Introduction to Algebraic Topology (Part I)
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Lecture 45
Regular Covering

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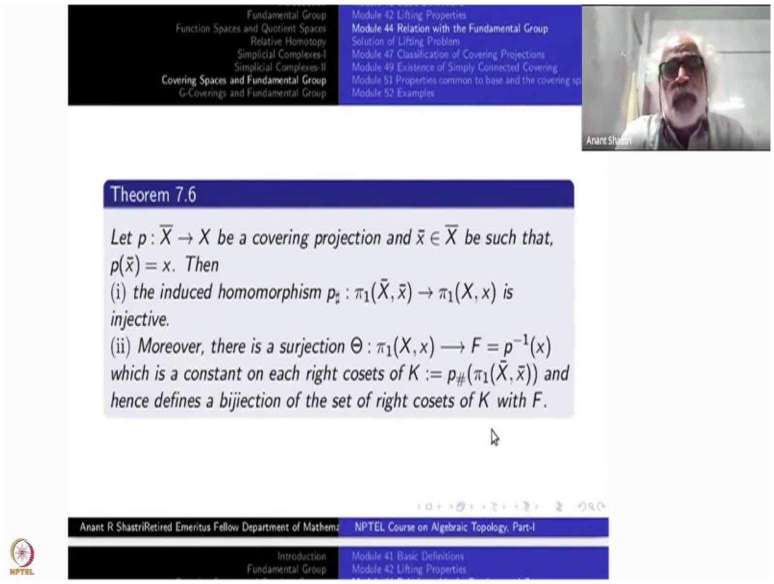

So, today we continue the investigation of the relation with the fundamental group and we will come to what are called regular coverings. Before that, we have to consolidate the results that we have proved last time and improve upon them a little bit. Let us go back a little bit here. So, this theorem, it says that induced homomorphism $p_\#$ is injective, the second part says that there is a surjection Θ from $\pi_1(X, x)$ onto the fibre $F = p^{-1}(x)$, which is a constant on each right coset of the sub group $K := p_\#(\pi_1(\bar{X}, \bar{x}))$, and hence, defines a bijection of the right cosets of K with the set F. So, this is purely algebra now, the only topology is the hypthesis that $p : \bar{X} \to X$ is a covering projection of path connected spaces. I have chosen base points, $\bar{x}$ on top of x. Once you fix these notation, there is a homomorphism $p_\#$ from the fundamental group of the top space to the fundamental group of the bottom space, and that homomorphism is injective it is a monomorphism.

So, the subgroup K can be identified, under $p_\#$, with the group $\pi_1(\bar{X}, \bar{x})$. Of course, identification has to take place under $p_\#$. This homomorphism is important. Moreover, if you take the right cosets of the subgroup, they correspond, in a nice fashion to the points of the fibre. So, this is a statement, proof is not very difficult, once you have done lemma 1, most important lemma 2 is purely geometry, everything follows very easily. So let us look at this one.

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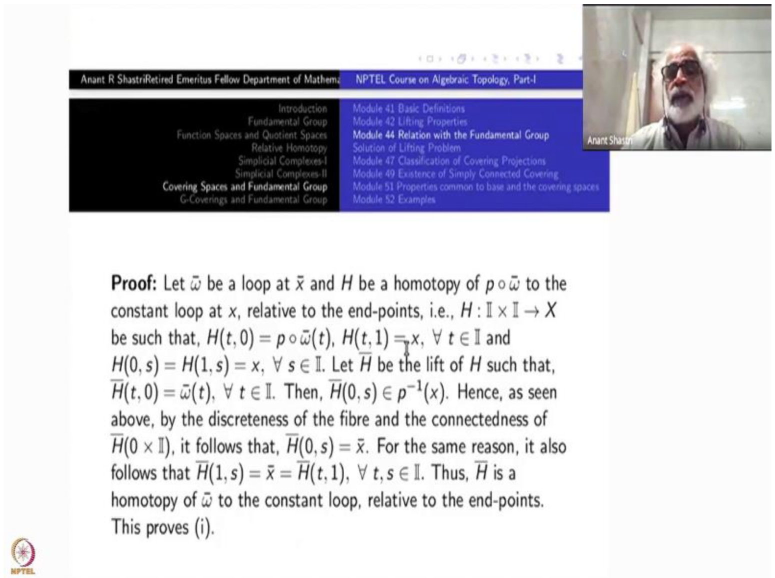

Let $\bar{\omega}$ be a loop at $\bar{x}$. I am proving the injectivity. Let H be a homotopy of $p \circ \bar{\omega}$, when you go down, to a constant loop. This just means that $p_{\#}[\bar{\omega}] = 1$. I want to show that $[\bar{\omega}] = 1 \in \pi_1(\bar{X}, \bar{x})$. That means, I have to show that that $\bar{\omega}$ is path- homotopic to the constant loop at $\bar{x}$ in $\bar{X}$.

When you go down to X, it is homotopic to a constant loop relative to endpoints. So, start with a homotopy, H such that the starting point is $p \circ \bar{\omega}$, end is a constant path. The homotopy is, remember, relative to the endpoints; i.e., two endpoints 0 and 1 are kept constant all the time.

Now, let $\bar{H}$ be the lift of H such that the starting path is $\bar{\omega}$. I do not say anything about the other end. This is just by the homotopy lifting property of p that I have used, that is all. Then you look at $\bar{H}(0 \times \mathbb{I})$. As we have seen before, it is inside $F = p^{-1}(x)$. That is because $H(0 \times \mathbb{I}) = \{x\}$. So, this is inside the fibre.

Therefore, as seen above, by the discreteness of the fibre and connectivity of the interval and henc that of $\bar{H}(0 \times \mathbb{I})$, it follows that this entire thing is one single point $\{\bar{x}\}$, because that is where you have lifted this whole thing. For the same reason, $\bar{H}(1 \times \mathbb{I}) = \{\bar{x}\} = \bar{H}(\mathbb{I} \times 1)$. ($\bar{\omega}$ is a loop $\bar{x}$. And $H(\mathbb{I} \times 1) = \{x\}$.) So, all of them are at this point, the H bar of t 1, for every t and s belong to s.

Therefore, $\bar{H}$ itself is a homotopy of $\bar{\omega}$, relative to 0 and 1 to a constant loop. Bottom is a constant loop; the top will be inside a fibre. But the fibre is what, discrete all the time I am using same thing. Therefore, it is a constant. So, that proves $p_{\#}$ is injective.

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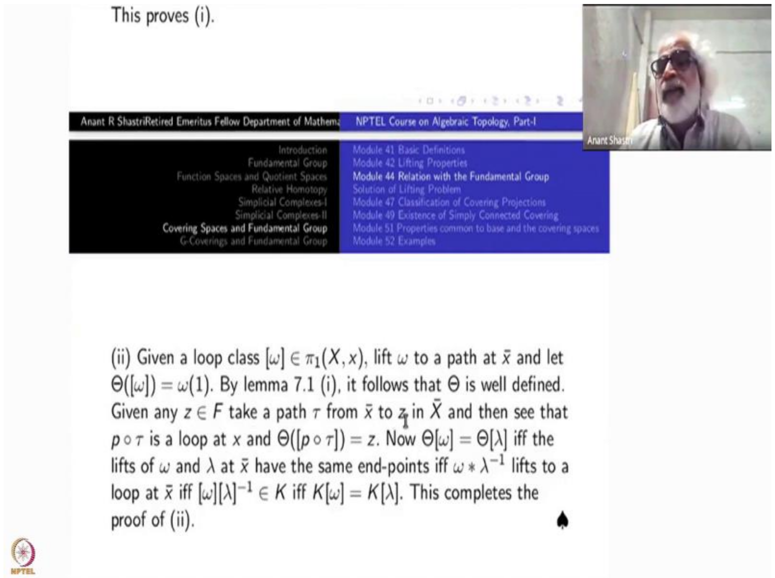

Now the next thing. I have defined a set theoretic function from the right cosets of $K = p_{\#}(\pi_1(\bar{X}, \bar{x})) \subset \pi_1(X, x)$ to the fibre $F = p^{-1}(x)$. This is done by first defining a function on the group itself. And then, restricted to cosets, observe that it is a constant. So, it gives you, after all, a function on the set right cosets. The set of right cosets is a quotient set of of the group, by the subgroup. So, this is what I am going to do.

Take a loop ω which represent an element here. (After all, elements of the fundamental group has to be represented by loops.) Lift this ω to a path $\bar{\omega}$ at $\bar{x}$, (this $\bar{x}$ is fixed, for the entire investigation). And let the function Θ be defined by $\Theta([\omega]) = \bar{\omega}(1)$. Recall that $\bar{\omega}(1)$ depends only on the class $[\omega]$, whatever loop I take in this class, it depends only on the class. Therefore, $\Theta([\omega])$ is well defined. By the first lemma that we proved, theta is well defined, well defined means what? If I change omega to omega prime here, omega prime 1 will be also equal to 1.

Now, take any $z \in F$, inside this fibre. Since $\bar{X}$ is path connected, we can join this one to $\bar{x}$. Say, τ is a path from $\bar{x}$ to z. Then see that $p \circ \tau$ is a loop at x. All these things, we have seen already I am just repeating it. Now what is $\Theta[p \circ \tau])$? How is it got?

You have to lift it; you have to lift it at x bar and look at the end point. But then point is that we started like that, there is only one lift remember and it is already a path there and you have taken the image of that path, there it will be a given path. So, it is tau itself and endpoint of tau is z. Therefore, $\Theta[p \circ \tau]) = \tau(1) = z.$

So, this shows that this Θ is surjective. Now, I have to show that this Θ takes same value on the right cosets of K. Now, I would prove it such that all steps are reversible, `if and only if' statements, so that automatically it will porve that Θ defines an injective mapping on the set of right cosets. And surjectivity we have already proved. So, the proof starts with $\Theta[\omega] = \Theta[\lambda]$ iff $\bar{\omega}(1) = \bar{\lambda}(1)$. What are omega and lambda? They represent elements of $\pi_1(X, x)$, some loops at x. $\bar{\omega}, \bar{\lambda}$ are the lifts of omega and lambda at $\bar{x}$. The end point are the same because that is the definition of Θ.

So, next one : $\bar{\omega}(1) = \bar{\lambda}(1)$ iff $\lambda * \omega^{-1}$ lifts to a loop at $\bar{x}$. So this also we have seen before. It is lemma 2 actually.

Next: $\lambda * \omega^{-1}$ lifts to a loop at $\bar{x}$ iff $[\lambda * \omega^{-1}] \in K.$ What is this K?

Remember $K = p_{\#}(\pi_1(\bar{X}, \bar{x}))$. So, this also we have seen.

Finally: $[\lambda * \omega^{-1}] \in K$ iff $K[\lambda] = K[\omega]$. This is an easy statement about subgroups. So, this is a group theory.

So, this completes the proof . Whatever happens to the geometry of the loops and lifts and so on is converted into an algebraic condition.

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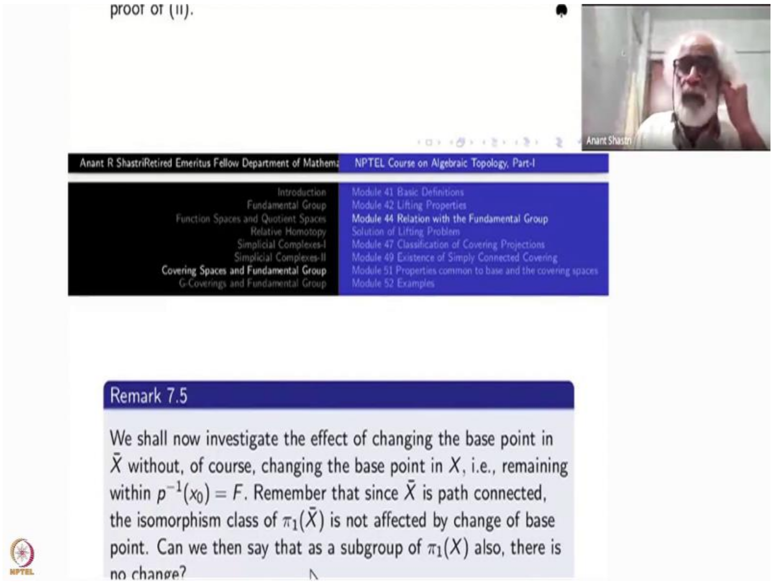

So, we shall now investigate the effect of changing the base point in $\bar{X}$. Of course, without changing the base point in X. That means that you look at the fibre F over x, and change the base point there, what happens? That is what we are interested in now. And once again we assume $\bar{X}$ is path connected; that is important. The isomorphism class of $\pi_1(\bar{X})$ is not changed by chaing base points, if $\bar{X}$ is path connected. This is what we have seen long back.

But the isomorphism class does not change does not mean that $p_\#$ is the same. So, can you also say that the subgroup K does not change? That is not clear. In fact, that is supposed to be not true also. And this is where whatever we did in the previous theorem comes into picture again. So that the answer is already there.

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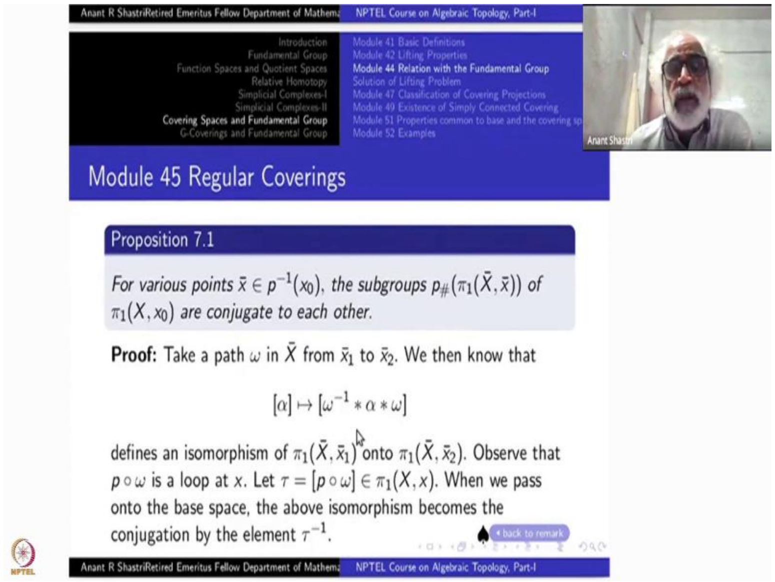

So, the regular coverings is the title of this topic. For various points $\bar{x} \in p^{-1}(x_0)$ (which may be finite or maybe infinite you do not know, p inverse of x0 is a discrete set, this much we know), look at $p_\#(\pi_1(\bar{X}, \bar{x})$, they are all subgroups that is what we have proved, subgroups of $\pi_1(X, x_0)$ The claim is that all these subgroups are conjugate to each other. The best relation that you can have : they are conjugate to each other.

Once again, the previous theorem contains a proof of this. Let us go through that one little bit. Take a path ω in $\bar{X}$ joining $\bar{x}_1, \bar{x}_2 \in F$. Its image under p will be a loop at x_0 in X. So, take $[\alpha] \mapsto [\omega^{-1} * \alpha * \omega]$. This defines an isomorphism $h_{[\omega]} : \pi_1(\bar{X}, \bar{x}_1) \to \pi_1(\bar{X}, \bar{x}_2)$ This omega is a loop, omega is a path from this one, but when you go down it is a loop. So, this will be a conjugate of alpha, sorry this is this is, this I am taking directly I have not applied p here. So, I am in X bar itself I am working.

So, start with alpha, a loop at $\bar{x}_1$. First, ω^{-1} is a path from $\bar{x}_2$ to $\bar{x}_1$. Then trace α which is a loop at $\bar{x}_1$ then go back via ω to $\bar{x}_2$. So, this will become a loop at $\bar{x}_2$. So, this defines an isomorphism pi 1 of X bar to on to pi 1, this we have seen already.

So, this is the isomorphism between any two fundamental groups at any two different points of a path connected space. This will depend upon the call $[\omega]$ of course, if I choose another path this may be different ismorphism. Observe that when you take $p \circ \omega$, that is a loop. So, let $\tau = [p \circ \omega] \in \pi_1(X, x_0)$. tau equal to p composite omega in pi 1 of x. Now

$p_{\#} \circ h_{[\omega]}([\alpha]) = [p \circ \omega^{-1}][p \circ \alpha][p \circ \omega] = \tau^{-1} p_{\#}[\alpha]\tau.$ That proves $p_{\#}(\pi_1(\bar{X}, \bar{x}_2) = \tau^{-1} p_{\#}(\pi_1 \bar{X}, \bar{x}_1))\tau.$

When you p of this, p of this one becomes tau inverse p omega p alpha something some element into this tau.

So, tau inverse tau, now tau is a loop, so its class is an element of pi 1 of X. So, when you come down this arbitrary isomorphism through path is becomes a conjugation, when you pass onto the base space the above isomorphism becomes the conjugation by an element of inverse here. Tau inverse alpha tau, is that, is that clear? I mean is there any doubt in this one.

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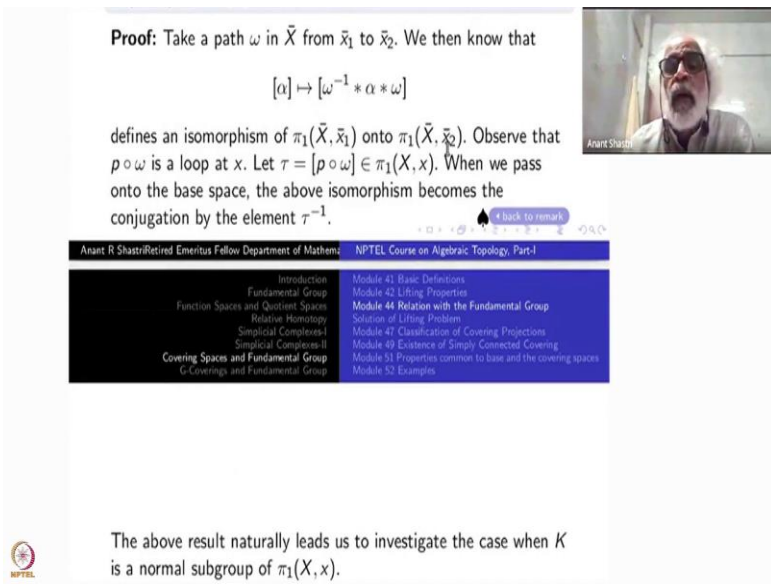

So, pi 1 of X bar x1 bar p check of this is some K1, p check of this is some K2, take a loop here that will look like p of some alpha, where alpha is a loop here, look at this this map it will give you a map, this will give you an isomorphism of this one, which another elements of pi 1 of X bar x. But p check of that will be K2, so but this will give tau K1 tau inverse K1 tau 2 is equal to K2, when you come down. So, let us say the conjugation is tau inverse.

So, various groups K1 K2 etcetera all of them their images of p check when you take the different points inside p inverse of X, there are all conjugate. So, this theorem automatically leads us to study of normal subgroups, because for a normal subgroup all the conjugates are the same, the conjugating element may be any element from the group, that is by definition. So, this leads to the notion of normal subgroup here. So, let us make a definition.

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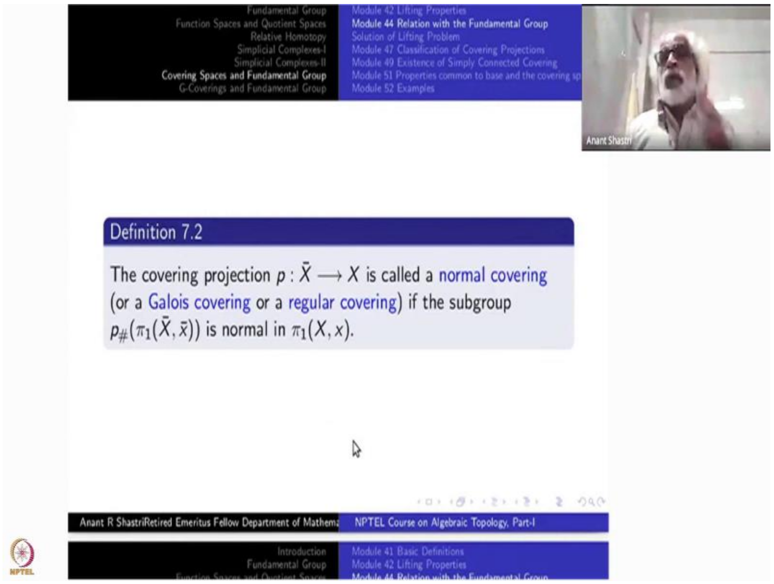

A covering projection is called a normal covering if the subgroup $p_{\#}(\pi_1(\bar{X}, \bar{x})) \subset \pi_1(X, x)$ is nroma. Obviously, this word `normal' is borrowed from group theory. I am not sure whether, whether group theory has borrowed it from the covering space theories or the other way round, because group theory was developed by Galois much later then Poincare, Poincare already studied this thing, so one is not sure whether it is normal covering was borrowed from group theory to here or the other way around.

It is also called Galois covering. That name is definitely after Galois. And there is other name regular covering also. Though all three different wordings are used by different authors. What is it? If the subgroup $p_{\#}(\pi_1(\bar{X}, \bar{x})) \subset \pi_1(X, x)$ is normal $\pi_1(X, x)$. (Here $\bar{x}$ is some point in $p^{-1}(x)$.) If it is normal for one point, for all other points in $p^{-1}(x)$ also, it will be normal.

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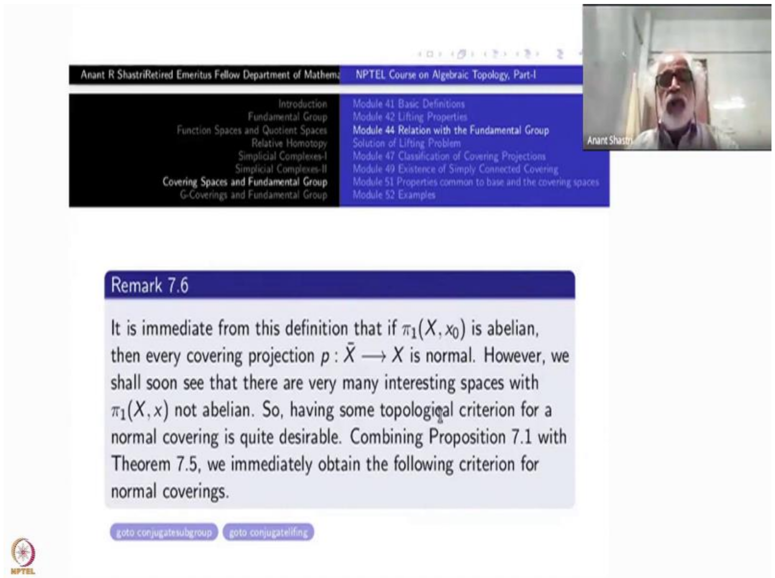

That is immediate from definition. Also, if the group $\pi_1(X, x)$ itself is abelian, then conjugation is identity no matter which element you conjugate. Therefore, every covering projection is normal, or you can call it Galois or you can call it regular covering. However, we shall soon see that there are many interesting spaces with $\pi_1(X, x)$ not abelian. If all $\pi_1(X, x)$ were abelian this would have been useless definition because everything is normal after all.

So, having some topological criteria for a normal covering is quite desirable, this is the definition of normal covering in terms of algebra. So, purely in terms of topology what does it mean? In terms of geometry, what does it mean? That is what we want to investigate, but this is more or less already answered. We have done the basic thing. All that I have do is combine proposition 7.1 and 7.5 immediately. So, this 7.1 says this one conjugation and the other one is already there for us. This theorem, so we will just put them together, regular covering.

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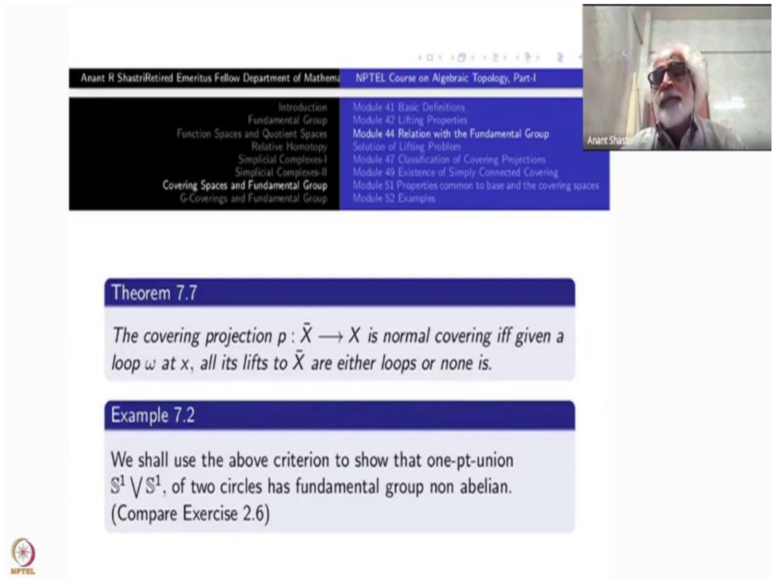

So, this is the theorem. The covering projection p is normal if and only if given any loop in X, all of its lifts are loops or none is. All lifts, means what? You have to only change the starting point of he lift, that is all, we have no other freedom. If you specify the starting point the lift is well defined. So, starting point you can take, any point of the fibre. Either all the lifts will be loops or none of them is a loop.

So, this is purely in terms of topology. If this geometric condtition is satisfied by p then p will be a normal covering, why? Can you see why this is true? What does it mean to say that p is noraml? K is a normal subgroup-- that is the definition. So, what is the meaning of normal subgroup? Conjugating by any element maps K into K. Take a loop omega in X, if some conjugate of $[\omega]$ is inside K then only, it has a loop as a lift. Yes, or no?

Student: Yes sir.

Professor: `If and only if' that is what we have seen. If no conjugate belongs to K, then none of the lifts of ω will be a loop. So as soon as a loop you can conjugate by that element you will get into inside K, that is what you have to see. So, because of that theorem this is the precisely the meaning of this one.

Some element some conjugate belongs to K as soon as that happens, all conjugates will be also inside the K, that is a group theory. If no, if even a one conjugate does not belong then no conjugate will belong, that belonging is converted into lifting into a loop or lifting not into not a loop, which is does not belong. So, group theory part is converted into this one, but this is purely in terms of topology now.

So, I will stop here. Next time, we shall use this one to to illustrate, not to develop any theory, but to illustrate that the fundamental group of wedge of two circles this is a figure 8, fundamental group is non-abelian. We will use that non-abelianness, to conclude that there must be some subgroup which is not normal, everything will be normal if it is abelian.

So, we will try to get that space and then use a covering and then show that some loops will lift to a loop, some other loops the same thing some other lift will not be a loop, same one single loop below in the covering at different points you lift them at one place it will be loop, another place will not be a loop. That will show that the covering is not normal, that means the fundamental group, the image of pi 1 of the above space inside the pi 1 of X is not normal, if you are in subgroup which is not normal the group cannot be abelian. So that is the way we have to do, but we will do it next time. Thank you.