Introduction to Algebraic Topology (Part I)
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Lecture 44
Relation with the Fundamental Group

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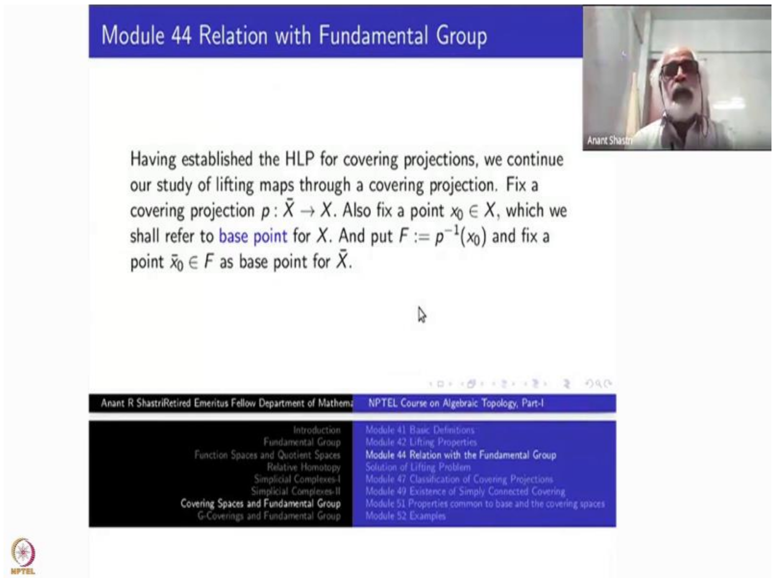

Today, we shall study the Relation with the Fundament Group of the covering space theory, having established the homotopy lifting property for covering projections, we will have to continue the study of lifting maps through a covering projection. So, throughout this lecture and for one or two more lectures, I fix the notation: $p : \bar{X} \to X$, is a covering projection, $x_0 \in X$ is a fixed point in X, which will be called the base point for X. We will put $F = p^{-1}\{x_0\}$; this F stands for fibre over x_0. inside this fibre, we will take a point $\bar{x}_0$ as a base point for $\bar{X}$. So, this $x_0, \bar{x}_0$ are fixed and $\bar{x}_0$ is mapped onto x_0 under p.

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So, to begin with we know that every path at x_0 inside X can be lifted to a path in $\bar{X}$ at $\bar{x}_0$. The question is, suppose I take a loop at x_0 as a path, we can lift it, but the lifted path may not be a loop. If it is not a loop, there is nothing you can do about it, because of the unique lifting property. Of course, if you take the base point to be different, there may be a lift which is a loop, but at that base point $\bar{x}_0$, there is no other way. So, we have to investigate this: when a lift of a loop will be a loop.

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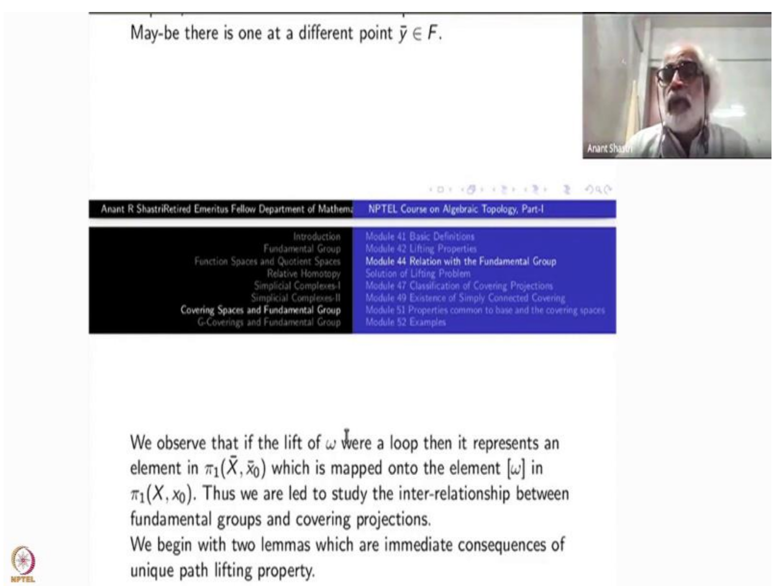

So, suppose the lift of a loop is a loop in $\bar{X}$. Then that loop will represent an element in $\pi_1(\bar{X}, \bar{X}_0)$ and its image under $p_\#$, ($p_\#$ is the induced group homomorphism the π_1 level,) is precisely the class represented by the loop that we started with. So, this is how the question of lifting a loop to a loop is related to something happening at the fundamental group level. So, the algebra enters here. So, for studying this problem, let us first state and prove fundamental lemmas and then keep using them to derive several algebraic properties.

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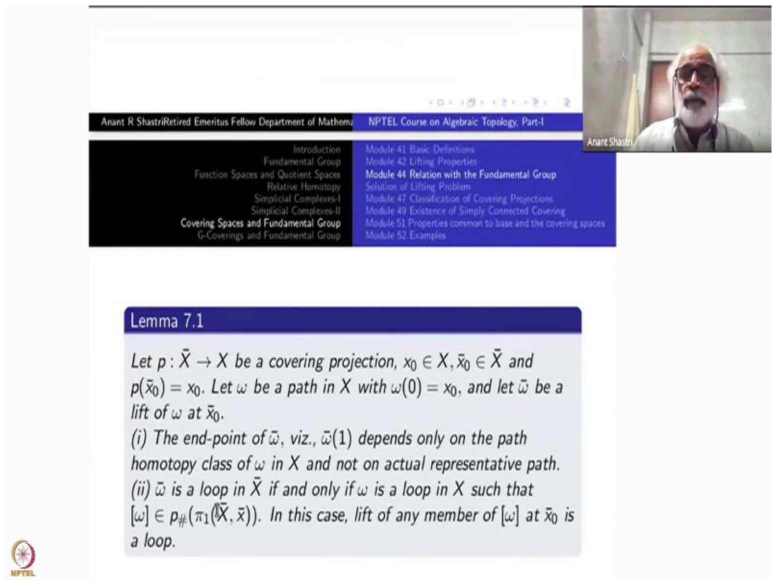

So, the notations are as before, p is a covering projection, x_0 is a base point at X and $\bar{x}_0$ is a base point in $\bar{X}$ sitting over x_0. Take a path ω in X starting at x_0, let $\bar{\omega}$ be the lift of ω at $\bar{x}_0$. First part of the lemma is: if you look at the endpoint of $\bar{\omega}$, this will not depend on the path ω itself but only on the homotopy class $[\omega]$. Whatever path you take in homotopy class of ω, the path may be different, but in the same homotopy class, then endpoint of its lift will be always the same. So, this is the first statement.

The second statement is: $\bar{\omega}$, the lifted path is a loop if and only if ω is a loop in X such that the represented the class $[\omega]$, the class represented by ω is in the image of $p_\#$. In this case lift of any member of $[\omega]$, that means, take any element of $p_\#(\pi_1(\bar{X}, \bar{x}_0))$, and any loop representing this class, for all of them, the lifts at $\bar{x}_0$ will be loops. So, this second part we have already observed, we have already observed and the second second part or second part will be easy consequence of the first part.

One lift, look at the end point, end point is the same thing as the starting point, so all other lifts will have to be loop because they should also have the same endpoints. So, this part follows easily. This we have seen. This is the meaning. An element a is in $p_{\#}(\pi_1(\bar{X}, \bar{x}_0))$, means $a = [p \circ \lambda]$ for some loop λ at $\bar{x}_0$, lifts of all members of a at $\bar{x}_0$ will be loops. So, second part easy.

The first part needs some explanation, why the endpoint is independent of the homotopy (homotopy class, this thing we have seen already in the case of exponential function. Remember how the degree of a loop in $\mathbb{S}^1$ was defined. You lifted it and then you looked at the end point, if you take another representive in the same class and lifted, the endpoint will be again the same, so that endpoint there happened to be an integer, because it is the exponential map.

Here there is no integer or anything, but we want to say that the endpoint is independent of the representative of the homotopy class, it only depends on the homotopy class. So, proof is more or less the same if we pretend as if we use the integers and exponential and so on. The fundamental property of exponential function was that it is covering projection, and that is what we are going to use here, namely for any covering, the fibre is a discrete space, discrete subset of the top space $\bar{X}$. So, that is what we are going to use. Once I have said that one, I have told the proof of the whole thing. However, let us go through it slowly and see once again why this is true.

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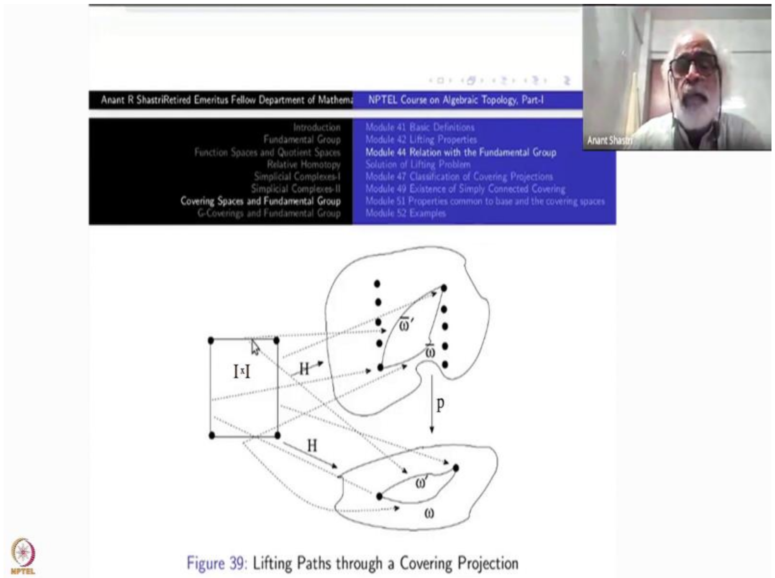

So, here is a full picture of this one, you can keep referring to this all the time. So, here is your X and here is your $\bar{X}$ and this is p, this ω is a path here from this point to this point, these dot dot dots here are sitting over this point, we could call this x_0 and these points are all in the fibre of x_0. So, here this is x_1 and these are all fibres above x_1 sitting over that x_1. So, this ω is lifted to a path here $\bar{\omega}$; ω' is homotopic to ω, (it is path homotopy). When you lift ω' this ends at the same point, that is the, that is the statement we have yet prove. So, this is the homotopy, $H : \mathbb{I} \times \mathbb{I} \to X$, this part gives ω, and that path gives ω'.

And when you lift the homomoty H, (this is $\bar{H}$ actually, it looks like H), $\bar{H} : \mathbb{I} \times \mathbb{I} \to \bar{X}$ defines from $\bar{\omega}$ to $\bar{\omega'}$. So, what have we done. We take this homotopy H from ω to ω' and take a lift $\bar{H}$ with the condition that $p \circ \bar{H}(t, 0) = \omega(t)$. Automatically $\bar{H}$ restricted $\mathbb{I} \times \{1\}$ will be a lift of ω' by the uniqueness of path-liftings. And what we want to show is that the endpoints are the same: $\bar{\omega}(1) = \bar{\omega'}(1)$.

Why the endpoints are the same? Look at this part, $\{1\} \times \mathbb{I}$, what is this part? Under H, because it is a path homotopy all this go to this single point point x_1, this whole thing goes to this point. Similarly, this is $\{0\} \times \mathbb{I}$ goes to x_0, This means when you lift it this entire thing the image must be contained inside these, these points. Similarly, image contained inside this point, we have chosen the image to be this point, once this point is there because this is connected space and this is a discrete space, so the entire thing must be this point here, there is no problem.

Same way, we have not chosen where $\bar{\omega}(1)$, $\bar{\omega'}(1)$ are, both are in $F = p^{-1}(\{x_1\})$. Also this entire path $\bar{H}(1 \times \mathbb{I})$ is a single point inside F, the same point $\bar{\omega}(1)$, because this is connected and this is a discrete space. So, that is the property that we use. So, this point is the same thing as endpoint of this one, over. So that is the proof of part one. And part two I have already shown, how how part two comes from part one, full detail is written down here with notations.

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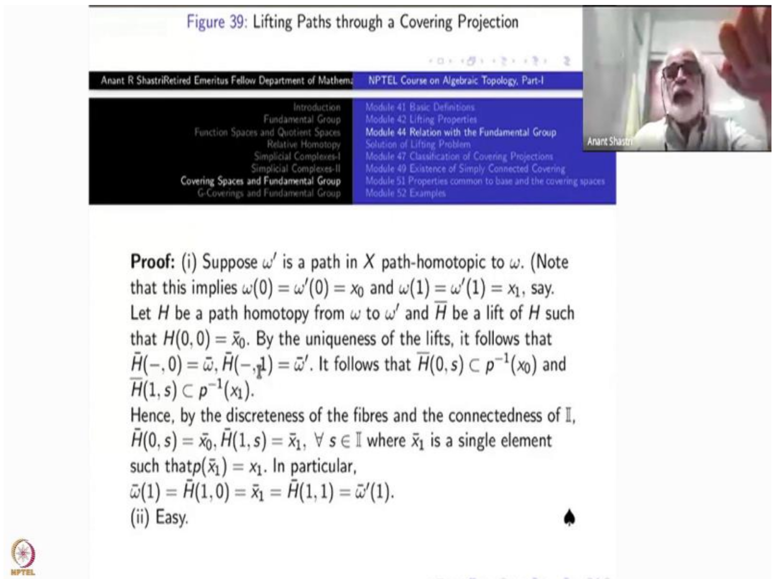

Actually, in this detail, there may be some typos, but what I have told you here is crystal clear. So, start with omega prime is path homotopic to omega, two different representatives of the same class. First of all, the starting points are the same, end points are the same, that is what you have require, we have to notice that. Now H is a path homotopy from omega to omega prime, $\bar{H}$ is a lift of H so that $\bar{H}(t,0) = \bar{\omega}(t)$. The rest of them are automatically defined.

Because of what? $\{t\} \times \mathbb{I}$ and $\mathbb{I} \times \{s\}$, by their uniqueness of the lifts, it follows that the first first horizontal interval $\mathbb{I} \times \{0\}$ goes to omega bar, the second one at level 1, goes to omega prime bar and $\bar{H}(0 \times \mathbb{I})$ is inside $p^{-1}(x_0)$ and $\bar{H}(\{1\} \times \mathbb{I}) \subset p^{-1}(x_1)$. These two are discrete, and this is connected. So, this image must be a single point. Let us, go to the next result now. This is the most fundamental. Just the discreteness of fibres are used here.

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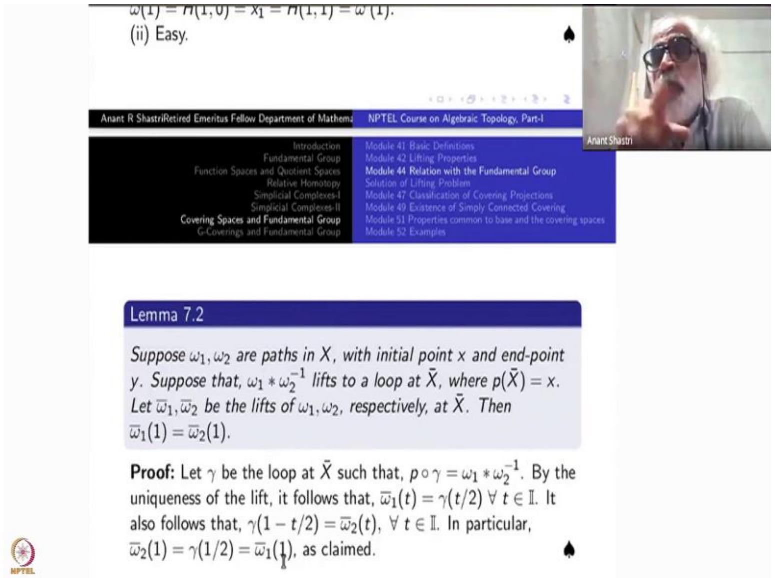

Now, suppose you have two paths ω_1, ω_2 in X, both with initial point x and endpoint y. (Instead of x1 and x2 or x0 and x1. I will change the notation, you should be able to do that without notation change, now there are some problems here with typo, I just changed it but still it has not come, sorry.) So, suppose you traverse ω_1 first and then come back by ω_2^{-1}, That is a loop at x. Suppose this loop lifts to a loop at $\bar{x}$ in $\bar{X}$, where $p(\bar{x}) = x$. Let $\bar{\omega}_1, \bar{\omega}_2$ be the lifts of ω_1, ω_2 respectively at the point $\bar{x}$. Then the endpoints of these ones are the same: $\bar{\omega}_1(1) = \bar{\omega}_2(1)$.

You see the first lemma said that ω_1, ω_2 are homotopic paths, then the endpointsof the two lifts are the same that was the conclusion. Here what I have? I just have that $\omega_1 * \omega_2^{-1}$ lifts to a loop. There is no homotopy or anything. Then how to get this one? Let us see. So, this is not all that difficult, but it looks somewhat challenging. Suppose γ is a loop at $\bar{x}$ such that $p \circ \gamma = \omega_1 * \omega_2^{-1}$.

So, this is the hypothesis. that omega 1 star omega 2 inverse lifts to a loop, that loop I am calling gamma. So, this is a loop at little $\bar{x}$. By the uniqueness of the lifts, it follows that the first half of γ is equal to $\bar{\omega}_1$. What is the definition of omega 1 star omega 2? $\omega_1 * \omega_2^{-1}(t) = \omega_1(2t), 0 \leq t \leq 1/2.$ Therefore $p \circ \gamma(t) = \omega_1(2t), 0 \leq t \leq 1/2.$ Similarly, $p \circ \gamma(t) = \omega_2^{-1}(2t - 1).$

Therefore, in the interval $0 \leq t \leq 1/2,$ we have, $p \circ \gamma^{-1}(t) = p \circ \gamma(1 - t) = \omega_2^{-1}(2(1 - t) - 1) = \omega_2^{-1}(1 - 2t) = \omega_2(2t).$ In particular, by the uniqueness of the lifts, it follows that $\gamma^{-1}(t) = \bar{\omega_2}(t), 0 \leq t \leq 1.$ This implies $\bar{\omega}_2(1) = \gamma^{-1}(1/2) = \gamma(1/2) = \bar{\omega}_1(1).$

So, this gamma wherever, if it cannot be looped if two different paths have been lifted, two loops have been lifted at one point and the end points are different, you start from 1 point to go, you cannot come back, you have to come back from the same path not from the other path. When you are coming from the other path means the endpoints of the other two must be the same. So, that is geometrically it is simple, but we have verified now. So, this is this looks like nothing to do with the covering space theory, this is simply geometry here.

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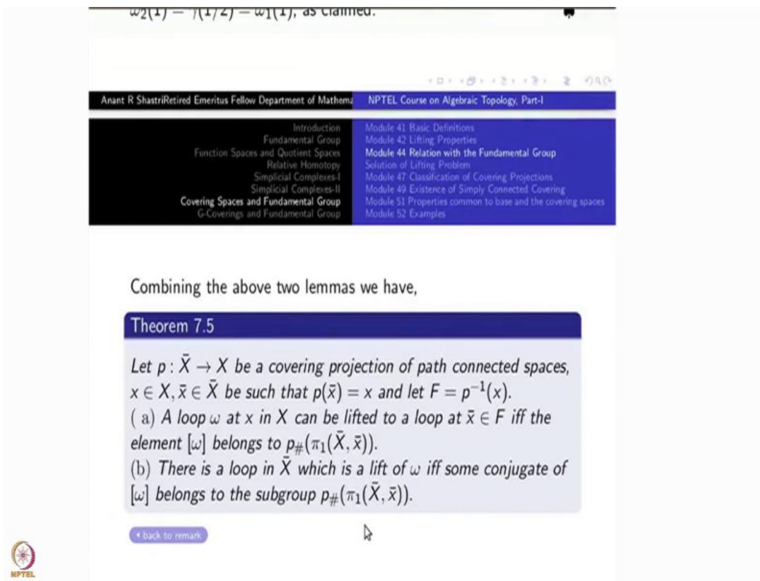

If you combine these two results, you get some wonderful result here. And this is our fundamental result now. So, start with a covering projection $p : \bar{X} \to X$ as usual, between path connected spaces, fix base points $x \in X, \bar{x} \in \bar{X}$ as usual, put $F = p^{-1}(x)$, $\bar{x} \in F$ etc. These notations I told you I have fixed, again I am repeating it. The first part here says, a loop ω at x can be lifted to a loop at $\bar{x}$ if and only if the element $[\omega] \in \pi_1(X, x)$ must be in the image of $p_\# : \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, x).$

So, this was the first part of the previous results. So, I am just summing it up, this is nothing more than that because I have combining the two lemmas here. Now, second part says, there is a loop in $\bar{X}$ which is a lift of ω if and only if some conjugate of $[\omega]$ belongs to the subgroup. So, in second to part, I am not mentioning the base point for $\bar{X}$. Just a loop in $\bar{X}$ which is a lift of omega, where is that loop taken?

However, the condition is that if and only if some conjugate of $[\omega]$ belongs to the subgroup $p_{\#}(\pi_1(\bar{X}, \bar{x})$. Here the base point $\bar{x}$ is fixed, it is sitting over x. The lifted loop may be at some other base point but the other base point will be also inside F. You cannot have any more freedom because p of that base point should be equal to x. Because, after all, ω is a loop at x. So, in $\bar{X}$, you are allowed to take a different base point. So, this part is what you have to work out here. The first part is only part of the first lemma, let us work out the proof of (b).

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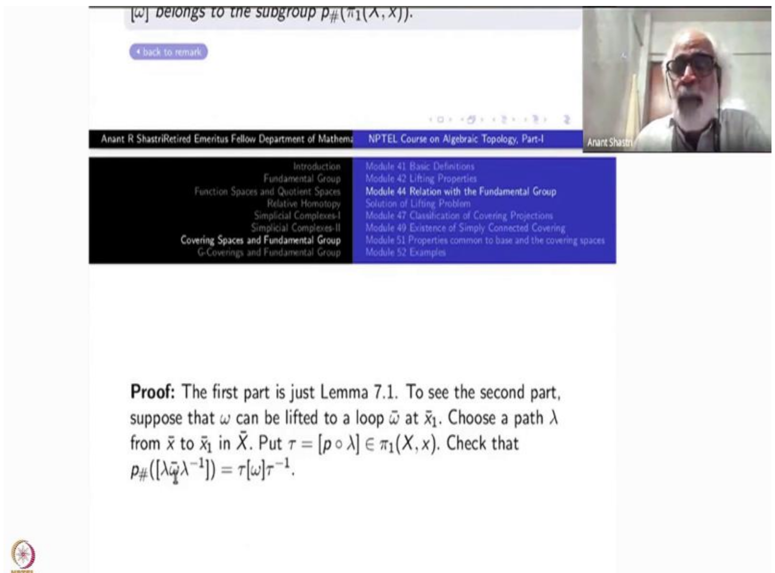

To see the part (b,) suppose ω can be lifted to a loop $\bar{\omega}$, based at some point $\bar{x}_1$ not necessarily equl to $\bar{x}$. Choose a path λ from $\bar{x}$ to $\bar{x}_1$ inside c $\bar{X}$. $\bar{X}$ is also path connected, the first time you are using that $\bar{X}$ is path connected. Now, put $\tau = p \circ \lambda$. Look at this. λ may notne a loop, but $p \circ \lambda$ is a loop at $x \in X$. That is because the end points of λ are in F. So, this is a loop, so look at its class in the fundamental group, denote it by $[\tau]$.

Now all that you have to check is: $p([\lambda * \bar{\omega} * \lambda^{-1}]) = [\tau][\omega][\tau]^{-1}$. Note th$\lambda * \bar{\omega} * \lambda^{-1}$ is a lopp at $\bar{x}$ because $\bar{\omega}$ is a loop at $\bar{x}_1$, so, first this λ is from $\bar{x}$ to $\bar{x}_1$, then this loop, again you come back by lambda inverse. So, this path becomes a loop $\bar{x}$. $\bar{\omega}$, a loop at $\bar{x}_1$ is converted is converted in to loop at $\bar{x}$. So, p composite of that is equal to $p \circ \lambda$, which is τ followed by $p \circ \bar{\omega}$, which is ω followed by $p \circ \lambda^{-1}$ which is τ^{-1}. Therefore $p_{\#}([\lambda * \bar{\omega} * \lambda^{-1}]) = [\tau][\omega][\tau]^{-1}$. The converse `if' is remaining, wahat we have seen is `only if' part. Let us workout the converse.

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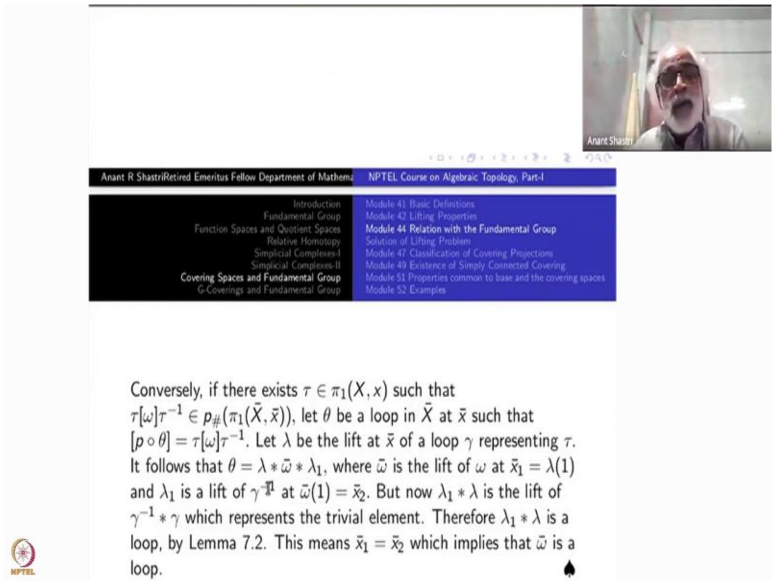

Suppose there is an element τ in the fundamental group, such that when you conjugate by that element the element that you started with, i.e., the element represented by a loop ω, you conjugate it and suppose that conjugate is in this inside this subgroup $p_{\#}(\pi_1(\bar{X}, \bar{x}))$. You start with some element, it is none of this is conjugate maybe in this subgroup, that is group theory.

Suppose this happens. Let θ be a loop in $\bar{X}$ at $\bar{x}$ such that $p_{\#}([\theta]) = [p \circ \theta] = \tau[\omega]\tau^{-1}$.

Now, let λ be the lift at $\bar{x}$ of a loop γ at x, representing the element $\tau \in \pi_1(X, x)$. This lift λ, of course, is only a path at $\bar{X}$. You may actually choose θ such that $p \circ \theta = \lambda * \omega * \lambda^{-1}$ So, it follows that once, by the uniqueness of path lifting that this θ the concatanation of λ followed by a lift $\bar{\omega}$ of ω at $\lambda(1)$, followed by a lift λ_1 of λ^{-1} at $\bar{\omega}(1)$. Since θ is a loop at $\bar{x}$, it follows that $\lambda_1(1) = \bar{x}$. Again by uniqueness of path liftings, it follows that $\lambda_1^{-1} = \lambda$. Therefore, $\bar{\omega}(0) = \lambda(1) = \lambda_1(0) = \bar{\omega}(1)$. That means ω is lifted to a loop $\bar{\omega}$. That completes the proof. You amy ignore the next two paragraphs.

So, where omega bar is lift of this one in $\bar{X}$ Lambda 1 is what? Lambda 1 is the lift of omega bar at the end point of this lift then omega bar of 1. If some other x2, x2 bar, but now look at lambda

1 composite lambda star, lambda 1 is somewhere x2 and ends up with x bar, so I can take lambda here, this is the lift of gamma inverse composite gamma, because lambda was a lift of, the gamma was a lift of lambda, lambda is the lift what is this lambda is a lift of tau, representing tau.

So, but this is whatever it is, this is gamma inverse gamma this is both of them are loops, this is null homotopy, it is a trivial element. Therefore, this lambda 1 composite lambda is a loop, because the lift of a trivial map is a trivial trivial path is trivial path, trivial path has both endpoints same. So, all its lifts must be in the homotopy class must be a loops. So, this is a loop that means this x1 bar is equal to x2 bar, once they are, these two points are same it just means that omega bar is a loop.

Note that given a loop at x if we are able to lift it in $\bar{X}$ as a loop then we have concluded that the given loop represents an element which is a conjugate of some element in the subgroup $p_{\#}(\pi_1(\bar{X},\bar{x}).$ However, if the lifted loop is based at $\bar{x}$, there is no need to take conjugate; the element represented by the loop is in the subgroup already.

If it is not in the subgroup, we may have to conjugate with something, then the lift will not be at that point but at some other point in the fibre. So, this is the thing, the geometrical statement is completely converted into algebra, what happens to the fundamental group level and what happens to that inside this subgroup $p_{\#}(\pi_1(\bar{X},\bar{x}).$ So, this subgroup becomes important for us.

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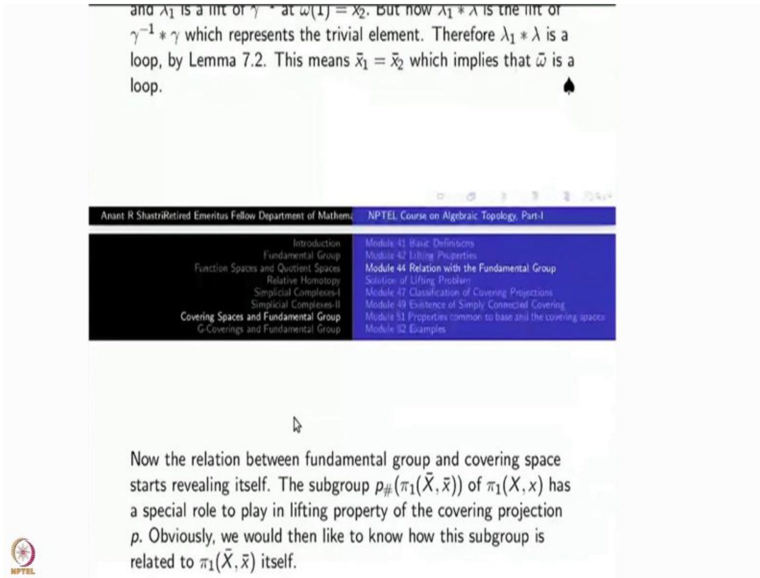

So, the relation between the fundamental group and the covering space starts revealing itself. The subgroup p check as a special role to play in the lifting property of covering projections. So obviously we would like to know how the subgroup is related to the fundamental group of pi 1 of X bar itself, more closely, it is image of that one under p check. So how is this homomorphism $p_\#$ behaves ---that is our next topic. So let us stop here and take up the study later on.