Introduction to Algebraic Topology
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture – 42
Lifting Properties

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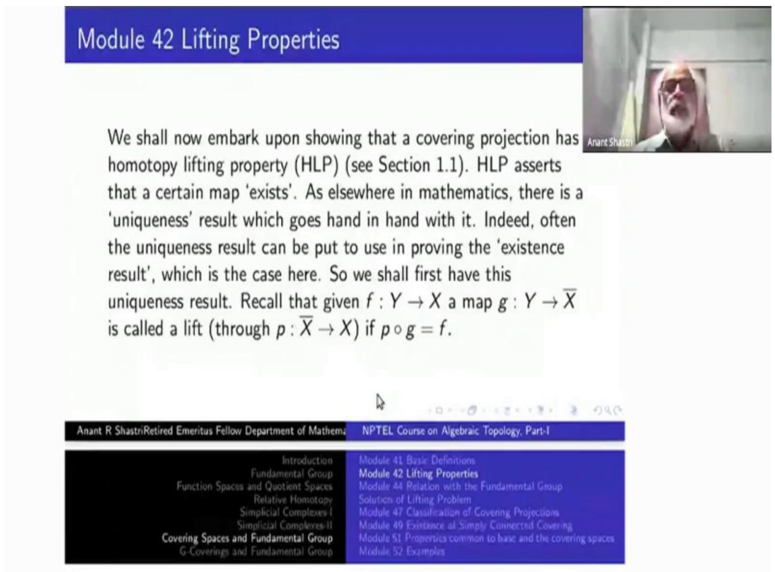

Today's topic is Lifting properties of covering projections. Last time we introduced the concept of covering projection, checked some basic properties such as: a covering projection is a local homeomorphism, and hence, its fibers are all discrete and it shares all the local properties of the base space and top space are the same. That is what we have seen. Before proceeding with lifting properties let me take up a few more properties bits of examples and so on. Then come back to and go ahead with lifting properties.

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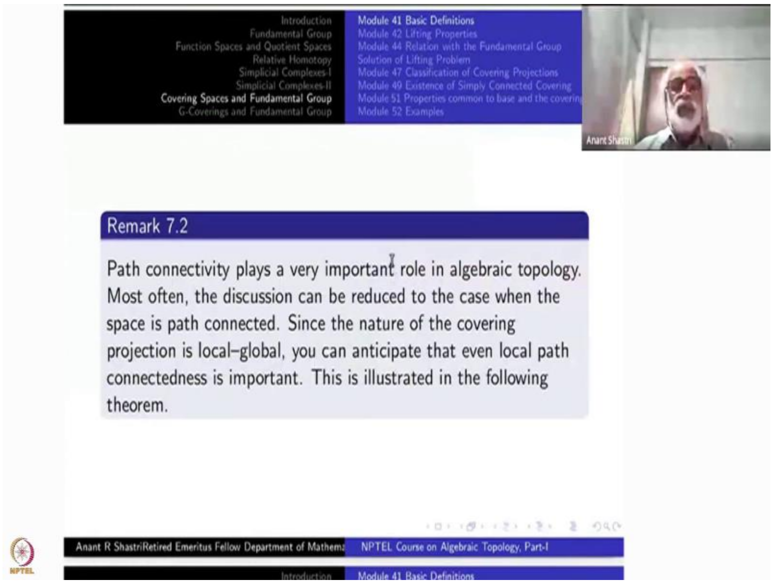

So, so we are about to establish that local connectivity, local path connectivity place a very important role in the study of covering spaces. So, often all authors blatantly assume that the base space is locally path connected before doing anything. But, just to keep the theory a little more general, we do not have to assume that. So, if you want to do some serious thing you may have to assume that.

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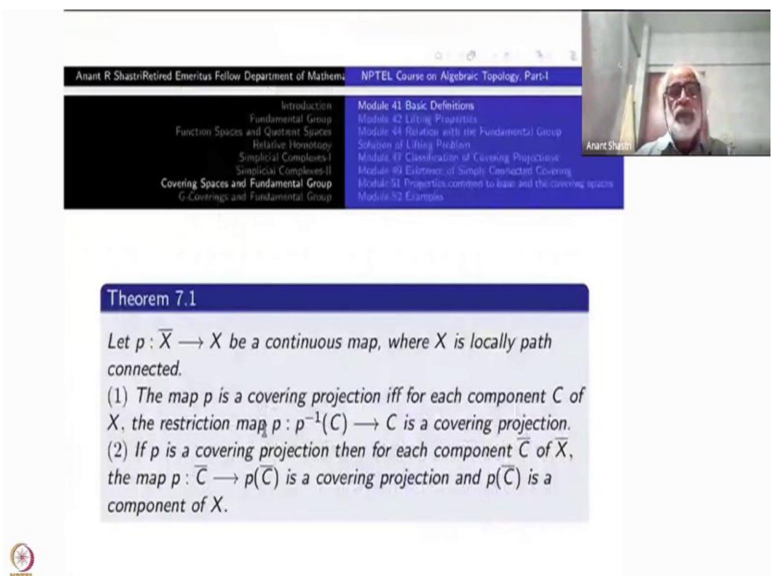

So, the key was this theorem namely when you have locally path connectedness then you can restrict the covering space to each path component of the top space. And there it will be a covering projection. So, you can study each of them separately and then put them put whatever observations you have made together. So, this is why this theorem is very important. So, this is what we proved last time.

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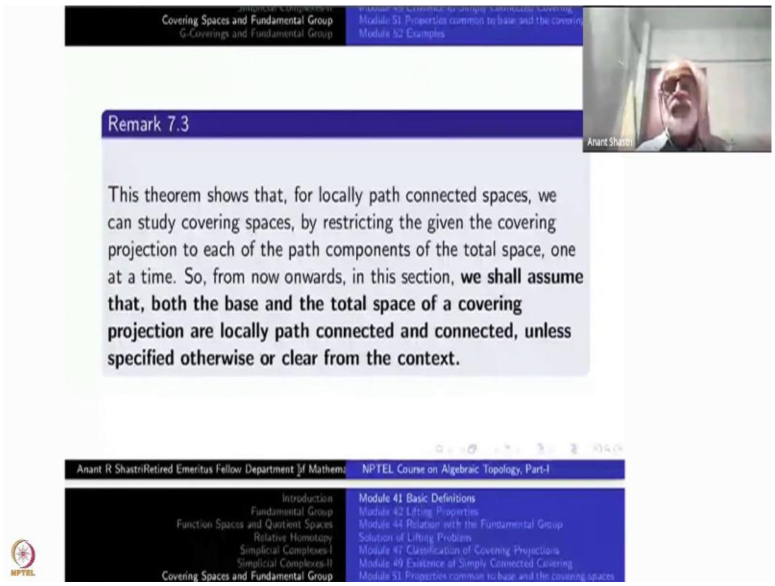

Now, this is what I tell you: we shall assume that both base and total space are the covering projection are path connected and connected unless specified otherwise or it is clear from the context. When you are discussing in some example earlier another example itself is not locally path connected then it is clear that it is not locally path connected that is all. If we left something then we are assuming this thing this you have to make it out form the context. this what I mean.

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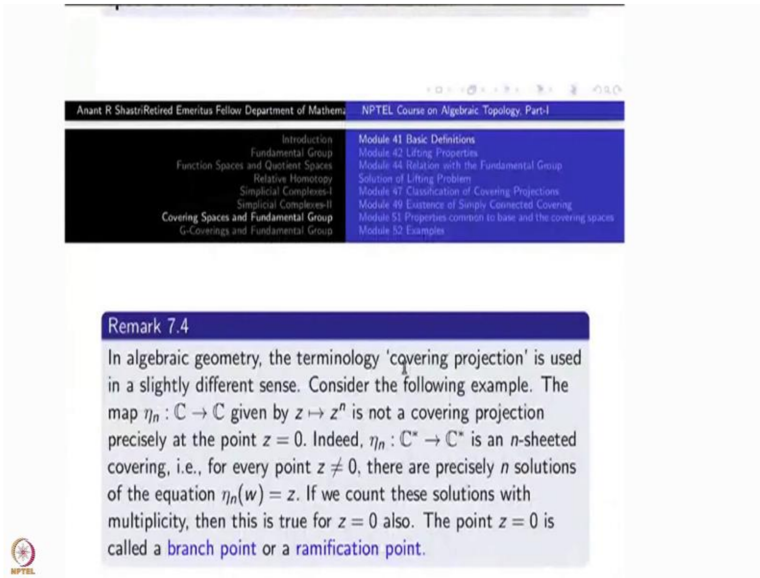

Now, for example this covering projection word is used by different people in slightly different ways, so one has to be careful. Especially in algebraic geometry, a covering projection need not be the covering projection the way we understand it.

But it is not too far. So, I will give you an example. Look at the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^n$. We have seen that $z = 0$ is a difficulty point. If you throw away that point then this map $\mathbb{C}^* \to \mathbb{C}^*$ is a covering projection. z is only point of trouble. On every other point, there are finitely many points in the inverse image. So, this kind of situation is acceptable in algebraic geometry. So, they allow a number of points, some subsets, some curves, some variety on which the given map may not be a covering projection. If we throw away that then on the rest of the space the function will a covering projection. Usually, on a `large' open set. Such things are called ramified coverings actually. In the above example the point $z = 0$ is a ramified point or a ramification point. So, they talk about ramification points and so on. But their covering itself may have ramification. So, they call it a covering projection. When they want to talk about the covering projection in our sense then they say unramified covering, i.e., there is no ramification. So, the more expressive thing unramified covering is used for a covering projection because covering projection for them belongs to a larger class of functions, frequented and used in algebraic geometry.

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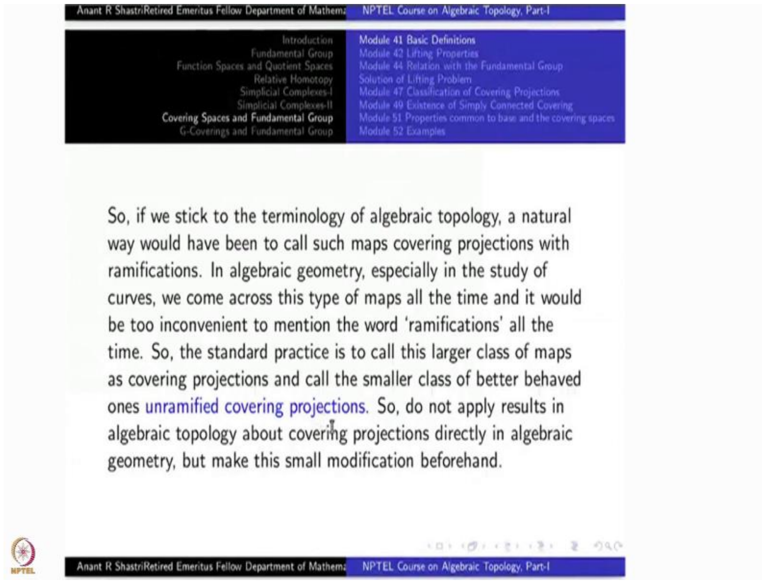

So, this unramified covering projection is the word they use for the actual covering projection that we use.

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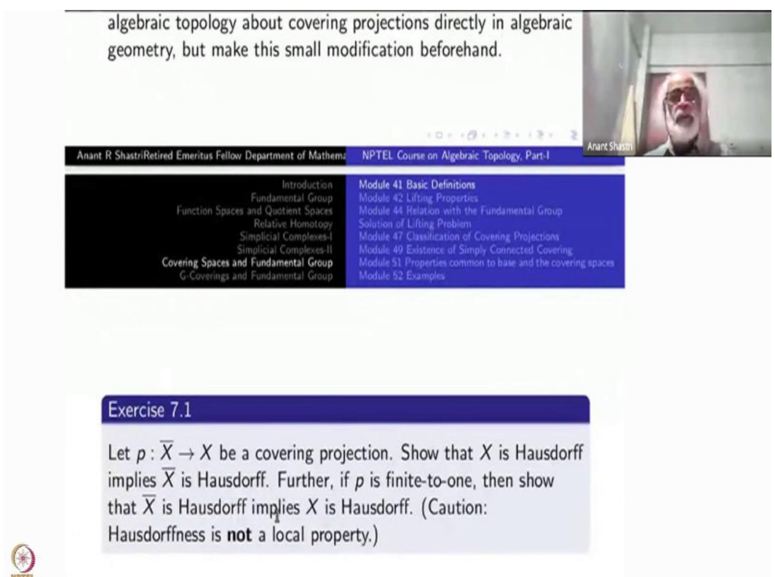

So, here are few exercises which are straight forward. You can try them but I will not use them till you have you have mastered them. So, take a covering projection $p:\bar{X}\to X$. If X is Hausdorff then $\bar{X}$ will be Hausdorff. It is the other way round here. You know that if $p:\bar{X}\to X$ is a covering projection then it is a quotient map.

So, under quotient map we know that if the top space $\bar{X}$ is Hausdorff it does not implies that X is Hausdorff. That is the case here also. But suppose you have a finite-to-one covering projection then $\bar{X}$ Hausdorff implies X is a Hausdorff. If it is a infinite covering then this is not quite true. So, we will have an example for this one but if it is a finite covering then you should show that $\bar{X}$ is Hausdorff implies X is Hausdorff. This is straight forward point set topological conclusion.

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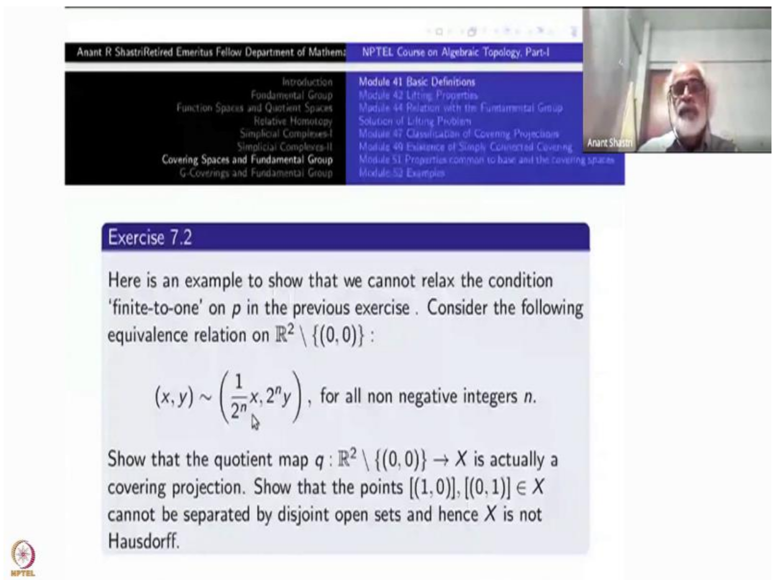

So, we have seen this example before. The top space is $\mathbb{R} \setminus \{0\}$ and there is an action of the infinite cyclic group on it. The generator takes (x, y) to $(x/2, 2y)$. So, more generally the n^{th}–power of the generator will take (x, y) to $(x/2^n, 2^n y)$. These are all equivalence classes now.

The quotient map is actually a covering projection. So, this happens because we have already introduced the terminology `even action' and this action happens to one such. So, the quotient map is a covering projection. Only thing is just like in the case of $\mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^n$, we throw away the point $(0, 0)$.

The origin is a single class so that creates a problem. However, after we throw that $(0, 0)$, it is a covering projection, Being a subspace of $\mathbb{R}^2$, the top space is Hausdorff. But the quotient space X is not. Look at the classes $[(1, 0], [(0, 1)]$. These two points cannot be separated by disjoint open subsets of X. I think we have seen this one before. If we have not seen yet you should check on this again. That shows X is not Hausdorff.

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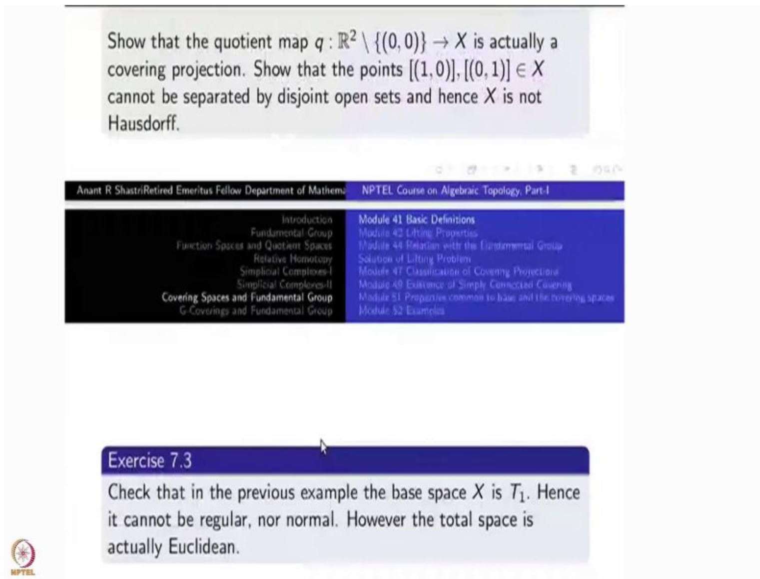

However, in this previous the space X is T1 space. Therefore, if it is regular then it will be Hausdorff also. So, it is not regular. Similarly, normal plus T1 will be implies T2. So, X is neither regular nor normal. However, for the top space, all these things are true $\mathbb{R}^2 \setminus \{0\}$, being a metrix space. But the quotient space, the bottom space fails to be satisfy all these the separation properties.

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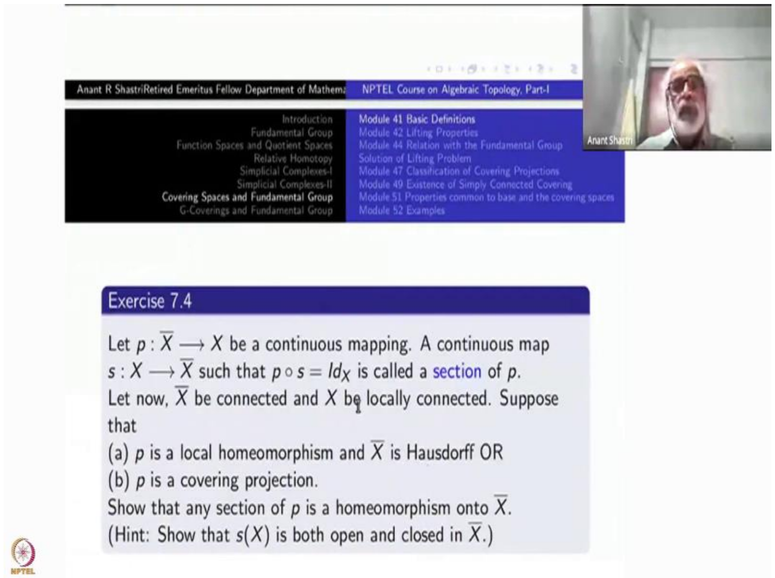

And here is another example(exercise) which you can do yourself. But when I want to use, I will prove this one. So, take any continuous function $p : \bar{X} \to X$. If we have a continuous function $s : X \to \bar{X}$ which is a right inverse to p, i.e., $p \circ s = Id_X$, then you say s is a section of p. (Sometimes people may not put the condition of continuity on s.) But, I want continuity because I am doing topology that is all. So, a section is a continuous right inverse it may not be left inverse. (If it is a left inverse also, p would be a homeomorphism.) Here it just implies p is a surjective map this is a left inverse. Also, there may be many sections of p.

In general, continuous right inverses are difficult to come by. Now suppose, you have a continuous right inverse that is called a section. Let now $\bar{X}$ be connected, X be locally connected or locally path connected. So, I am specifically saying these things here because this for this exercise, and is important that is all.

Now, in part (a) suppose that p is a local homeomorphism and $\bar{X}$ is Hausdorff. So, I am not assuming that p is a covering projection. The second part (b), I am assuming p is a covering projection but no Hausdorffness. Either local homeomorphism plus Hausdorffness of $\bar{X}$ or just p is a covering projection.

Then the conclusion is: any section s of p is automatically a homeomorphism onto $\bar{X}$. This just means that s is now a left inverse also, it is a two-sided inverse. So, p becomes a homeomorphism, s is its inverse. The hint is to show that $s(X)$ is both open and closed in $\bar{X}$. Because, $\bar{X}$ is connected it must be the whole of $\bar{X}$.

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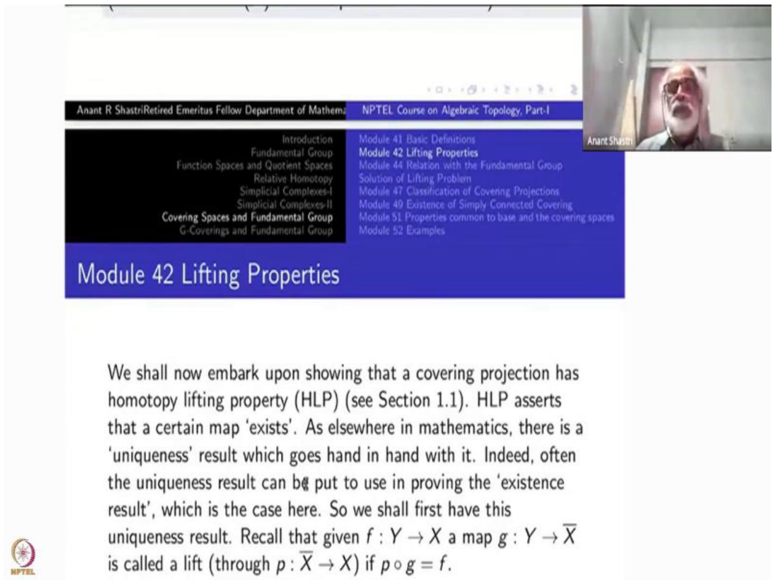

Now, let us carry on with lifting properties. Homotopy lifting property if you recall asserts that certain maps exists. What are they? What are they? Namely, if one map can be lifted out of a homotopy then the whole homotopy can be lifted.

Lift means p composite whatever new map you get is the old map. So, that is what we have we have seen that homotopy lifting property I am just recalling this. In mathematics there is in a uniqueness whenever you have uniqueness result which goes hand in hand with unique existence result. Quite often. Like first order differential equations solutions.

In a small neighborhood there exist and in a smaller neighborhood it is unique. Such things are always nicer and more much more applicable then just existence theorem. Indeed, the uniqueness part actually solves actually helps to solve the existence part. Truly the existence part.

So, this is the case here also. So, we shall first have this uniqueness result then we will use this uniqueness result to prove the existence of this lifts homotopy lifting. So, recall that if $f : Y \to X$ and $g : Y \to \bar{X}$ are maps, g is a lift of f through p means what? $p \circ g = f$. That is the meaning of lifts. Now, we are always concentrating upon the map $p : \bar{X} \to X$, talking lifts through p.

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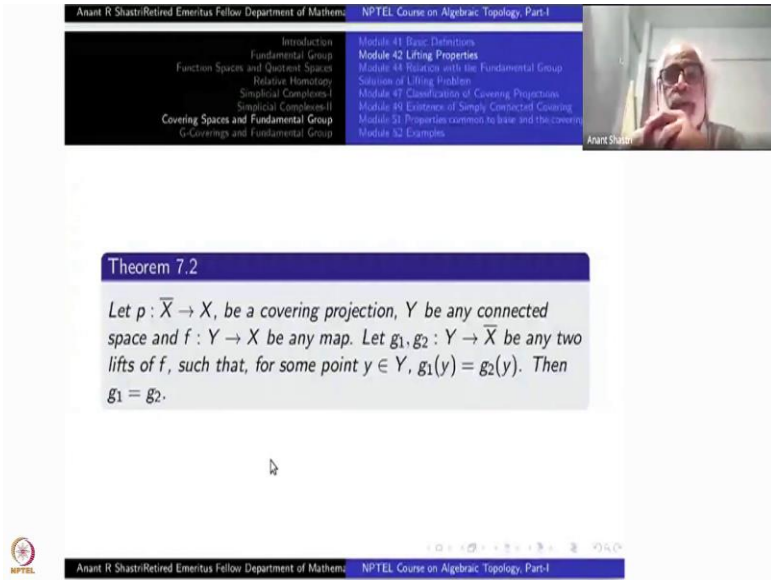

So, here is a neat theorem. Take p to be a covering projection now, a connected space Y and a map $f : Y \to X$. Suppose by chance we have two lifts of f, viz., $g_1, g_2 : Y \to \bar{X}$. Suppose further that they agree at one point $y \in Y$, i.e., $g_1(y) = g_2(y)$. At one point they coincide. Then the conclusion is $g_1 = g_2$ on the whole of Y.

So, this is similar to the uniqueness of the solution first order differential equation. If you specify the initial condition than it is unique. This is like a initial condition at one point they agree. Then the whole thing get easy just like integration theory. Integrals are defined up to a constant additive constant. There is no addition, multiplication here. At one point they agree continuity is there that is all. They already agree. So, the key is that Y is connected. That is all. Y is connected and of course p has to be a covering projection. Otherwise, for arbitrary maps and spaces, this will not be true.

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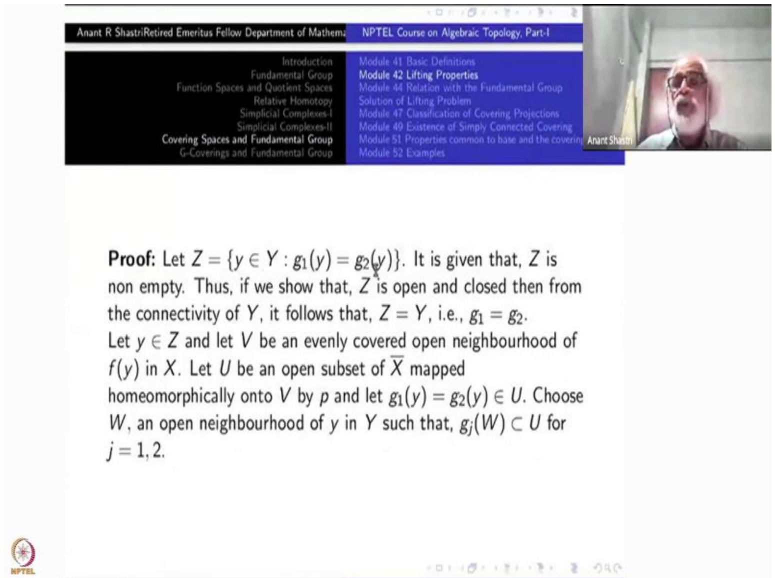

So, let us go through the proof which is very straight forward. Look at the set Z of all points in Y at which g_1 and g_2 coincide. That is a subset of Y. It is given that this subset is non empty there is at least one point. So, if we show that Z is open and closed in Y then because of the connectivity of Y it follows that Z is the entire Y.

Z is equal to Y means g_1 will be equal to g_2 at all the points of Y. Therefore, what we want to prove now is that Z is both open and closed. Take a point y in Z and let V be an evenly covered open neighborhood of $f(y) \in X$, $y \in Y, f(y) \in X$. And X is covered by evenly covered open sets, because p is a covering projection.

So, take a neighborhood V of $f(y)$, which is evenly covered. And this neighborhood is neighborhood of fy. $p^{-1}(V) = \coprod_i U_i,$ each U_i is an open subset of $\bar{X}$ and is mapped homeomorphically onto V by p. So, you choose one of them, say, $U = U_i$ be an open subset of $\bar{X}$ mapped homeomorphically onto V by p. Look at the point $g_1(y) = g_2(y)$. It must be in one of the U_i's and choose U that one.

Now, choose an open subset $W, y \in W \subset Y$, such that both g_1 and g_2 map W inside U. Since g $g_1(y) = g_2(y) \in U$, by continuity of g_i first you get two open neighbourhoods W_i such that $g_i(W_i) \subset U$. Now you take $W = W_1 \cap W_2$. That will give you W as required.

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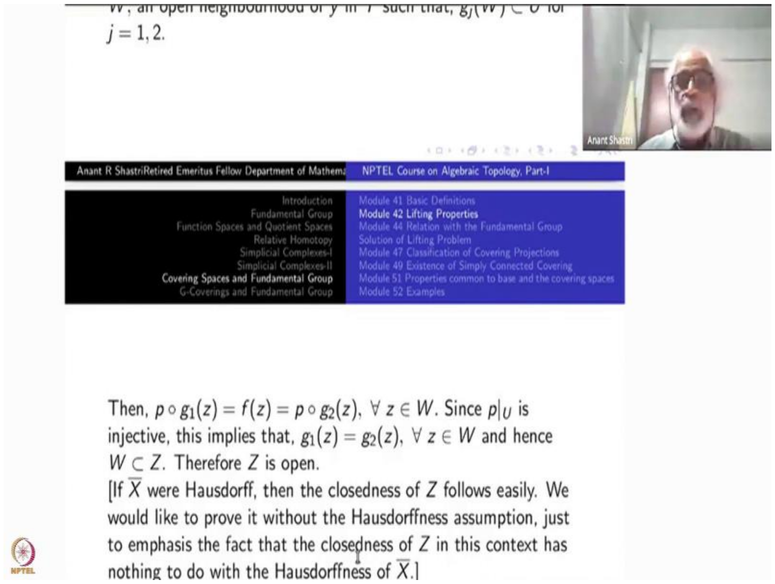

Then look at this: $p \circ g_1(z) = f(z) = p \circ g_2(z)$, for every $z \in W$. But p restricted U is an injective mapping. It is a homeomorphism. We have $g_1(z), g_2(z) \in U$ and $p(g_1(z)) = p(g_2(z))$. Therefore $g_1(z) = g_2(z), z \in W$.

By definition this W is contained inside Z. By choice W is an open subset around singleton point whatever point you have taken in Z. So, what we have proved is Z is open.

Now, if $\bar{X}$ were Hausdorff, then set of points were in the two continuous functions agree will be always closed. But we do not want to do that, we can do without the assumption that $\bar{X}$ is Hausdorff. We are going prove that the set $Z$is closed, by showing that its complement in Y is open.

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Take the set of all points z in Y such that $g_1(z) \neq g_2(z)$. We have to show that this is also open. See this is compliment of capital Z. So, I want to show that compliment is also open now. So, to show that what do I do? As before let V be an evenly covered open neighborhood of $f(z) = p \circ g_1(z) = p \circ g_2(z)$. Of course, $g_1(z) \neq g_2(z)$, but $p(g_1(z)) = p(g_2(z))$. Therefore, these two points $g_i(z)$ are in the same fibre of p. So, we can find open neighborhood $U_i, i = 1, 2$ around these points respectively, and we know $U_1 \cap U_2 = \emptyset$. Of course both $g_1(z), g_2(z) \in p^{-1}(V)$,but since $p : U_i \to V$ is injective, these tow different points cannot be in the same U_i. And U_i are mutually disjoint. By continuity, g_1 and g_2 are continuous function you will find a neighborhood W of z such that $g_i(W) \subset U_i, i = 1, 2$. Same thing again, you first get different W_i and then take the intersection. It follows that W is an open neighborhood of z not intersecting Z at all. Because $U_1 \cap U_2 = \emptyset$, g_1 of any point in W will not be equal g_2 of that same point. Because they will be going to two disjoint open sets here. So, the entire W is in the compliment of Z. That whole compliment of Z is open hence Z is closed. This proves the theorem.

So, covering projection is the only thing which is used critically here.

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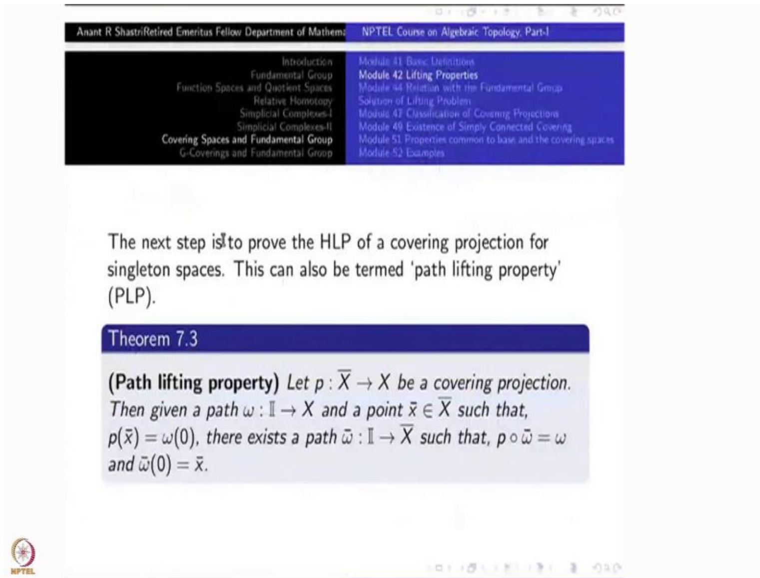

The next step is to prove homotopy lifting property of covering projections for singleton spaces. For a singleton space what is a meaning of a homotopy? We have seen that it is a path so this is called path lifting property. Path lifting property for our covering projections. First, we will prove that one. First point wise we will prove the pointwise version before taking up general version.

Take a covering projection $p : \bar{X} \to X$. Given a path $\omega : \mathbb{I} \to X$ and a point $\bar{x} \in \bar{X}$ such that $p(\bar{x}) = \omega(0)$. This is the starting point it is that initial value condition. Take a point above. Always points can be lifted because p is a surjective mapping. So, starting point you lift it up in $\bar{X}$. Then there exists a path $\bar{\omega} : \mathbb{I} \to \bar{X}$ such that $p \circ \bar{\omega} = \omega$. p composite of omega bar is equal to omega. So, omega is lifted.

The starting point of this omega bar is a point which I have chosen as x bar. We have already proved that such a thing will be unique if there is another path omega 1 with the same properties then agreeing at one point then omega 1 will equal to omega at all points. But we do not know whether this exists. But the uniqueness of this will help to prove the existence now. And it is not difficult. Let us go through this one.

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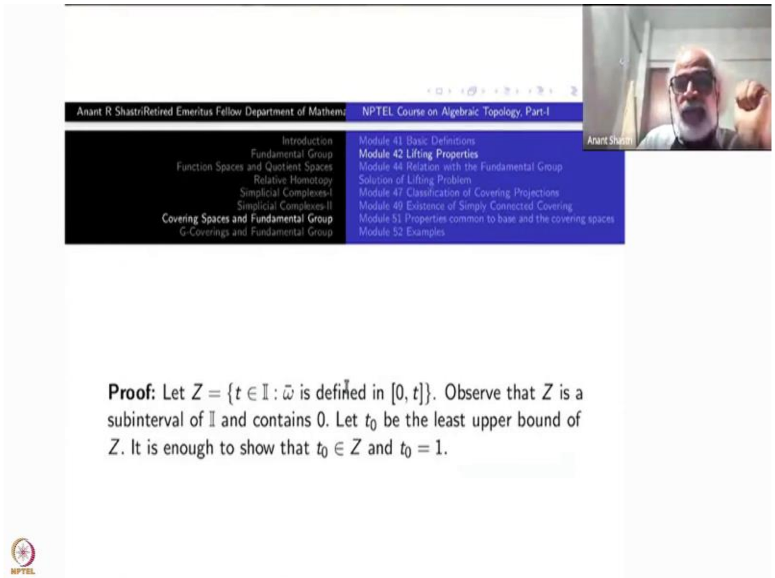

Similar to the proof of uniqueness, like this one we used several times, we define Z to be the set of all points inside $\mathbb{I}$ such that $\bar{\omega}$ is defined up to t, on $[0, t]$ the closed interval. $\bar{\omega}(0) = \bar{x}$ the point that we have chosen. Then it is defined up to t means what? That it is a map $\bar{\omega} : [0, t] \to \bar{X}$ such that $p \circ \bar{\omega}(s) = \omega(s),\ 0 \leq s \leq t.$ Observe that Z happens to be sub-interval containing 0 and contained in $\mathbb{I}$. So, interval and contains 0, 0 is already there because x bar is there. 0 to 0 is just closed interval it is a omega is defined already.

Now, let t_0 be the least upper bound of Z. It exists because after all this Z is a subset of the closed interval $\mathbb{I}$ and so it is a bounded above. So, take t_0 as the least upper bound or what is called as supremum. I want to show that $t_0 \in Z$, first thing.

Second thing is to show that $t_0 = 1.$ Essentially, this is equivalent to showing that $Z = \mathbb{I}$ the whole interval by showing that Z is both and and closed and using that $\mathbb{I}$ is connected. But we will aviod that and make it easier here, just use the existence of supremum and then that supremum has to be inside Z which implies that $\mathbb{I}$.

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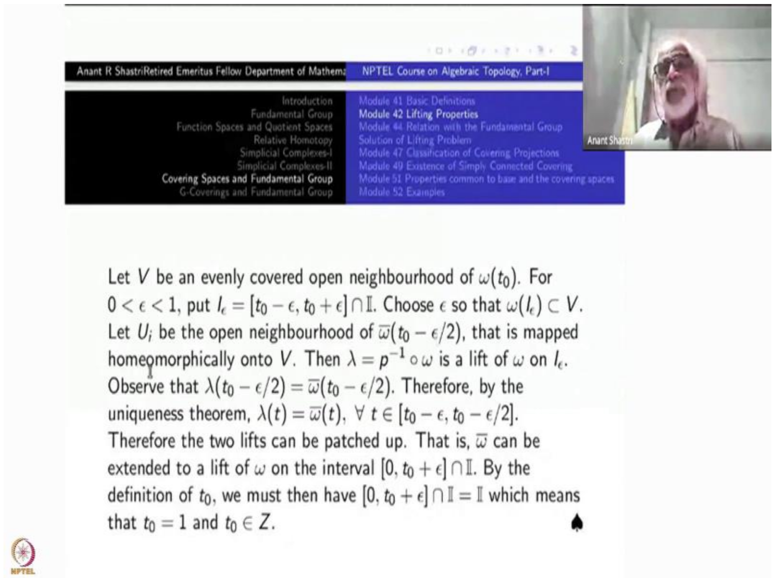

Start with an evenly covered neighborhood V of $\omega(t_0) \in X.$ Up to t_0, the map is defined. t_0 is the supremum means that ff you take any point $s < t_0$, $\bar{\omega}$ is defined on $[0, s]$ Now, I want to se what is happening at $t_0.$ So, take an evenly covered neighborhood $V$of $\omega(t_0) \in X.$

For each $0 < \epsilon < 1$, you put this notation namely, $I_\epsilon = (t_0 - \epsilon, t_0 + \epsilon) \cap \mathbb{I}.$ (I am only interested actually in an open interval around t_0 but I do not know, one part may be closed etc. is equal to and so on. So, I will take a close interval or a half closed interval, no problem. Epsilon is positive that is good enough for us. I epsilon is t naught minus epsilon comma t naught plus epsilon this interval may go out of I. It may get out of [0,1] at some part so I do not want that. So, intersect it with [0,1].

Now, by continuity of ω, choose epsilon positive so that omega this is notation now choose epsilon so that this omega of this is I epsilon which is a neighborhood of t naught that is contained inside V. $\omega(I_\epsilon) \subset V$. V is an open subset containing omega t naught, so by continuity of omega some neighborhood will be contained inside V that is all. This is this just from the fact that omega is continuous.

Now, let U be the open set in $\bar{X}$ containing $\bar{\omega}(s)$ for some $s \in I_\epsilon,\ s < t_0.$ Take s, just to the left of t_0 then $\bar{\omega}(s)$ is already defined because t_0 is the supremum. so omega bar makes sense there. So, take this open neighborhood U of this such that U is mapped homeomorphically onto V by p.

Because, V is evenly covered and $\bar{\omega}(s) \in p^{-1}(V)$, such a choice of U is possible. Put $\lambda = p^{-1} \circ \omega : I_\epsilon \to \bar{X}$ Then λ becomes a lift of ω on I_ϵ . See this is a homeomorphism $p|_U : U \to V$, p restricted to U from U to V is a homeomorphism. So, I can take the inverse of that. p inverse is taken on this subset. Then composite with omega. If you take p of this it will be omega therefore this is a lift of omega inside the whole of I epsilon. Because this whole of I epsilon contained inside V. $\omega(I_\epsilon) \subset V.$ So, composing with p^{-1} makes sense.

Now, look at $\lambda(s).\ p(\lambda(s)) = \omega(s) \in V.$ Up to s there we have already path $\bar{\omega}$. Therefore, by the uniqueness $\bar{\omega}(t) = \lambda(t)$ for all $s \leq t < t_0.$ This omega bar is already defined.

I have got two lifts, they agree at one point s, so, in this whole interval $[s, t_0)$ which is a connected subset, by the uniqueness theorem they should agree on the whole of $[s, t_0)$. Therefore, omega bar can be extended on the union $[0, t_0) \cup I_\epsilon.$ Therefore $[0, t_0) \cup I_\epsilon \subset Z$. In particular t_0 belongs to Z. But t_0 is the supremum of Z. So I_ϵ does not contain any number bigger than t_0. Therefore, it follows that $t_0 = 1.$ This completes the proof of the existence of lifts of paths.

So, let us stop here and the full lifting property will be done next time. Thank you.