**Introduction to Algebraic Topology (Part I)**
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**Lecture 43**
**Homotopy Lifting**

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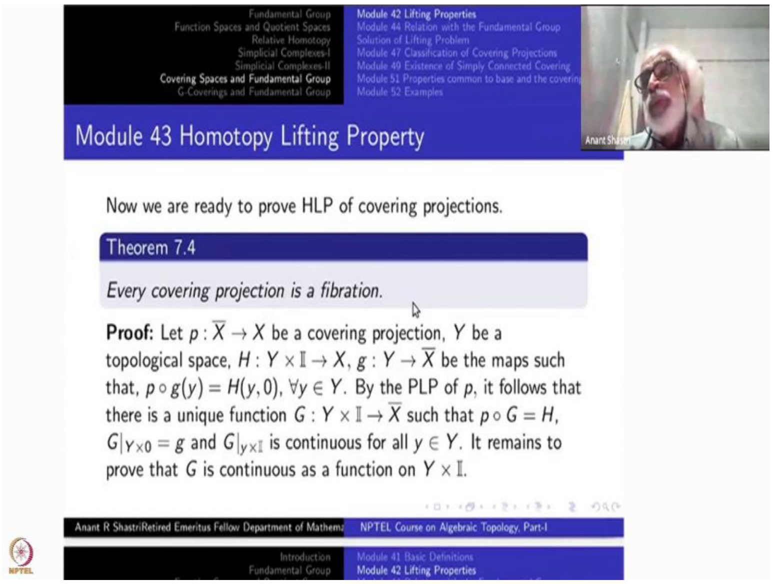


Today's topic is Homotopy Lifting property. So, the big theorem here is that every covering projection is a fibration. The proof is not at all new in the sense that the fundamental idea involved in has been already seen while studying the computation of the fundamental group of the circle. The exponential map had the property that paths can be lifted and homotopy of paths can also lifted, that is homotopy path lifting property that we have. Whatever idea is there it is the same idea that will work here; the only thing is that instead of instead of the exponential map explicitly we are taking any covering projection.

And the second thing is instead of homotopy of paths now we have homotopy of functions from arbitrary spaces. So, you will see that you have to work a little harder that is all. Start with the covering projection $p : \bar{X} \to X$, $Y$ is any topological space and H is the homotopy on $Y$ into $X$, $g : Y \to \bar{X}$ such that $p \circ g(y) = H(y, 0)$, just the starting map of the homotopy. This is the homotopy lifting the data if you recall.

To show that $p$ homotopy lifting property, what we have to proof is that there is a function $G: Y\times \mathbb{I}\to \bar{X}$, a continuous function such that $p\circ G = H$, and $G(y,0) = g(y)$. That is what we have to find. But path lifting property you have already proved, not homotopy lifting property.

Now, you fix one point here $y\in Y$, at a time, $H:\{y\}\times \mathbb{I}\to X$ is a path that can be lifted. starting point what you take? You take it to be $g(y)$. It is given to you already this point $g(y)$, $p(g(y)) = H(y,0)$. Y restricted to no no H restricted to little y cross I is a path, so that can be lifted already, do it for all $y\in Y$. What you get is a function $G$ with this property that $p\circ G = H$, and $G(y,0) = g(y)$. But this function $G$ may not be continuous as a function from $Y\times \mathbb{I}$ to $\bar{X}$. However, it is continuous whenever you restrict the first coordinate $Y$ to a single point. So, that much you have already got.

And once you require, by this requirement as a function $G$ is unique, there is no choice but we have to prove that this very $G$ whatever we have got is actually continuous function $Y\times \mathbb{I}\to \bar{X}$ . If this is not true then the theorem will be false that is all. So we have to prove this one, there is no other choice you cannot be any other function, but this G has to be this one. So, this is the point of path lifting property, the homotopy of each point here is nothing but a path. So that is what we have to do. So, what we have to do is G is continuous as a function from Y cross I this all we have to prove.

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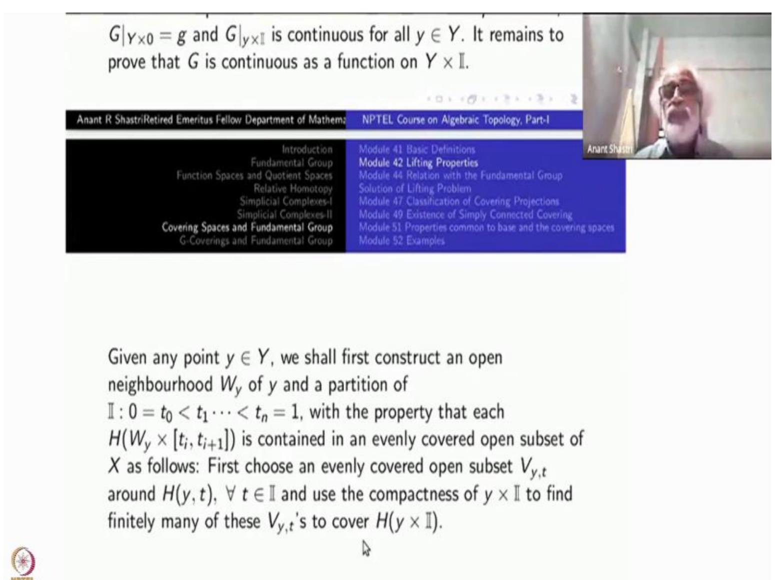

Given any point in $y \in Y$, we shall first construct an open neighbourhood $W_y$ of $y$ such that, such that H restricted to sorry G restricted to $W_y \times \mathbb{I}$ is continuous, then we are done. Because continuity is after all local property for each neighbourhood if it is true, then we are done. So, I am finding a neighbourhood $W_y$ of $y$ such that there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval [0,1] with the following property that each $W_y \times [t_i, t_{i+1}]$ is mapped by $H$, $H$ is a continuous function, $H$ of this one is contained in an evenly covered open subsets of $X$. So, this is what I want to do.

So, how do we do that? First of all choose an evenly covered open subset $V_{y,t}$ around $H(y,t)$, for each $t \in \mathbb{I}$, ($y \in Y$ is fixed) . Use the compactness of $H(\{y\} \times \mathbb{I})$ to find a partition of $\mathbb{I}$ as above, so that each $H(\{y\} \times t_i, t_{i+1}])$ is contained in an evenly coverd open set. All these things are happeing inside $X$ now.

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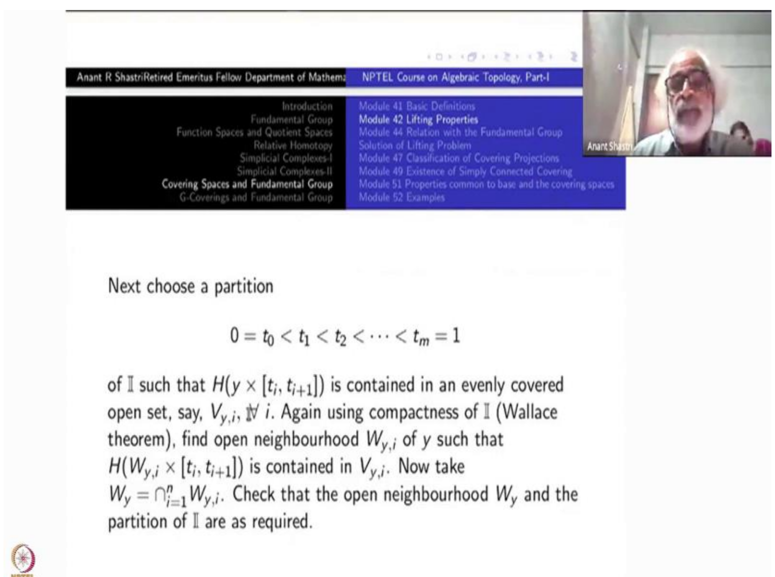


The partition of $\mathbb{I}$ as above follows by standard agruements with Lebesgue number etc. So, the interval can be divided into finitely many sub interval such that consecutive intervals y of y cross ti ti plus 1, H of that will go into one of the evenly covered open sets. So, this is similar to what we have done with the path lifting property also, path even to lift path lifting we are done this one.

Again, using compactness of the intervals $[t_i, t_{i+1}]$, and continuity of $H$, for each i, we get neighbourhoods $W_{y,i}$ of $y \in Y$ such that $H(W_{y,i} \times [t_i, t_{i+1}])$ are contaiend in evenly covered open sets. That is the Wallace theorem. You can find an open neighbourhood Wy,i of y such that this H of Wy i cross ti to ti plus 1 is contained V y i for every.

Now, take $W_y = \cap_{i=1}^{n} W_{y,i}$. So, you have got a uniform neighbourhood $W_y$ of $y \in Y$ and a partition of $\mathbb{I}$ such that $H(W_{y \times t_i, t_{i+1}}])$ is contained inside an evenly covered open set. That was our first aim here, the one single neighbourhood of y such that these intervals keep changing consecutively all of them are contained inside they, from first we had various intervals here, next we might make them finitely many, then you took the intersection of this finitely many to get into one single neighbourhood of the point y. Now, the hard work is over.

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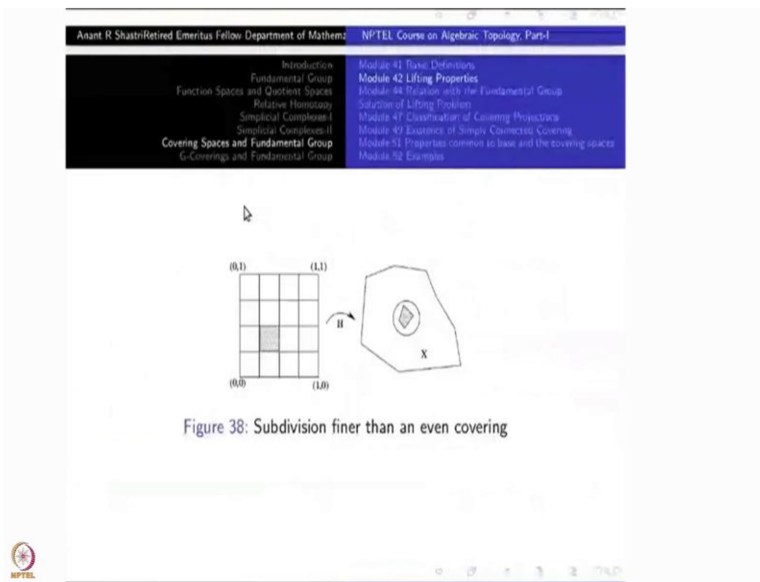


So, here I am again quoting same picture that you had while path lifting property for the exponential function, this is your interval and this is your X here that is all, X was also interval, so here, now you have to think of this as X, so when X is now covered by open sub sorry Y, Y is covered by open subsets, Wy, instead of rectangles like this and so on, but on along the Y axis you have you have subdivisions along the I the interval you have subdivisions here.

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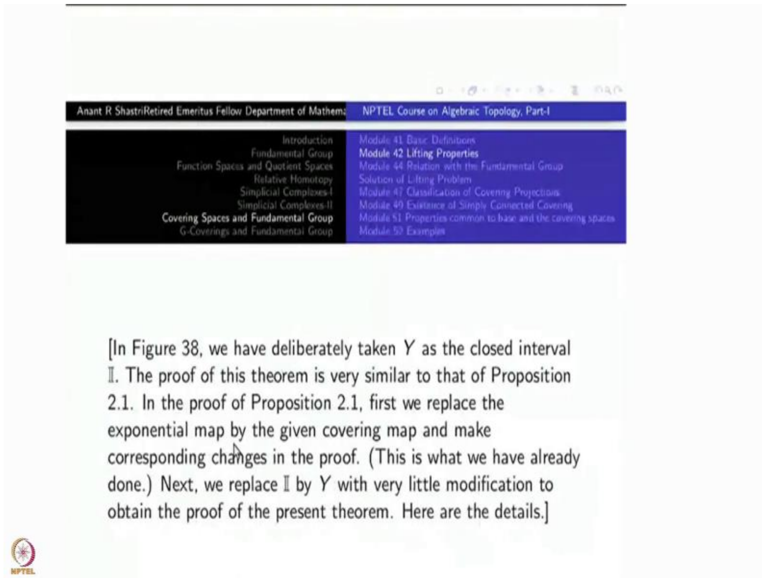


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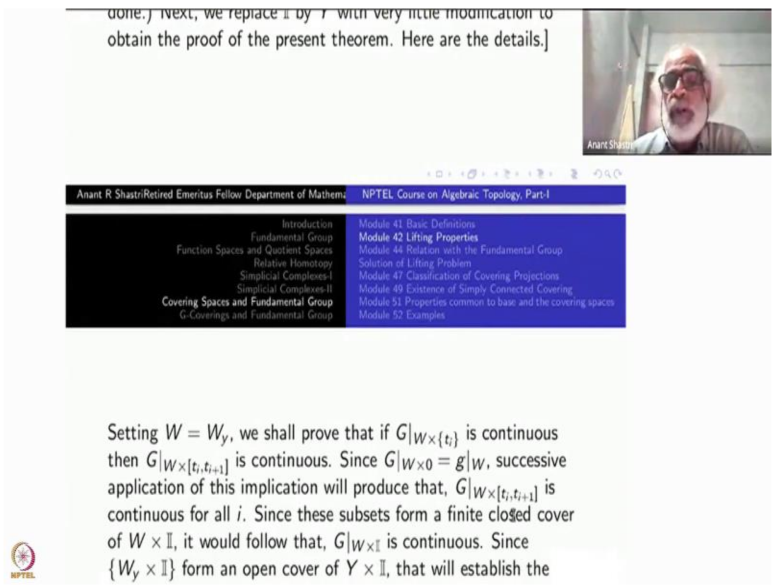


So, this is what I have explained already.

So, now put $W = W_y$. We shall prove that if $G|_{W\times\{t_i\}}$ implies $G|_{W\times[t_i,t_{i+1}]}$ is continuous. Since for $i=0$, $G|_{W\times\{0\}} = g$ is given to be continuous, successive application of this will produce imply that $G|_{W\times[t_i,t_{i+1}]}$ is continuous for all i and hence $G$ is continous on $W\times\mathbb{I}$. That will complete the proof the theorem.

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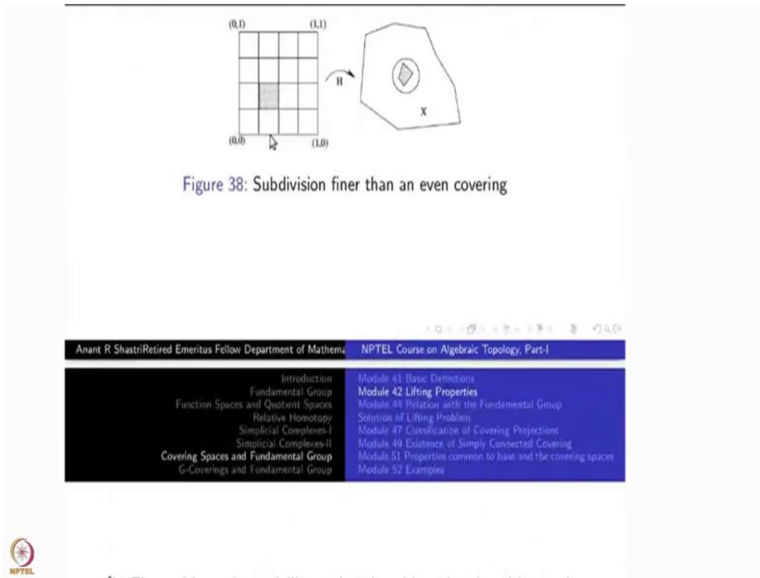


Given $z \in W$, choose an open set $U \subset \bar{X}$ such that $G(z, t_i) \in U$ and $p : U \to V$ is a homeomorphism, where $H(W \times [t_i, t_{i+1}]) \subset V \subset X$. By continuity of $G$, there exists an opens set $W_z \subset W$, such that $z \in W_z$ and $G(W_z \times t_i) \subset U$. Now take $F = p^{-1} \circ H : W_z \times [t_i, t_{i+1}] \to \bar{X}$. Then $F$ is continuous and on each $y' \in W_z$, we have $p \circ F(y', t) = H(y', t) = p \circ G(y', t)$. Moreover, the very choice, we have $F(y', t_i) = G(y', t_i)$. By uniqueness of the path lifting that we have proved, it follows that $F = G$ on $W_z \times [t_i, t_{i+1}]$. In particular, this means $G$ restrictedd to $W_z \times [t_i, t_{i+1}]$ is continuous. Since $z \in W$ was arbitrary, we have proved that $G$ is continous on $W \times [t_i, t_{i+1}]$.

So, what I am trying to say is to start with, on this one your function continuous. Therefore, you get a continuous function along this one, because this whole thing is contained inside a single even neighbourhood, so the inverse image makes sense, because it is evenly covered neighbourhood. But uniqueness whatever G you have got already there is no question this must be same thing as that one. So, continuity on these parts like this one-by-one follows. So, on each of them we have $G = p^{-1} \circ H$. Now, you have continuity along say this is G cross some ti, in short G cross y cross 0 to y cross ti, use that again along these blocks to go to the next stage.

So, ultimately and in finitely with the stages you will come but the whole thing is continuous, continuity on each of them these are closed subsets it is of Y cross I after all, being actually open subsets, so finitely many open subsets actually all along this way but this way did not need not finitely many because there are Wi's which cover the whole of y. So, on each open subset it is continuous, so whole thing is continuous. So, this part is same as in the case of exponential function and a path homotopy lifting.

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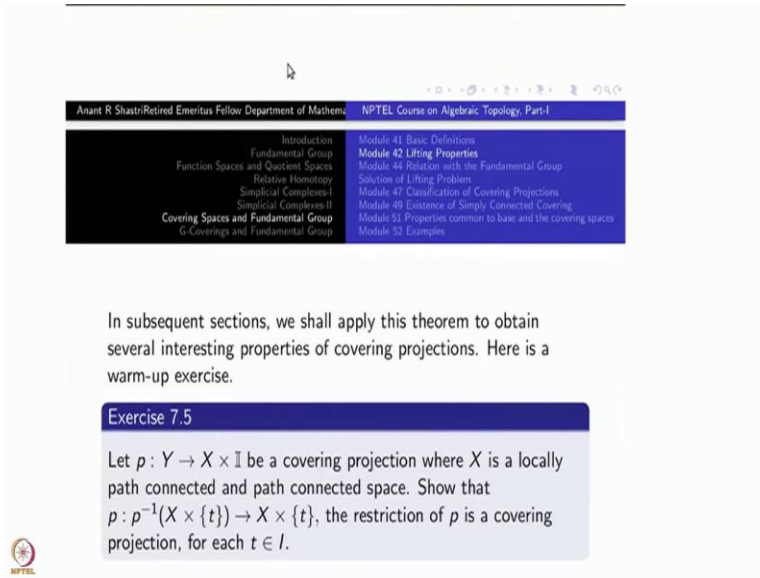


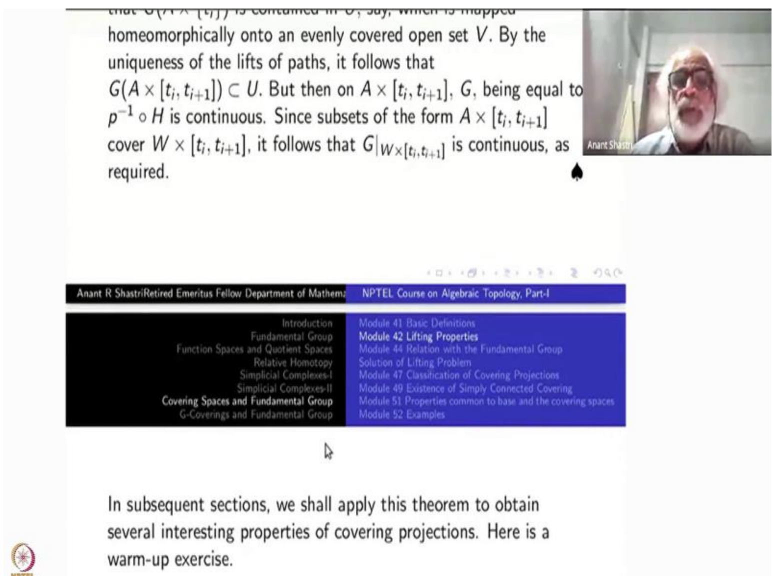


So, let us go to some little bit of exercises and so on. In what we are going to do in subsequent sections, this theorem will be applied again and again and then we will, then we will produce lot of interesting results. The fundamental result is that given homotopy $H$ from any space $Y$ to $X$ if one of the maps $H_t$ can be lifted, then the whole entire homotopy can be lifted. That is homotopy lifting property of the covering projection, this to be used again and again.

So, let us look at this exercise, take a covering projection $p$ over the space $X \times \mathbb{I}$, where $X$ is locally path connecting etcetera you have assume, and you can assume path connectivity also. You look at the restriction of $p$, $p : p^{-1}(X \times t) \to X \times t$ for each level $t \in \mathbb{I}$. This restriction map is a covering projection for each t, is what you have to show.

Remember if you take any arbitrary subset of the base and the universe image, fully inverse image, then that will be a covering projection. So, this is not a very difficult thing to prove, the point of this one is all these covering projections will be in some sense to b emade precise later on, will be same because X cross t is the same, X cross t or X cross s, they are homeomorphic to each other, they are copies of X, above spaces will be also homeomorphic to each other in such a way that the projection map it commutes with that homeomorphisim. So, that will come later, right now we have to just say this much, this is the second part.

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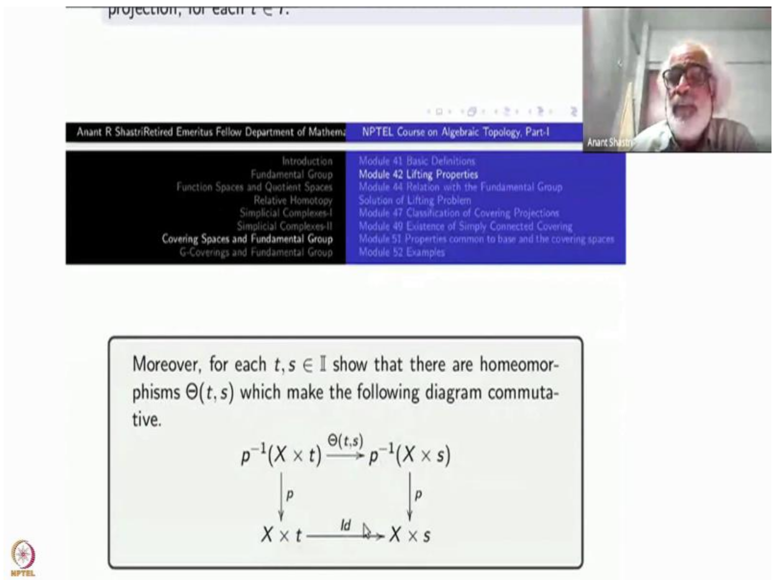


For each $t, s \in \mathbb{I}$, you have to show that there are homeomorphisms $\Theta_{t,s}$, depending upon t and s from $p^{-1}(X \times t)$ to $p^{-1}(X \times s)$ fitting the above diagram. This are homeomorphisms, this is identity map, identity map in the sense $(x, t) \mapsto (x, s)$. Only t is chaged to s . t goes to s that is That is all. So, such a diagram is possible, this is what you have to show. Looks crazy, but you do not need anything more than what you have learnt so far, do not even need homotopy lifting properties, just the definition of covering projections. So, let us stop here. Thank you.