood W of $z$ in $Y$, so that $f(W) \subset V$. First, by continuity, you can choose $W$such that $f(W) \subset V$, but then inside this $W$ whatever you have chosen you can choose another one $W'$ such that this one is path connected neighbourhood of $z$, because $Y$ is locally path connected. So, by renaming $W'$ as $W$, you get a path connected neighbourhood $W$ such that $f(W) \subset V$.

So, this W contains the point $z$, it is a neighbourhood of $z$. Let $\omega$ be the path from $y$ to $z$ chosen to define $\hat{f}(z)$. Recall how you defined $\hat{f}(z)$. You have fixed some paths then take f of that and take the lift of that at $\bar{x}$ and take the endpoint of that lift. So, look at this $\omega$; it is coming from $y$ all the way to $z$ which is this $W$, which is path connected. Therefore, I can take a path completely lying inside $W$ from $z$ to any other point in $W$. For each point $a \in W$, we can choose a path $\gamma_a$ completely lying inside $W$ and joining $z$ to $a$. Then use the path.

Now gamma followed by sorry, $\omega * \gamma_a$ is a path in $Y$ joining $y$ to $a$. So, I can take f of that, lift it, and take the endpoint. That will be the definition $\hat{f}(a)$. Remember in defining $\hat{f}(a)$, we are free to choose any path from $y$ to $a$ So, we have chosen this path. Now if $\bar{\gamma}_a$ denotes the lift of $f \circ \gamma_a$ at the point $\hat{f}(z)$, then $\bar{\omega} * \bar{\gamma}_a$ is the lift of $f \circ (\omega * \gamma_a)$ at the point $\bar{x}$. Therefore $\hat{f}(a) = \bar{\gamma}_a(1)$.

Note that since $\gamma_a$ is inside $W$, we have $f \circ \gamma_a$ is completely inside $V$. So, where is $\bar{\gamma}_a(1)$? Where is this point is the question. If this point is inside our $U$ then I am done, then I have shown that $\hat{f}(W) \subset U$. Now recall that I have chosen $W$ so that $f(W) \subset V$. I want to show that $\hat{f}(W) \subset U$.

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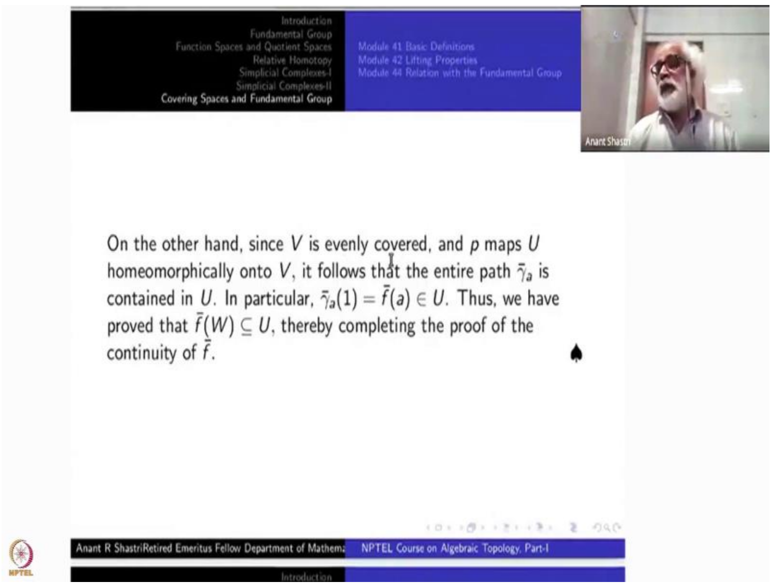


On the other hand, since $V$ is evenly covered (that is what we have to use now), and $p$ maps $U$ homeomorphically onto $V$ , it follows that the entire path $\bar{\gamma}_a$, which is connected, is inside a single component of $p^{-1}(V)$. It cannot be a portion is here and a portion there and and so on. So look where the starting point $\bar{\gamma}_a$ , which is nothing but $\bar{\omega}(1) = \hat{f}(z)$. So whatever component contains it that must contain the entire $\bar{\gamma}_a$ and hence its end point $\hat{f}(a)$. But our choice is that this components is precisely $U$. $U$is one of the components homeomorphically mapped onto $V$. So, it must be inside this one. So, that completes the proof of that $\hat{f}$ continuous.

Rest of them you already done. So, you see effortlessly, almost effortlessly, taking small, small steps we have proved a big theorem now, namely, when an arbitrary continuous map can be lifted.

From now onwards there will be several important results derived from this one, just like we derived several things by just looking at what happens to a lift of a  single path.

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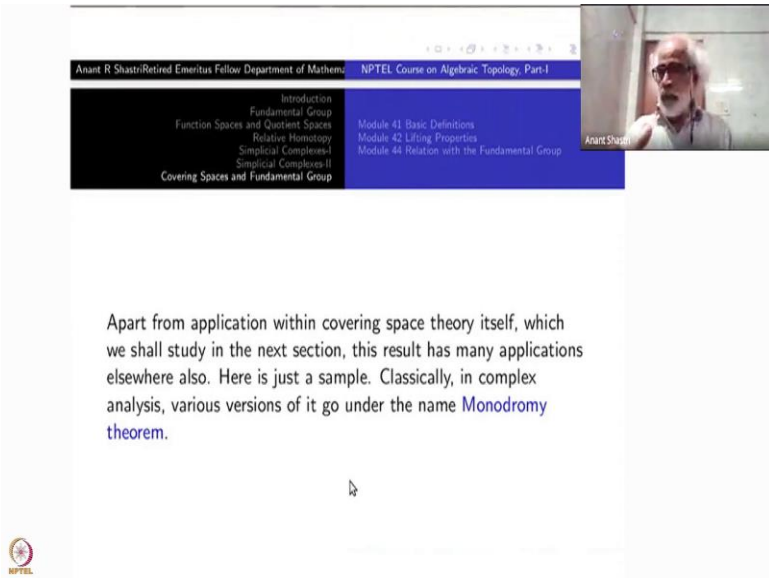


So, this is what I am telling you. Apart from application within covering space theory which we will study in the next section, this result has many applications elsewhere also. Here is just a sample. Classically in complex analysis, various statements there  go under the name Monodromy theorem, and that is the application I am going to give. So, the Monodromy theorem you may read may look like slightly  different but if you understand this one correctly, what I am going to say here,  all other versions the same in principle. There are different versions of this one that is what I wanted to tell you. I cannot cover all of them separately.

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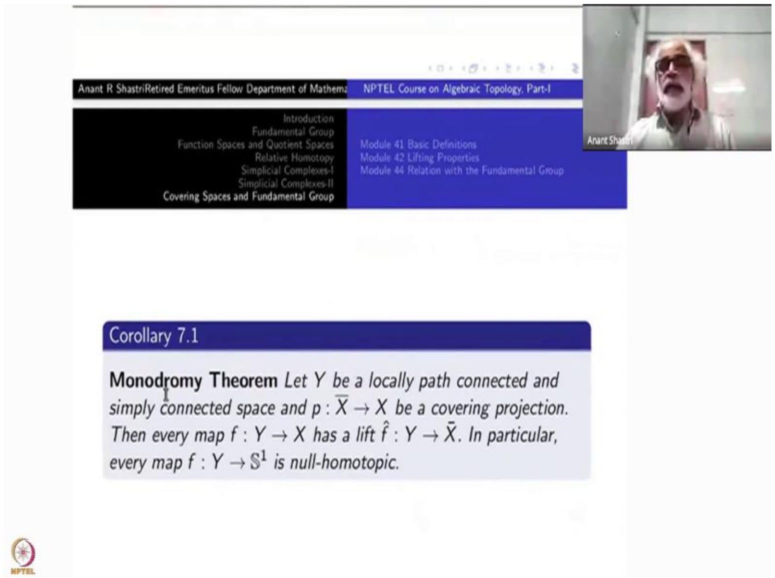


So, I will state one thing which immediately follows from what we have seen. Take $Y$ to be a locally path connected and simply connected space. (When you take open subsets, which are discs or some such things or simply connected domains in complex analysis, this condition will be automatically satisfied, because open subsets of $\mathbb{R}^n$ are local path connected.)

Take $p : \bar{X} \to X$, a covering projection, any cover projection. Then every map $f : Y \to X$ has a lift $\hat{f} : Y \to \bar{X}$. Why? Because $Y$ is simply connected, i.e., $\pi_1(Y)$=(1), $f_{\#}$ f of that is trivial, and the trivial group is contained inside every subgroup, and hence it is contained $K = p_{\#}(\pi_1(\bar{X}, \bar{x}))$. So, whatever criteria we need is trivially satisfied. Therefore, every map can be lifted to the covering.

In particular look at a map from $Y$ to $\mathbb{S}^1$. It can be lifted through the exponential function to $\mathbb{R}$. That is the meaning of this because $Y$ is simply connected. It can be lifted through the exponential function. Now $\mathbb{R}$ is contractible. So, any map into $\mathbb{R}$ is what? Is null homotopic. Take a null homotopy, compose it with exponential function, you get a null homotopy of the map here. So, every map from simply connected space to the circle is null homotopic. Of course, I am assuming locally path connected and connected and so on. So, this is what it is.

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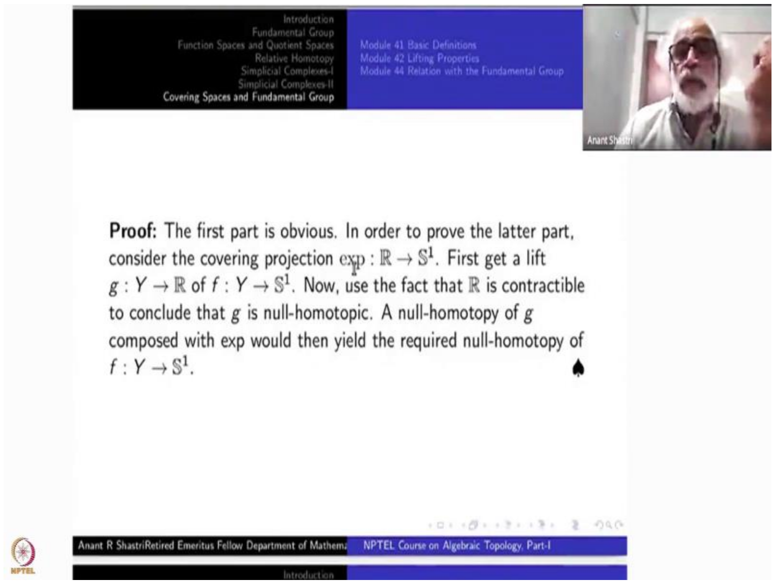


The first part is obvious because by simply connectedness. The trivial group is contained inside every subgroup, so criterion for lifting is satisfied. Second part, you lift it to through exponential function, and then use the fact that $\mathbb{R}$ is contractible. So that is null homotopic, push it down to \ $\mathbb{S}^1$ by taking the exponential function.

So, in the remaining time, I will do a little bit of new constructions-- how to construct non-trivial coverings.

If you take identity map, it is a covering, exponential map is a covering, but you might not have seen many examples of covering projections. I have given you only some group action and so on. There are other ways to construct a covering projection. Start with a space and constructing coverings of it-- not the other way round. Our group actions and the quotients etc. Give covering projection in a different way.

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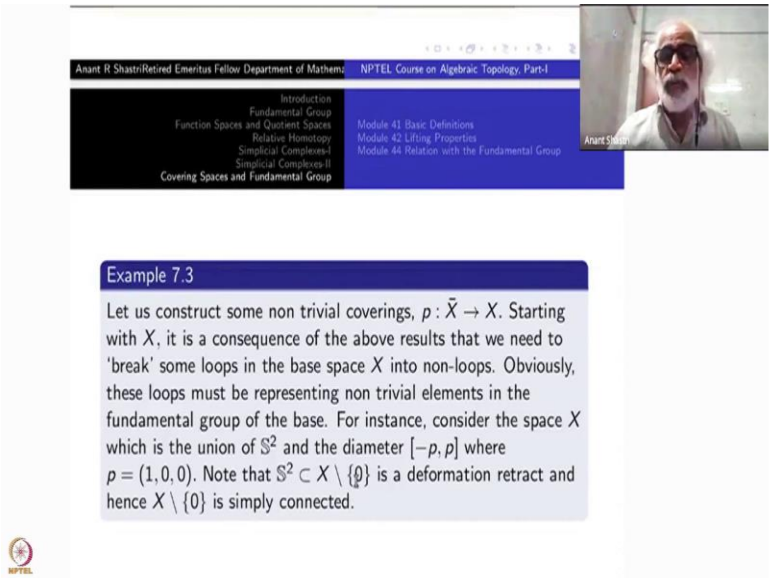


Let us construct some non-trivial covering maps $p : \bar{X} \to X$. To begin with note that $p : \bar{X} \to X$ is non-trivial covering means that every loop in $X$ cannot be lifted. If every loop can be lifted, then it will not be a non-trivial covering. You should not be able to lift at least some loop at some poitn or the other. That means that you have to break certain loops in the space $X$. So, that is now heuristic saying. Obviously you must look at loops representing non-trivial elements the fundamental group of $X$.

For instance, let us do a very simple thing, namely, take the space $\mathbb{S}^2$, the 2-sphere along with one of the diameters namely, let us take $p = (1, 0, 0), -p(-1, 0, 0) \in \mathbb{S}^2$. And join them. So, that is the diameter. So, now your space $X$ is 2-sphere and a diameter, in the centre you have a handle it is like that. If you throw away $(0, 0, 0) \in \mathbb{R}^3$, which is a centre of this diameter, what do you get?

You get $\mathbb{S}^2$ union two open arcs, half open arcs. Those arcs can be deformed back to $\mathbb{S}^2$. This just means that $X \setminus \{(0, 0, 0)\}$ deforms to $\mathbb{S}^2$; $\mathbb{S}^2$ is a strong deformation retract of $X \setminus \{(0, 0, 0)\}$ But we know that $\mathbb{S}^2$ is simply connected. Therefore, $X \setminus \{(0, 0, 0)\}$ is also simply connected, because these two have same homotopy type. If you remove one particular form $X$, it becomes simply connected.

Now, your theorem says that if you take the inclusion map think of this as $Y$=$X \setminus \{(0, 0, 0)\}$. $Y$is locally path connected and simply connected. Therefore the inclusion map, you must be able to lift it up, you choose a point somewhere say above $(1, 0, 0)$, there will be many points because it is going to be a non trival covering projection, at each point you will have a copy of $X \setminus \{(0,0,0)\}$ sitting inside $\bar{X}$.

So, this is the picture you have of $\bar{X}$. Then the missing points will be also there, a number of missing points will be there, if you fill them up you get the full picture of $\bar{X}$. So how is this to be visualized? So, I will give you a simple example of this one, this much is the logic, now we will need some visualization. So, first take only two copies of $X \setminus \{(0,0,0)\}$ and then you fill up points lying above $(0,0,0)$ somehow and so that you a single connected space $\bar{X}$ and a covering projection $p : \bar{X} \to X$. So, this is what you have to achieve. So, it is not very difficult, so, all this I have done.

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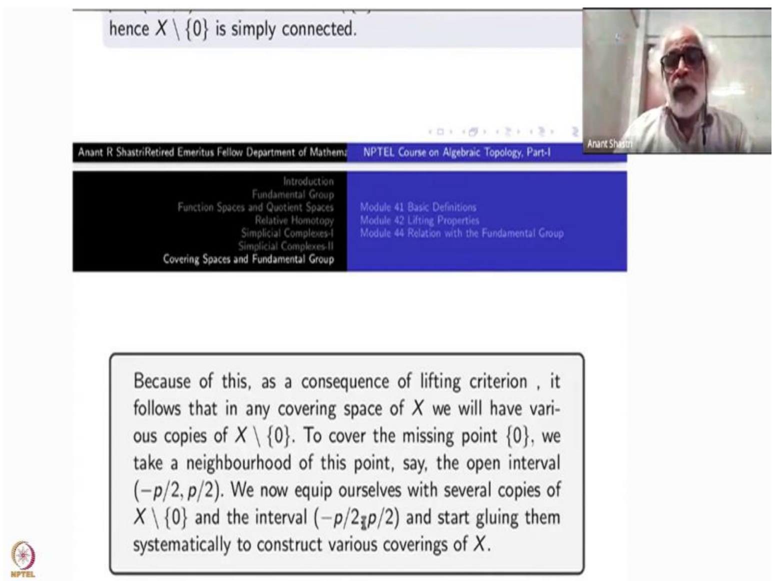


So, while filling missing points you should be careful also, namely just filling points may not be right. The neighbourhood of the point $(0,0,0)$ in $X$ is an open arc, that is also simply connected. So, all these missing points above in $\bar{X}$ will have neighbourhoods equal to an open arc. So, what do you have to do is: you have to take a number copies of $X \setminus \{(0,0,0)\}$ and equal number of copies of open intervals, then glue all them together, neatly. When you glue them together only the extra point in that neighbourhood must be at the centre point 0 of the arc. So that is what you have to do. So, I have taken these arc to be copies of $(-p/2, p/2)$, which is an open subset the whole $X$ and is neighbourhood of $(0,0,0)$. So, I will show you the picture here now.

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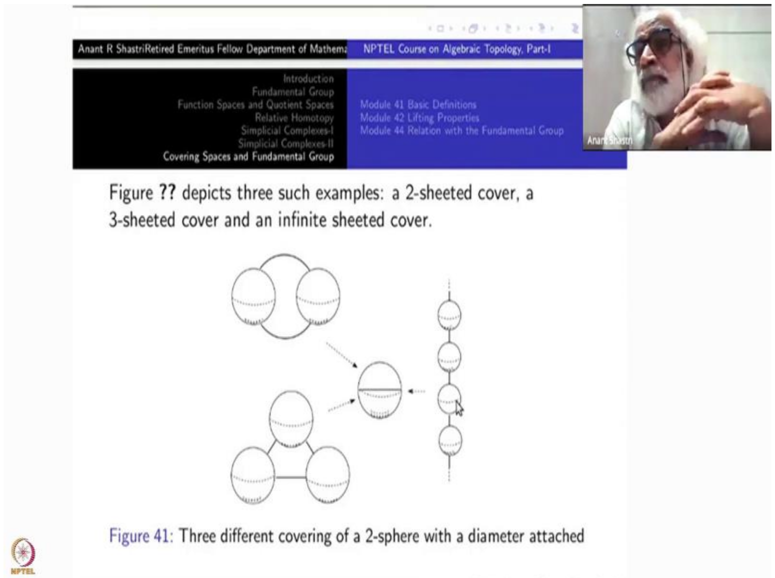


This is the picture and this is the starting space $X$, $\mathbb{S}^2$ with a diameter. So here what I have done? I have taken several copies of this one, this is one copy, the inside portion I have drawn outside that is all. I will join it to next one, another one will be joined to next one, and so on. So, this is an infinite cyclic covering, infinitely many copies just like in the case of infinite covering $exp : \mathbb{R} \to \mathbb{S}^1$, exactly similar to that.

Instead of these spheres here if you put a single point here-- say that single point is an integer and these are intervals of length 1, what you get is a copy of $\mathbb{R}$ here. So, I could have constructed this space as follows: take the exponential function $exp : \mathbb{R} \to \mathbb{S}^1$, replace each integer by a copy of the 2-sphere and at the bottom what you have to do, take the circle $\mathbb{S}^1$, look at the point $(1, 0) \in \mathbb{S}^1$, replace it by a small 2-sphere. So that is the picture. So, that is a infinite covering projection.

I can do just a double covering projection. Take two copies of $X \setminus \{(0,0,0)\}$, draw them as a 2-sphere with two whiskers here, two whiskers here, then join them together like this neatly. So, both these spheres go to the same sphere below here by a homeomorphism, in that homeomorphism this path goes precisely to the central line here, the diameter. Similarly, this will be also going to central line.

Now, this is three of them, this a triple covering. You can do a triple covering also, you can do a four-fold covering, you can do a n-fold covering. Do you know why this is so simple? The fundamental group of this picture is nothing but infinite cycling, this covering space. Pretend as if these are all coverings of $\mathbb{S}^1$. Only one of the point is replaced by a big 2-sphere there, that is the exactly what is happening. Let me do one more construction which I had promised last time, did not have time to do. Let me do that one also now.

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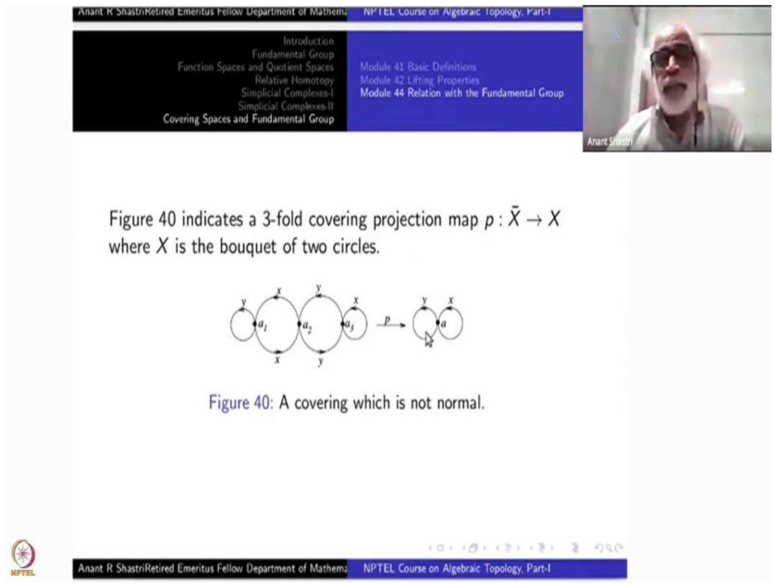


Yeah this is the one. Obtaining a covering which is not a regular covering, that will show that the fundamental group is not abelian. So, this was the topic last-last time, so I start with $X = \mathbb{S}^1 \bigvee \mathbb{S}^1$. This can be taken as the figure 8. There is one common point for both circles, labeled $a$. I orient this one and label it $x$, orient the other circle, labelled it y. So, how is the covering got? This point $a$ is copies three times here and labeled $a_1, a_2, a_3$. This one circle is copied here and leabled $x$ itself. Another one copied here and labeled $y$. But I have taken two arcs each labeled them as $x, y$ respectively, So, each of these open arcs is mapped to single circle here. Note that the points $a_3, a_2, a_1$ are all mapped to the same point $a$. So, this arc goes to the circle $y$ and the bottom arc here also goes to $y$, but these two arcs go to $x$, there are three arcs and sorry two arcs and a circle mapping onto this circle. Similarly, two arcs and a circle mapping onto this circle, that is a covering projection.

Now, by the very nature, this loop at $a \in X$ denoted by $x$, has a lift as a loop at $a_3$ but at $a_1$ it lifts to a open arc. This is not a loop. There is another loop here at $a_2$ it goes back to $a_1$ that is also an open arc. At $a_2$ if you lift it, it comes to $a_1$ and at $a_1$, it comes back to $a_2$. but at $a_3$ it comes back to $a_3$ itself. So, this is a loop, whereas these two are not loops, that means that this element is not in the normal subgroup of the image, the image is not a normal subgroup actually.

So, it happens that the the image $K$ of the fundamental group of $\bar{X}$ is actually a subgroup of order 3. I could not have constructed such a covering of order two, because index two subgroups are normal automatically. So, I need at least three points in the covering projection, three sheeted covering to get a non normal covring. Is that clear? That this is a covering projection which is not a normal covering. Why? Because a particular loop lifts to a loop as well as two open arcs here. Thanks, we will continue this kind of discussion next time.