

**Introduction to Algebraic Topology (Part I)**  
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**Lecture 44**  
**Relation with the Fundamental Group**

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Module 44 Relation with Fundamental Group

Having established the HLP for covering projections, we continue our study of lifting maps through a covering projection. Fix a covering projection  $p : \bar{X} \rightarrow X$ . Also fix a point  $x_0 \in X$ , which we shall refer to **base point** for  $X$ . And put  $F := p^{-1}(x_0)$  and fix a point  $\bar{x}_0 \in F$  as base point for  $\bar{X}$ .

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Today, we shall study the Relation with the Fundamental Group of the covering space theory, having established the homotopy lifting property for covering projections, we will have to continue the study of lifting maps through a covering projection. So, throughout this lecture and for one or two more lectures, I fix the notation:  $p : \bar{X} \rightarrow X$ , is a covering projection,  $x_0 \in X$  is a fixed point in  $X$ , which will be called the base point for  $X$ . We will put  $F = p^{-1}\{x_0\}$ ; this  $F$  stands for fibre over  $x_0$ . Inside this fibre, we will take a point  $\bar{x}_0$  as a base point for  $\bar{X}$ . So, this  $x_0, \bar{x}_0$  are fixed and  $\bar{x}_0$  is mapped onto  $x_0$  under  $p$ .

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To begin with, we know that every path can be lifted. What about loops? Given a loop  $\omega : \mathbb{I} \rightarrow X$  at  $x_0$  say, we can lift it at  $\bar{x} \in F$  as a path. By the ULP, if this path is not a loop, then we are helpless, in the sense that there is no loop at  $\bar{x}$  which is a lift of  $\omega$ . May-be there is one at a different point  $\bar{y} \in F$ .

So, to begin with we know that every path at  $x_0$  inside  $X$  can be lifted to a path in  $\bar{X}$  at  $\bar{x}_0$ . The question is, suppose I take a loop at  $x_0$  as a path, we can lift it, but the lifted path may not be a loop. If it is not a loop, there is nothing you can do about it, because of the unique lifting property. Of course, if you take the base point to be different, there may be a lift which is a loop, but at that base point  $\bar{x}_0$ , there is no other way. So, we have to investigate this: when a lift of a loop will be a loop.

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May-be there is one at a different point  $\bar{y} \in F$ .



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We observe that if the lift of  $\omega$  were a loop then it represents an element in  $\pi_1(\bar{X}, \bar{x}_0)$  which is mapped onto the element  $[\omega]$  in  $\pi_1(X, x_0)$ . Thus we are led to study the inter-relationship between fundamental groups and covering projections. We begin with two lemmas which are immediate consequences of unique path lifting property.

So, suppose the lift of a loop is a loop in  $\bar{X}$ . Then that loop will represent an element in  $\pi_1(\bar{X}, \bar{x}_0)$  and its image under  $p\#$ , ( $p\#$  is the induced group homomorphism the  $\pi_1$  level,) is precisely the class represented by the loop that we started with. So, this is how the question of lifting a loop to a loop is related to something happening at the fundamental group level. So, the algebra enters here. So, for studying this problem, let us first state and prove fundamental lemmas and then keep using them to derive several algebraic properties.

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**Lemma 7.1**

Let  $p: \bar{X} \rightarrow X$  be a covering projection,  $x_0 \in X$ ,  $\bar{x}_0 \in \bar{X}$  and  $p(\bar{x}_0) = x_0$ . Let  $\omega$  be a path in  $X$  with  $\omega(0) = x_0$ , and let  $\bar{\omega}$  be a lift of  $\omega$  at  $\bar{x}_0$ .

(i) The end-point of  $\bar{\omega}$ , viz.,  $\bar{\omega}(1)$  depends only on the path homotopy class of  $\omega$  in  $X$  and not on actual representative path.

(ii)  $\bar{\omega}$  is a loop in  $\bar{X}$  if and only if  $\omega$  is a loop in  $X$  such that  $[\omega] \in p\#(\pi_1(\bar{X}, \bar{x}))$ . In this case, lift of any member of  $[\omega]$  at  $\bar{x}_0$  is a loop.

So, the notations are as before,  $p$  is a covering projection,  $x_0$  is a base point at  $X$  and  $\bar{x}_0$  is a base point in  $\bar{X}$  sitting over  $x_0$ . Take a path  $\omega$  in  $X$  starting at  $x_0$ , let  $\bar{\omega}$  be the lift of  $\omega$  at  $\bar{x}_0$ . First part of the lemma is: if you look at the endpoint of  $\bar{\omega}$ , this will not depend on the path  $\omega$  itself but only on the homotopy class  $[\omega]$ . Whatever path you take in homotopy class of  $\omega$ , the path may be different, but in the same homotopy class, then endpoint of its lift will be always the same. So, this is the first statement.

The second statement is:  $\bar{\omega}$ , the lifted path is a loop if and only if  $\omega$  is a loop in  $X$  such that the represented the class  $[\omega]$ , the class represented by  $\omega$  is in the image of  $p\#$ . In this case lift of any member of  $[\omega]$ , that means, take any element of  $p\#(\pi_1(\bar{X}, \bar{x}_0))$ , and any loop representing this class, for all of them, the lifts at  $\bar{x}_0$  will be loops. So, this second part we have already observed, we have already observed and the second second part or second part will be easy consequence of the first part.

One lift, look at the end point, end point is the same thing as the starting point, so all other lifts will have to be loop because they should also have the same endpoints. So, this part follows easily. This we have seen. This is the meaning. An element  $a$  is in  $p_{\#}(\pi_1(\bar{X}, \bar{x}_0))$ , means  $a = [p \circ \lambda]$  for some loop  $\lambda$  at  $\bar{x}_0$ , lifts of all members of  $a$  at  $\bar{x}_0$  will be loops. So, second part easy.

The first part needs some explanation, why the endpoint is independent of the homotopy (homotopy class, this thing we have seen already in the case of exponential function. Remember how the degree of a loop in  $S^1$  was defined. You lifted it and then you looked at the end point, if you take another representative in the same class and lifted, the endpoint will be again the same, so that endpoint there happened to be an integer, because it is the exponential map.

Here there is no integer or anything, but we want to say that the endpoint is independent of the representative of the homotopy class, it only depends on the homotopy class. So, proof is more or less the same if we pretend as if we use the integers and exponential and so on. The fundamental property of exponential function was that it is covering projection, and that is what we are going to use here, namely for any covering, the fibre is a discrete space, discrete subset of the top space  $\bar{X}$ . So, that is what we are going to use. Once I have said that one, I have told the proof of the whole thing. However, let us go through it slowly and see once again why this is true.

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Figure 39: Lifting Paths through a Covering Projection

So, here is a full picture of this one, you can keep referring to this all the time. So, here is your  $X$  and here is your  $\bar{X}$  and this is  $p$ , this  $\omega$  is a path here from this point to this point, these dot dot dots here are sitting over this point, we could call this  $x_0$  and these points are all in the fibre of  $x_0$ . So, here this is  $x_1$  and these are all fibres above  $x_1$  sitting over that  $x_1$ . So, this  $\omega$  is lifted to a path here  $\bar{\omega}$ ;  $\omega'$  is homotopic to  $\omega$ , (it is path homotopy). When you lift  $\omega'$  this ends at the same point, that is the, that is the statement we have yet prove. So, this is the homotopy,  $H : \mathbb{I} \times \mathbb{I} \rightarrow X$ , this part gives  $\omega$ , and that path gives  $\omega'$ .

And when you lift the homotopy  $H$ , ( this is  $\bar{H}$  actually, it looks like  $H$ ),  $\bar{H} : \mathbb{I} \times \mathbb{I} \rightarrow \bar{X}$  defines from  $\bar{\omega}$  to  $\bar{\omega}'$ . So, what have we done. We take this homotopy  $H$  from  $\omega$  to  $\omega'$  and take a lift  $\bar{H}$  with the condition that  $p \circ \bar{H}(t, 0) = \omega(t)$ . Automatically  $\bar{H}$  restricted  $\mathbb{I} \times \{1\}$  will be a lift of  $\omega'$  by the uniqueness of path-liftings. And what we want to show is that the endpoints are the same:  $\bar{\omega}(1) = \bar{\omega}'(1)$ .

Why the endpoints are the same? Look at this part,  $\{1\} \times \mathbb{I}$ , what is this part? Under  $H$ , because it is a path homotopy all this go to this single point point  $x_1$ , this whole thing goes to this point. Similarly, this is  $\{0\} \times \mathbb{I}$  goes to  $x_0$ , This means when you lift it this entire thing the image must be contained inside these, these points. Similarly, image contained inside this point, we have chosen the image to be this point, once this point is there because this is connected space and this is a discrete space, so the entire thing must be this point here, there is no problem.

Same way, we have not chosen where  $\bar{\omega}(1)$ ,  $\bar{\omega}'(1)$  are, both are in  $F = p^{-1}(\{x_1\})$ . Also this entire path  $\bar{H}(1 \times \mathbb{I})$  is a single point inside  $F$ , the same point  $\bar{\omega}(1)$ , because this is connected and this is a discrete space. So, that is the property that we use. So, this point is the same thing as endpoint of this one, over. So that is the proof of part one. And part two I have already shown, how part two comes from part one, full detail is written down here with notations.

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**Figure 39: Lifting Paths through a Covering Projection**

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**Proof:** (i) Suppose  $\omega'$  is a path in  $X$  path-homotopic to  $\omega$ . (Note that this implies  $\omega(0) = \omega'(0) = x_0$  and  $\omega(1) = \omega'(1) = x_1$ , say. Let  $H$  be a path homotopy from  $\omega$  to  $\omega'$  and  $\bar{H}$  be a lift of  $H$  such that  $\bar{H}(0, 0) = \bar{x}_0$ . By the uniqueness of the lifts, it follows that  $\bar{H}(-, 0) = \bar{\omega}$ ,  $\bar{H}(-, 1) = \bar{\omega}'$ . It follows that  $\bar{H}(0, s) \subset p^{-1}(x_0)$  and  $\bar{H}(1, s) \subset p^{-1}(x_1)$ . Hence, by the discreteness of the fibres and the connectedness of  $\mathbb{I}$ ,  $\bar{H}(0, s) = \bar{x}_0$ ,  $\bar{H}(1, s) = \bar{x}_1$ ,  $\forall s \in \mathbb{I}$  where  $\bar{x}_1$  is a single element such that  $p(\bar{x}_1) = x_1$ . In particular,  $\bar{\omega}(1) = \bar{H}(1, 0) = \bar{x}_1 = \bar{H}(1, 1) = \bar{\omega}'(1)$ .  
(ii) Easy.

Actually, in this detail, there may be some typos, but what I have told you here is crystal clear. So, start with  $\omega'$  is path homotopic to  $\omega$ , two different representatives of the same class. First of all, the starting points are the same, end points are the same, that is what you have require, we have to notice that. Now  $H$  is a path homotopy from  $\omega$  to  $\omega'$ ,  $\bar{H}$  is a lift of  $H$  so that  $\bar{H}(t, 0) = \bar{\omega}(t)$ . The rest of them are automatically defined.

Because of what?  $\{t\} \times \mathbb{I}$  and  $\mathbb{I} \times \{s\}$ , by their uniqueness of the lifts, it follows that the first horizontal interval  $\mathbb{I} \times \{0\}$  goes to  $\bar{\omega}$ , the second one at level 1, goes to  $\bar{\omega}'$  and  $\bar{H}(0 \times \mathbb{I})$  is inside  $p^{-1}(x_0)$  and  $\bar{H}(\{1\} \times \mathbb{I}) \subset p^{-1}(x_1)$ . These two are discrete, and this is connected. So, this image must be a single point. Let us, go to the next result now. This is the most fundamental. Just the discreteness of fibres are used here.

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$\omega(1) = p(1,0) = x_1 = p(1,1) = \omega(1)$ .  
(ii) Easy.

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**Lemma 7.2**

Suppose  $\omega_1, \omega_2$  are paths in  $X$ , with initial point  $x$  and end-point  $y$ . Suppose that,  $\omega_1 * \omega_2^{-1}$  lifts to a loop at  $\bar{X}$ , where  $p(\bar{X}) = x$ . Let  $\bar{\omega}_1, \bar{\omega}_2$  be the lifts of  $\omega_1, \omega_2$ , respectively, at  $\bar{X}$ . Then  $\bar{\omega}_1(1) = \bar{\omega}_2(1)$ .

**Proof:** Let  $\gamma$  be the loop at  $\bar{X}$  such that,  $p \circ \gamma = \omega_1 * \omega_2^{-1}$ . By the uniqueness of the lift, it follows that,  $\bar{\omega}_1(t) = \gamma(t/2) \forall t \in \mathbb{I}$ . It also follows that,  $\gamma(1-t/2) = \bar{\omega}_2(t), \forall t \in \mathbb{I}$ . In particular,  $\bar{\omega}_2(1) = \gamma(1/2) = \bar{\omega}_1(1)$ , as claimed.

Now, suppose you have two paths  $\omega_1, \omega_2$  in  $X$ , both with initial point  $x$  and endpoint  $y$ . ( Instead of  $x_1$  and  $x_2$  or  $x_0$  and  $x_1$ . I will change the notation, you should be able to do that without notation change, now there are some problems here with typo, I just changed it but still it has not come, sorry.) So, suppose you traverse  $\omega_1$  first and then come back by  $\omega_2^{-1}$ , That is a loop at  $x$ . Suppose this loop lifts to a loop at  $\bar{x}$  in  $\bar{X}$ , where  $p(\bar{x}) = x$ . Let  $\bar{\omega}_1, \bar{\omega}_2$  be the lifts of  $\omega_1, \omega_2$  respectively at the point  $\bar{x}$ . Then the endpoints of these ones are the same:  $\bar{\omega}_1(1) = \bar{\omega}_2(1)$ .

You see the first lemma said that  $\omega_1, \omega_2$  are homotopic paths, then the endpoints of the two lifts are the same that was the conclusion. Here what I have? I just have that  $\omega_1 * \omega_2^{-1}$  lifts to a loop. There is no homotopy or anything. Then how to get this one? Let us see. So, this is not all that difficult, but it looks somewhat challenging. Suppose  $\gamma$  is a loop at  $\bar{x}$  such that  $p \circ \gamma = \omega_1 * \omega_2^{-1}$ .

So, this is the hypothesis. that  $\omega_1 * \omega_2^{-1}$  lifts to a loop, that loop I am calling  $\gamma$ . So, this is a loop at  $\bar{x}$ . By the uniqueness of the lifts, it follows that the first half of  $\gamma$  is equal to  $\bar{\omega}_1$ . What is the definition of  $\omega_1 * \omega_2^{-1}$ ?  $\omega_1 * \omega_2^{-1}(t) = \omega_1(2t), 0 \leq t \leq 1/2$ . Therefore  $p \circ \gamma(t) = \omega_1(2t), 0 \leq t \leq 1/2$ . Similarly,  $p \circ \gamma(t) = \omega_2^{-1}(2t - 1)$ .

Therefore, in the interval  $0 \leq t \leq 1/2$ , we have,  $p \circ \gamma^{-1}(t) = p \circ \gamma(1 - t) = \omega_2^{-1}(2(1 - t) - 1) = \omega_2^{-1}(1 - 2t) = \omega_2(2t)$ . In particular, by the

uniqueness of the lifts, it follows that  $\gamma^{-1}(t) = \bar{\omega}_2(t), 0 \leq t \leq 1$ . This implies  $\bar{\omega}_2(1) = \gamma^{-1}(1/2) = \gamma(1/2) = \bar{\omega}_1(1)$ .

So, this gamma wherever, if it cannot be looped if two different paths have been lifted, two loops have been lifted at one point and the end points are different, you start from 1 point to go, you cannot come back, you have to come back from the same path not from the other path. When you are coming from the other path means the endpoints of the other two must be the same. So, that is geometrically it is simple, but we have verified now. So, this is this looks like nothing to do with the covering space theory, this is simply geometry here.

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The screenshot shows a presentation slide with a table of contents at the top and a theorem statement below. The table of contents includes:

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Below the table, the text reads: "Combining the above two lemmas we have," followed by a blue box containing:

**Theorem 7.5**

Let  $p : \bar{X} \rightarrow X$  be a covering projection of path connected spaces,  $x \in X, \bar{x} \in \bar{X}$  be such that  $p(\bar{x}) = x$  and let  $F = p^{-1}(x)$ .

(a) A loop  $\omega$  at  $x$  in  $X$  can be lifted to a loop at  $\bar{x}$  in  $\bar{X}$  iff the element  $[\omega]$  belongs to  $p_{\#}(\pi_1(\bar{X}, \bar{x}))$ .

(b) There is a loop in  $\bar{X}$  which is a lift of  $\omega$  iff some conjugate of  $[\omega]$  belongs to the subgroup  $p_{\#}(\pi_1(\bar{X}, \bar{x}))$ .

If you combine these two results, you get some wonderful result here. And this is our fundamental result now. So, start with a covering projection  $p : \bar{X} \rightarrow X$  as usual, between path connected spaces, fix base points  $x \in X, \bar{x} \in \bar{X}$  as usual, put  $F = p^{-1}(x), \bar{x} \in F$  etc. These notations I told you I have fixed, again I am repeating it. The first part here says, a loop  $\omega$  at  $x$  can be lifted to a loop at  $\bar{x}$  if and only if the element  $[\omega] \in \pi_1(X, x)$  must be in the image of  $p_{\#} : \pi_1(\bar{X}, \bar{x}) \rightarrow \pi_1(X, x)$ .

So, this was the first part of the previous results. So, I am just summing it up, this is nothing more than that because I have combining the two lemmas here. Now, second part says, there is a loop in  $\bar{X}$  which is a lift of  $\omega$  if and only if some conjugate of  $[\omega]$  belongs to the subgroup. So, in second



to part, I am not mentioning the base point for  $\bar{X}$ . Just a loop in  $\bar{X}$  which is a lift of  $\omega$ , where is that loop taken?

However, the condition is that if and only if some conjugate of  $[\omega]$  belongs to the subgroup  $p_{\#}(\pi_1(\bar{X}, \bar{x}))$ . Here the base point  $\bar{x}$  is fixed, it is sitting over  $x$ . The lifted loop may be at some other base point but the other base point will be also inside  $F$ . You cannot have any more freedom because  $p$  of that base point should be equal to  $x$ . Because, after all,  $\omega$  is a loop at  $x$ . So, in  $\bar{X}$ , you are allowed to take a different base point. So, this part is what you have to work out here. The first part is only part of the first lemma, let us work out the proof of (b).

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$[\omega]$  belongs to the subgroup  $p_{\#}(\pi_1(\lambda, X))$ .

back to remark

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**Proof:** The first part is just Lemma 7.1. To see the second part, suppose that  $\omega$  can be lifted to a loop  $\bar{\omega}$  at  $\bar{x}_1$ . Choose a path  $\lambda$  from  $\bar{x}$  to  $\bar{x}_1$  in  $\bar{X}$ . Put  $\tau = [p \circ \lambda] \in \pi_1(X, x)$ . Check that  $p_{\#}([\lambda \bar{\omega} \lambda^{-1}]) = \tau[\omega]\tau^{-1}$ .

To see the part (b,) suppose  $\omega$  can be lifted to a loop  $\bar{\omega}$ , based at some point  $\bar{x}_1$  not necessarily equal to  $\bar{x}$ . Choose a path  $\lambda$  from  $\bar{x}$  to  $\bar{x}_1$  inside  $\bar{X}$ .  $\bar{X}$  is also path connected, the first time you are using that  $\bar{X}$  is path connected. Now, put  $\tau = p \circ \lambda$ . Look at this.  $\lambda$  may not be a loop, but  $p \circ \lambda$  is a loop at  $x \in X$ . That is because the end points of  $\lambda$  are in  $F$ . So, this is a loop, so look at its class in the fundamental group, denote it by  $[\tau]$ .

Now all that you have to check is:  $p([\lambda * \bar{\omega} * \lambda^{-1}]) = [\tau][\omega][\tau]^{-1}$ . Note that  $\lambda * \bar{\omega} * \lambda^{-1}$  is a loop at  $\bar{x}$  because  $\bar{\omega}$  is a loop at  $\bar{x}_1$ , so, first this  $\lambda$  is from  $\bar{x}$  to  $\bar{x}_1$ , then this loop, again you come back by  $\lambda$  inverse. So, this path becomes a loop  $\bar{x}$ .  $\bar{\omega}$ , a loop at  $\bar{x}_1$  is converted to a loop at  $\bar{x}$ .

to loop at  $\bar{x}$ . So,  $p$  composite of that is equal to  $p \circ \lambda$ , which is  $\tau$  followed by  $p \circ \bar{\omega}$ , which is  $\omega$  followed by  $p \circ \lambda^{-1}$  which is  $\tau^{-1}$ . Therefore  $p_{\#}([\lambda * \bar{\omega} * \lambda^{-1}]) = [\tau][\omega][\tau]^{-1}$ . The converse 'if' is remaining, what we have seen is 'only if' part. Let us work out the converse.

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Conversely, if there exists  $\tau \in \pi_1(X, x)$  such that  $\tau[\omega]\tau^{-1} \in p_{\#}(\pi_1(\bar{X}, \bar{x}))$ , let  $\theta$  be a loop in  $\bar{X}$  at  $\bar{x}$  such that  $[p \circ \theta] = \tau[\omega]\tau^{-1}$ . Let  $\lambda$  be the lift at  $\bar{x}$  of a loop  $\gamma$  representing  $\tau$ . It follows that  $\theta = \lambda * \bar{\omega} * \lambda_1$ , where  $\bar{\omega}$  is the lift of  $\omega$  at  $\bar{x}_1 = \lambda(1)$  and  $\lambda_1$  is a lift of  $\gamma^{-1}$  at  $\bar{\omega}(1) = \bar{x}_2$ . But now  $\lambda_1 * \lambda$  is the lift of  $\gamma^{-1} * \gamma$  which represents the trivial element. Therefore  $\lambda_1 * \lambda$  is a loop, by Lemma 7.2. This means  $\bar{x}_1 = \bar{x}_2$  which implies that  $\bar{\omega}$  is a loop.

Suppose there is an element  $\tau$  in the fundamental group, such that when you conjugate by that element the element that you started with, i.e., the element represented by a loop  $\omega$ , you conjugate it and suppose that conjugate is in this inside this subgroup  $p_{\#}(\pi_1(\bar{X}, \bar{x}))$ . You start with some element, it is none of this is conjugate maybe in this subgroup, that is group theory.

Suppose this happens. Let  $\theta$  be a loop in  $\bar{X}$  at  $\bar{x}$  such that  $p_{\#}([\theta]) = [p \circ \theta] = \tau[\omega]\tau^{-1}$ .

Now, let  $\lambda$  be the lift at  $\bar{x}$  of a loop  $\gamma$  at  $x$ , representing the element  $\tau \in \pi_1(X, x)$ . This lift  $\lambda$ , of course, is only a path at  $\bar{X}$ . You may actually choose  $\theta$  such that  $p \circ \theta = \lambda * \omega * \lambda^{-1}$ . So, it follows that once, by the uniqueness of path lifting that this  $\theta$  the concatenation of  $\lambda$  followed by a lift  $\bar{\omega}$  of  $\omega$  at  $\lambda(1)$ , followed by a lift  $\lambda_1$  of  $\lambda^{-1}$  at  $\bar{\omega}(1)$ . Since  $\theta$  is a loop at  $\bar{x}$ , it follows that  $\lambda_1(1) = \bar{x}$ . Again by uniqueness of path liftings, it follows that  $\lambda_1^{-1} = \lambda$ . Therefore,  $\bar{\omega}(0) = \lambda(1) = \lambda_1(0) = \bar{\omega}(1)$ . That means  $\omega$  is lifted to a loop  $\bar{\omega}$ . That completes the proof. You may ignore the next two paragraphs.

So, where  $\omega$  bar is lift of this one in  $\bar{X}$   $\lambda(1)$  is what?  $\lambda(1)$  is the lift of  $\omega$  bar at the end point of this lift then  $\omega$  bar of 1. If some other  $x_2, x_2$  bar, but now look at  $\lambda$

1 composite lambda star, lambda 1 is somewhere x2 and ends up with x bar, so I can take lambda here, this is the lift of gamma inverse composite gamma, because lambda was a lift of, the gamma was a lift of lambda, lambda is the lift what is this lambda is a lift of tau, representing tau.

So, but this is whatever it is, this is gamma inverse gamma this is both of them are loops, this is null homotopy, it is a trivial element. Therefore, this lambda 1 composite lambda is a loop, because the lift of a trivial map is a trivial trivial path is trivial path, trivial path has both endpoints same. So, all its lifts must be in the homotopy class must be a loops. So, this is a loop that means this x1 bar is equal to x2 bar, once they are, these two points are same it just means that omega bar is a loop.

Note that given a loop at  $x$  if we are able to lift it in  $\bar{X}$  as a loop then we have concluded that the given loop represents an element which is a conjugate of some element in the subgroup  $p_{\#}(\pi_1(\bar{X}, \bar{x}))$ . However, if the lifted loop is based at  $\bar{x}$ , there is no need to take conjugate; the element represented by the loop is in the subgroup already.

If it is not in the subgroup, we may have to conjugate with something, then the lift will not be at that point but at some other point in the fibre. So, this is the thing, the geometrical statement is completely converted into algebra, what happens to the fundamental group level and what happens to that inside this subgroup  $p_{\#}(\pi_1(\bar{X}, \bar{x}))$ . So, this subgroup becomes important for us.

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and  $\lambda_1$  is a lift of  $\gamma^{-1}$  at  $\omega(1) = x_2$ . But now  $\lambda_1 * \lambda$  is the lift of  $\gamma^{-1} * \gamma$  which represents the trivial element. Therefore  $\lambda_1 * \lambda$  is a loop, by Lemma 7.2. This means  $\bar{x}_1 = \bar{x}_2$  which implies that  $\bar{\omega}$  is a loop.

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Now the relation between fundamental group and covering space starts revealing itself. The subgroup  $p_{\#}(\pi_1(\bar{X}, \bar{x}))$  of  $\pi_1(X, x)$  has a special role to play in lifting property of the covering projection  $p$ . Obviously, we would then like to know how this subgroup is related to  $\pi_1(\bar{X}, \bar{x})$  itself.

So, the relation between the fundamental group and the covering space starts revealing itself. The subgroup  $p_{\#}$  check as a special role to play in the lifting property of covering projections. So obviously we would like to know how the subgroup is related to the fundamental group of  $\pi_1$  of  $X$  bar itself, more closely, it is image of that one under  $p_{\#}$ . So how is this homomorphism  $p_{\#}$  behaves ---that is our next topic. So let us stop here and take up the study later on.