Introduction to Algebraic Topology
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Lecture 41
Covering Spaces and Fundamental Groups

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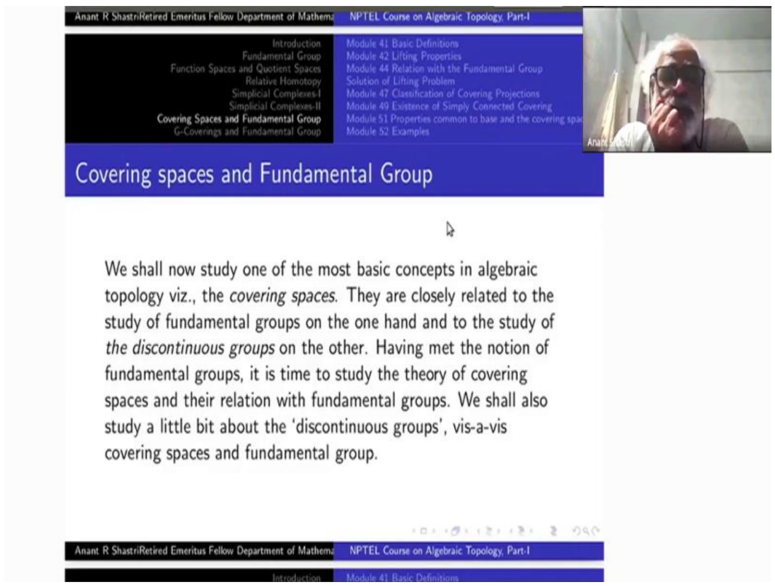

We start a new chapter, the new topic: Covering Spaces and Fundamental Groups. Of course, fundamental group we seen before. Now we want to study fundamental group in relation with covering spaces. On the other hand, covering spaces is one of the oldest or the simplest or the very beginning concepts in algebraic topology. It is a link between point set topology and algebraic topology. It has lot of point set topologies which slowly passes on to give you the algebraic topology result as we have already seen in the calculation of the fundamental group of $\mathbb{S}^1$.

What we have used is the exponential map and that is a prototype, a beautiful example of a covering space. Covering space here is closely related to another concept, namely, this is the classical approach what they used to call `discontinuous groups'. So, the discontinuous group action you know we will be also studied here to some extent. Why only to some extent? Each subject here is very vast, and applied in almost all branches of mathematics.

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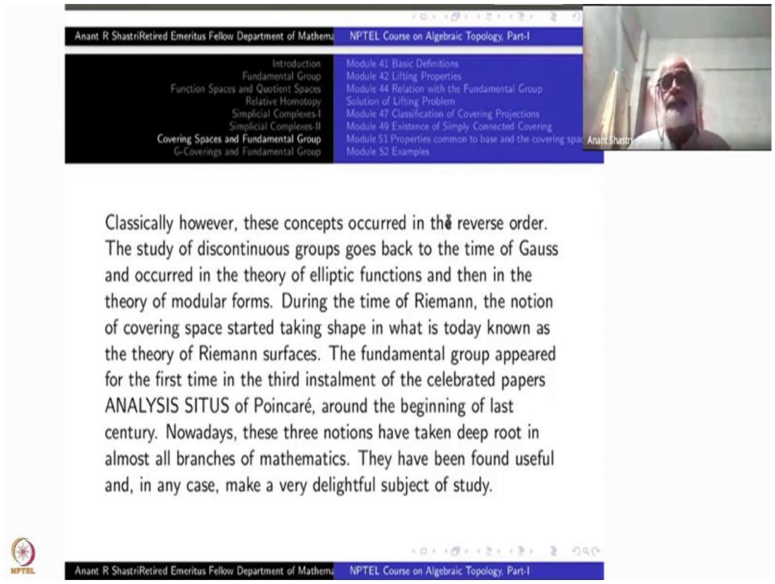

Classically again, these concepts occurred in the reverse order. Namely, the discontinuous groups were the ones which we have studied. The notion of covering spaces etc., came much later. So, like you know one can cite examples of Gauss's work, and then theory of elliptic functions by Abel and Weierstrass's and so on.

During the time of Riemann, the notion of covering spaces started taking place. In fact, you may say Riemann is the one who introduced even the manifolds. And then this concept of covering space also in his Riemann surfaces of some algebraic functions and so on. The fundamental group appeared much later in Poincare's work.

So, one can give big credit to Poincare to put various things together and invent some other way of looking at it at all. Nowadays, these three notions have taken deep root among all branches of mathematics. They have been found useful and, in any case, make a very very enjoyable, delightful study of mathematics.

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So, covering spaces. So, let us begin with the simple definition and then consequences of this definition and so on. Later we will give some examples also. So, that is what we will do for today.

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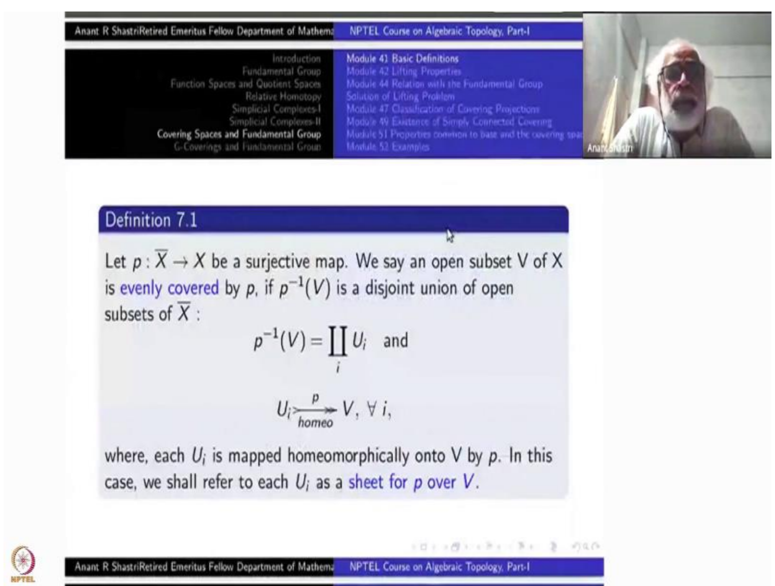

So, start with a continuous function from one space to another space which is also surjective. So, the first space I am denoting by $\bar{X}$. (The bar has nothing to do with closer and so on. This is just a notation.) And is space while $\bar{X}$ is another space, p is a map from $\bar{X}$ to X. This is the standard notation for a covering projection.

Take a surjective function. We say a subset V of X (usually this X is called the bottom space, and $\bar{X}$ is called the total space by the way. I will introduce those terminology also.) So, in the bottom space you take a subset V an open subset this will be called evenly covered by p, if $p^{-1}(V)$ (this word `cover' is not the usual cover, everything is very strange wordings here,) inverse of V you have to come to $\bar{X}$ here, $p^{-1}(V)$ is the disjoint union of open subset of $\bar{X}$. So, let us write p inverse of V as disjoint union, $p^{-1}(V) = \coprod_i U_i$, indexed by some set. Now, you can restrict p to each of the U_i and come back to V, because this whole thing is inside p inverse of V. So, so when you apply p to any of U_i, you comeback to V. This map restriction map $p : U_i \to V$, this must be a homeomorphism, for every i. In other words, each U_i is a homeomorphic copy of this V and all U_i are disjoint from each other. Union of all of them will be the full inverse image of V.

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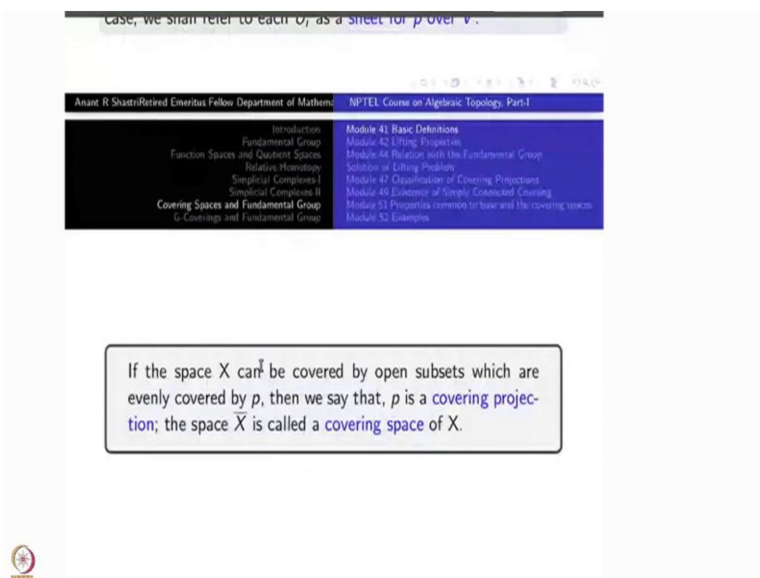

So, if this happens, V is called evenly covered by p. And look at the cardinality of this indexing set, that will be called number of sheets of p. Each U_i is called as sheet. Sheet means what? It is just a copy of V, copy in the sense it is homeomorphic. If X can be covered by open subsets each of which is evenly covered by p, then we call p a covering projection.

Now, this covering is precisely that what I mean see this word covering here, this is that the union of all such V is equal to X. So, this is just usual covering of a space by open subsets. These open subsets each of them should be evenly covered by p. So, that means that inverse image of each of this set under p is the disjoint union like this. This condition should be satisfied. So, if you vary V with this condition satisfying for all the V's and if you get the whole of X, then this p will be recalled a covering projection.

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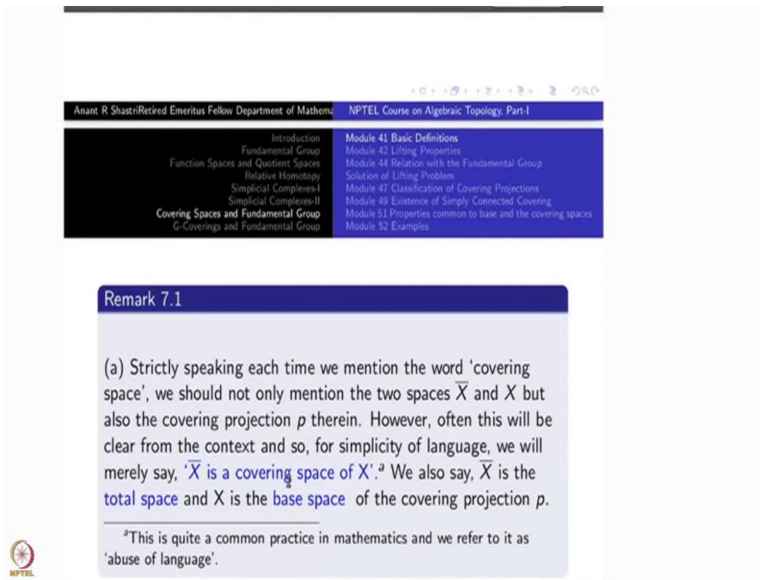

Strictly speaking each time, we mention the word covering projection or a covering space, we should not only mention the two spaces $\bar{X}$ and X, but also the covering projection. It is like similar to a quotient space. When you say quotient space you have to specifically tell what is that quotient map from $\bar{X} \to X$. So, it is like that; p should be mentioned.

So, strictly speaking it should be denoted by $(\bar{X}, p, X)$. The triple is a covering. It is a covering projection. But that is too much to write here. Like even we do not write $(X, \mathcal{T})$ for a topological space, the topology is not usually mentions, let X be a topological space, we say. Similarly, we have to do with this shortening terminology. That is all.

However, often this will be clear from the context which function we are taking and so on. So, for simplicity of language, we merely say $\bar{X}$ is a covering space of X. We also say that, $\bar{X}$ is the total space and X is the base space. This I have already told you. Whenever you are covering a projection p it comes with a total space and a base space.

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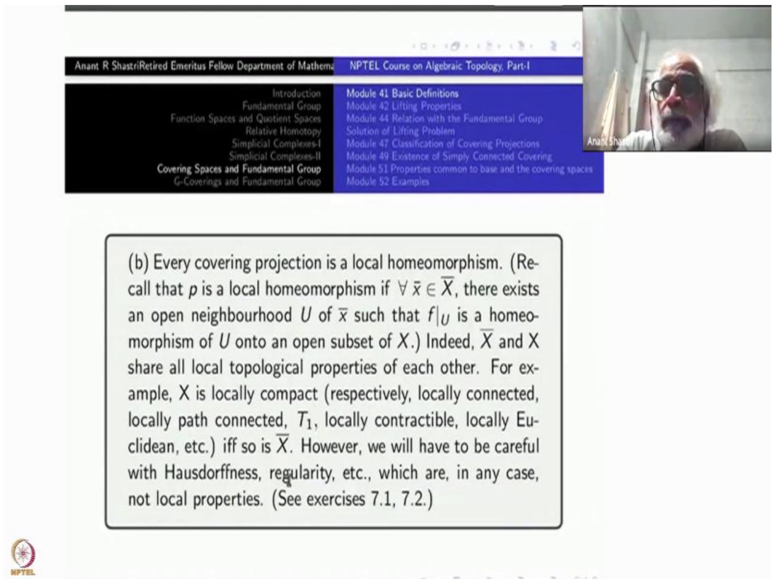

Every covering projection turns out to be a local homeomorphism. Recall what is the local homeomorphism $f : \bar{X} \to X$. For every x in the domain namely, $x \in \bar{X}$, you must have an open neighborhood U of x such that f restricted to U is a homeomorphism onto $f(U)$, which is an open subset of X. If this is happens for every $x \in \bar{X}$ then you call f is a local homeomorphism. And that is precisely happening here in the case of p.

If X is covered by open set like this inverse image will be covered by open set like this. And these are disjoint union of U_i so x will be inside one of them and from $p : U_i \to V$ is a homeomorphism. And to begin with these $V's$ are open in X. Therefore, every covering projection is a local homeomorphism. This property, local homeomorphism, has a tremendous influence. Whatever local property of X is there, it will be there on $\bar{X}$ also.

For example, if X is locally path connected, then $\bar{X}$ will be locally path connected and conversely. If this is locally compact then $\bar{X}$ will be locally compact and conversely. If this is, say what is that, first countable then this will be also first countable and conversely and so on. More than that there are structure types; this is a smooth manifold then this will be a smooth manifold and conversely. Such, things are also true.

So, you take properties such as local compactness, locally connectedness, locally path connectedness, T1 ness, locally contractible, locally Euclidean(that is manifold etc.,) each of them holds for X if and only if it holds for $\bar{X}$. So however, you have to be very careful in extending this kind of list. I said T1, that is fine. But the next one T2? it is not true. So, this will actually tell you that T2 ness is not a local property; its local-global. It is about two points, whereas T1 ness is about one single point, at each point something happens.

Like Hausdorfness, the next regularity, normality etc., none of these is a local property. If $\bar{X}$ has it, X may not have it; X has it, $\bar{X}$ may not have it. Either way it may or may not, we do not know, we cannot say. So, there are exercise about that one.

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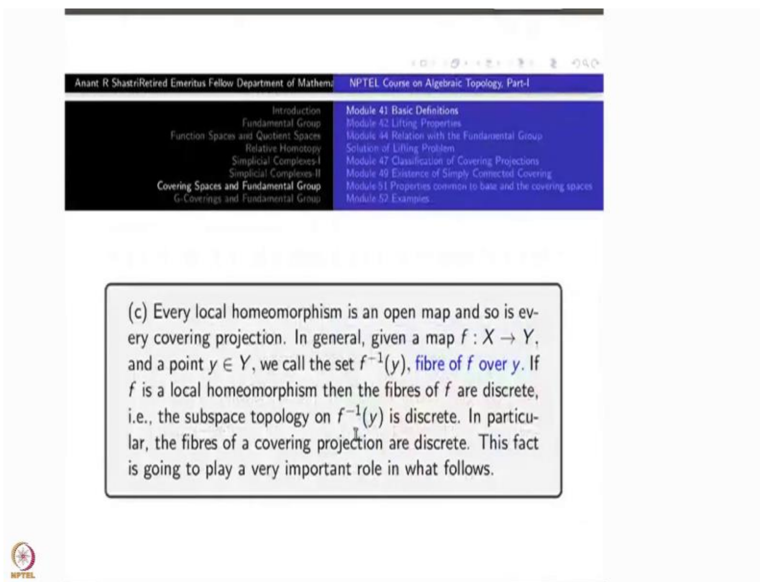

Every local homeomorphism is automatically an open mapping. Is that clear? Because take an open set for each point we have neighborhoods whose image is open. So, this open set is union of such open sets, so image will be union of those open sets. Therefore, image of every open set is open. So, it is automatically an open mapping. Recall that a continuous surjection which is an open mapping is also a quotient map.

Therefore, X will be a quotient space of $\bar{X}$. In general, if you have $f : X \to Y$ a function, and a point $y \in Y$, we call the set $f^{-1}(y)$ a fibre over f. This is a general notation; a general terminology. Fibre of a map means inverse image of a point under that map. If f is a local homeomorphism, the fibres of f are always discrete. Look at the fibre, take two points there. Each of them has a neighborhood homeomorphic to a neighborhood below of the point y, each of them.

If they had a non empty intersection, then this would not have happened because under f two of them are coming to the same thing that will be a problem. The same f restricted to this open set, restricted to this open set is a homeomorphism. So, therefore the two open subset that you have taken above must be disjoint automatically. So, this happens for every pair of points which just means that you know you can do for single point and everywhere else. So, this just means that f inverse is f inverse of the single point y is a discrete set.

[Editor’s note: Actually, the proof is simpler than as described above. Given $x \in f^{-1}(y)$ and an open set U of X such that $f : U \to f(U)$ is injective, it follows that $f : U \cap f^{-1}(y) \to \{y\}$ is also injective. Therefore $U \cap f^{-1}(y) = \{x\}$.]

So, in the subspace topology $f^{-1}(y)$ is a discrete space. In particular, the fibres of the covering projection are discrete. This fact is going to play a very important role in what we are going to study. These are some few immediate consequences of definition that we have done namely the entire X is covered by evenly covered open sets. So, I am repeating a few of them here.

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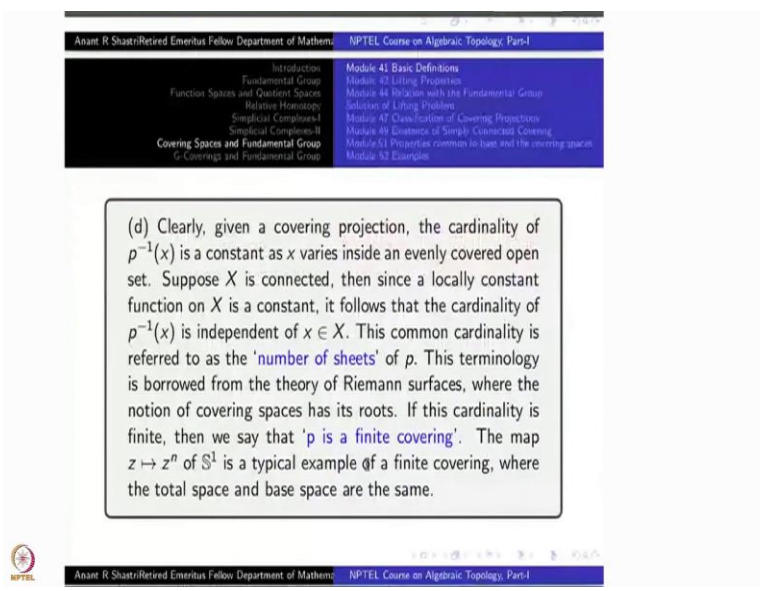

Given a covering projection p, the cardinality of $p^{-1}(x)$ is a constant as x varies inside any evenly covered open set. Because, for that open set V, you look at the inverse image it is disjoint union of open sets, each of them coming bijectively to the set V. Therefore, for each point there are exactly as many points as the indexing set for this disjoint union.

So, what does this imply immediately? The cardinality of $p^{-1}(x)$ is locally a constant function. In an open subset it is a constant. Suppose now , X is connected. This kind of topology you must be familiar with already, that every locally constant function on a connected space is a constant. Connectivity has to be used strongly here.

A locally constant function will be constant if X is connected. So, in particular this just means that if you start with a connected space X then take a covering $p : \bar{X} \to X$, then inverse image of every point has the same cardinality. That cardinality is called number of sheets of p. If this cardinality happens to be a finite number , then we call p a finite covering.

So, here is an example: $z \mapsto z^n$. We have studied it earlier as a example of a quotient map perhaps, self-quotient of $\mathbb{S}^1$ onto itself. This is a typical example of a finite covering, where the total space and base spaces are the same. This map is n to 1, n points in $\mathbb{S}^1$ go to the same point. No matter what point you take.

Take any point in $\mathbb{S}^1$. The inverse image has exactly n points. That does not prove that it is a covering. But you can show that $z \mapsto z^n$ is a covering. So, now let us workout few more examples properly.

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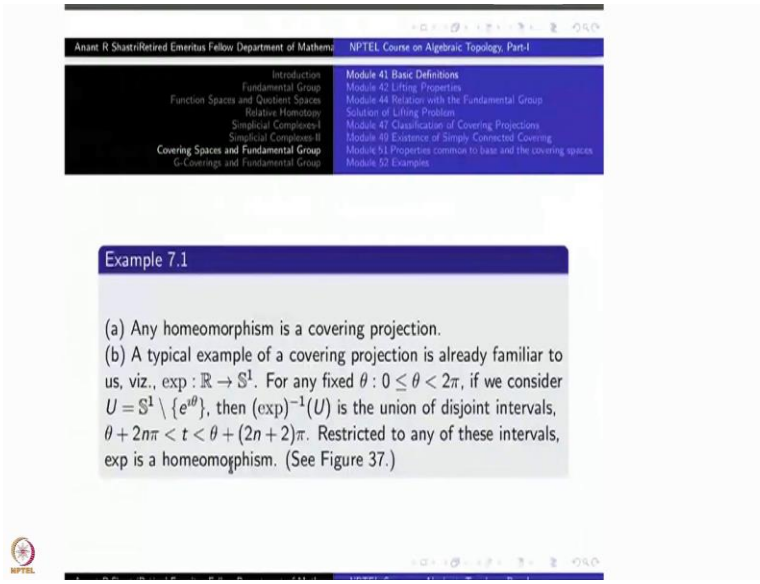

One simple thing is, if you take identity map, it is a covering. Next, you can take any homeomorphism, that is also a covering. Those things are trivial coverings, they do not give you much. A typical example of covering projection is already familiar to you namely the exponential map, $exp : \mathbb{R} \to \mathbb{S}^1$, which you can write as $t \mapsto e^{2\pi\imath t}$, (or $\theta \mapsto e^{\imath\theta}$, according to your fancy; multiplying by $2\pi\imath$ is just a normalizing factor, that is all).

Fix a $\theta \in [0, 2\pi)$, 2π omitted. Then take $U_\theta = \mathbb{S}^1 \setminus \{e^{\imath\theta}\}$. I have fixed θ, and removing its image point under the exponential function. Now the inverse image of U is the disjoint union of open intervals $(\theta + 2n\pi, \theta + 2(n+1)\pi), n \in \mathbb{Z}$. It is of period 2π. Here I have just taken $exp(\theta) = e^{\imath\theta}$. That is why I get multiples of 2π here. If I take the first map $t \mapsto e^{2\pi\imath t}$, then these interval would be $(t + n, t + n + 1), n \in \mathbb{Z}$.

Restricted to any of these interval's, exp is a homeomorphism. Therefore $U_\theta = \mathbb{S}^1 \setminus \{e^{\imath\theta}\}$ is evenly covered. Keep varying this $\theta \in [0, 2\pi)$, $U_\theta' s$ will give you the covering of $\mathbb{S}^1$. (In fact, just two different values of $\theta$are enough.) Therefore, this will tell you, if you have verified it, actually we have verified this one already and we have used this property before, so, this shows that exponential function is a covering projection.

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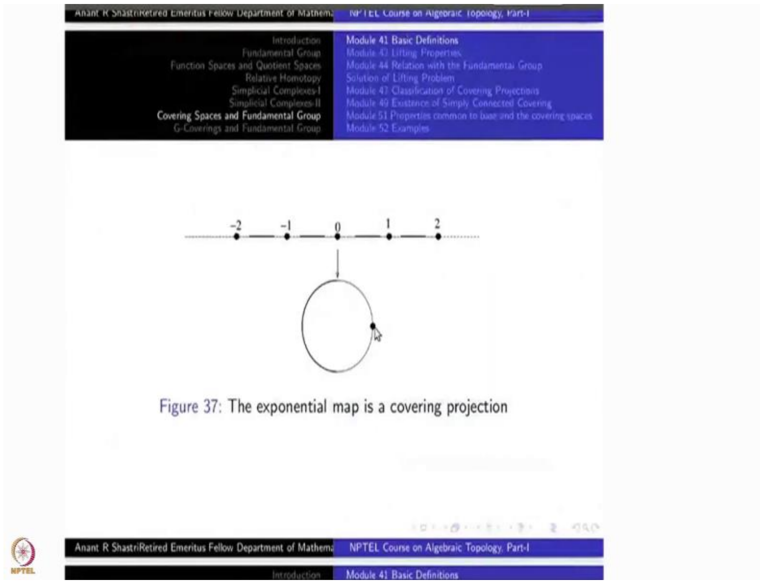

And this picture also perhaps seems familiar to you. Here for the function $t \mapsto e^{2\pi \imath t}$, I have thrown away this point $1 \in \mathbb{S}^1$, let us say. In fact, I have taken only half circle here, what are the inverse image of this arc? It will be $(n + \frac{1}{4}, n{+}4)$, n$\in \mathbb{Z}$, jumping by intervals of length.

The inverse image of this point, this is $\{1\} \in \mathbb{S}^1$, the inverse image will be the set of all integers. So, when you throw away all the integers from $\mathbb{R}$, the exponential functions restricted to each of these intervals to this arc is a homeomorphism. The full inverse image of this arc will be union of all these intervals, this part we have already seen.

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In similar way, and it is not hard to do this one namely, for the function $z \mapsto z^n$; to show that it is a covering projection from $\mathbb{C}^* \to \mathbb{C}^*$ itself. $\mathbb{C}^*$ is what? non-zero complex numbers. Restricted to circle $\mathbb{S}^1$, it will go into the circle itself, namely, unit circle mod z is 1, mod z power n is also 1. If you do not put that condition, any non-zero thing will go to a non-zero thing and it will be n to 1 mapping. You can work out neighborhoods, how they, how small neighborhood should be taken here and then look at the inverse image there will be n copies of that each of them mapped onto the same open subset here.

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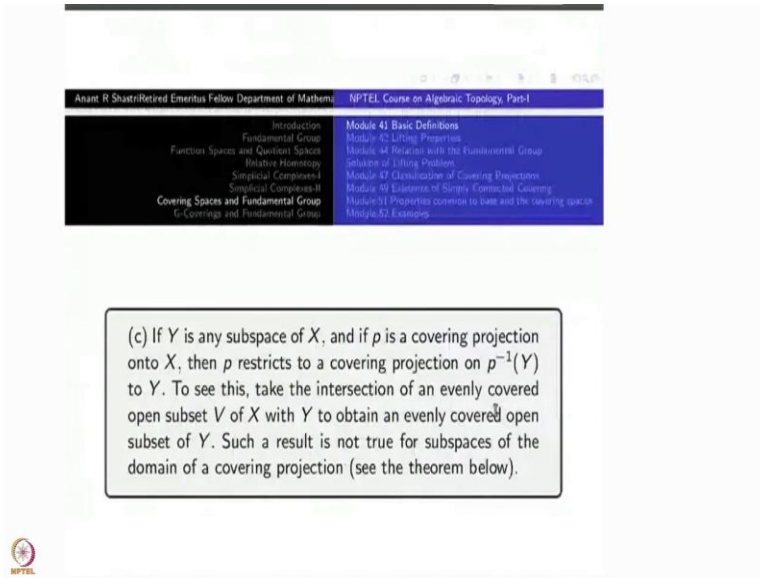

If Y is any subspace of X, I can take inverse image of Y under p that will be a subset of $\bar{X}$. Now you take p restricted to $p^{-1}(Y) \to Y$. That itself will be a covering projection. What you have to do? Take an evenly covered open set V inside X. Take intersection of that with Y, that is all. $U \cap Y$ will be even covered by this restricted function.

On the other hand, you cannot do this by taking a subspace of the top space $\bar{X}$. Then you have to be careful. In general, if I take $p : \bar{X} \to X$ a covering projection and $\bar{Y} \subset \bar{X}$, the restriction map may not be a covering projection. So, what will happen? This is an interesting question. We will see it later on.

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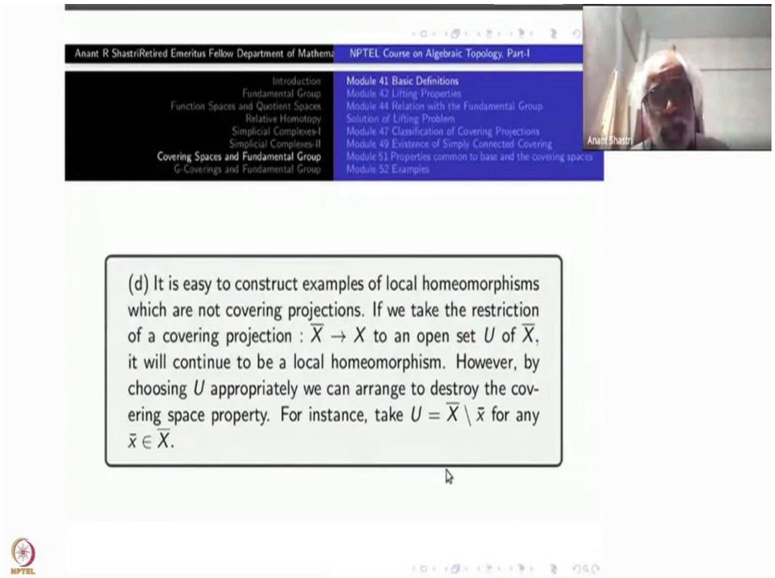

To construct local homeomorphism, we said a covering projection is always a local homeomorphism, but to construct a local homeomorphism which is not a covering projection is very easy. Just local homeomorphism onto subset would not give you covering space or covering projection. So, if we take the restriction of a covering projection $p : \bar{X} \to X$ to any open set U inside $\bar{X}$ it will be automatically a locally homeomorphism.

However, by choosing U very badly in some way you can destroy the covering space property. The evenly covering property you can destroy in many ways. To retain it is more difficult, destroying is automatically happens, namely, all that you have to do is for example, just omit one point from $\bar{X}$, that is an open set (provided X is T_1). So now, if you restrict, this will never be a covering projection. Starting with a non trivial covering projection $p : \bar{X} \to X$, the restricted map $p : \bar{X} \setminus \{\bar{x}\} \to X \setminus \{p(\bar{x})\}$ will never be a covering projection. Verify this one, then you will understand better the covering projections.

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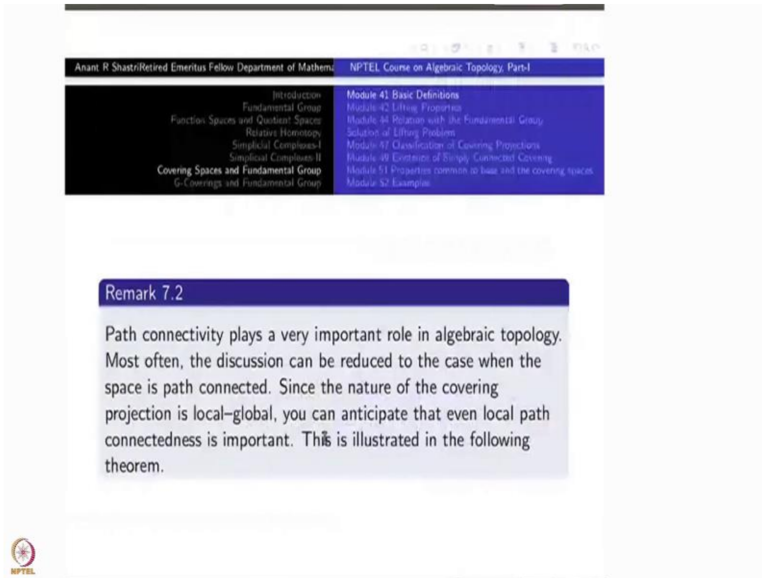

So, I want to tell you some more thing about the role of path connectivity here. Path connectivity, local path connectivity are part and parcel of algebraic topology. We assume that this one condition always. So, here local path connectivity is very important in the case of covering projections so that is what I want to tell you.

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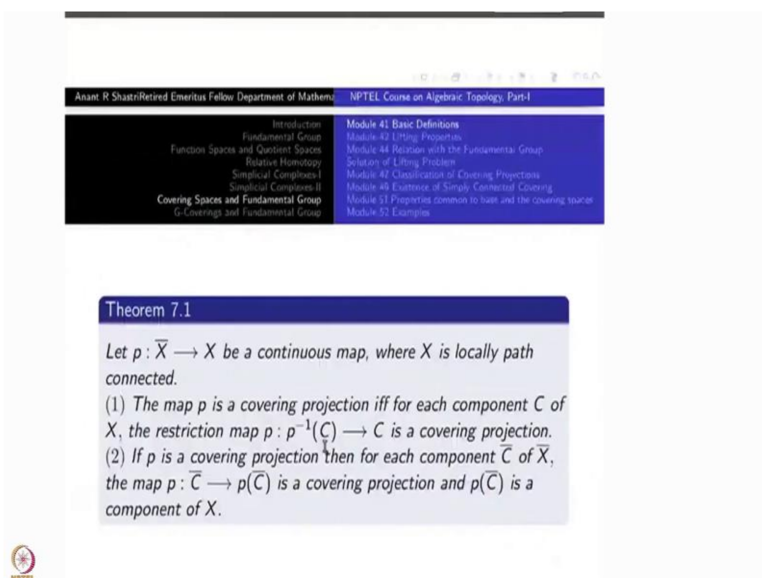

So, this is the theorem. Take a continuous function $p : \bar{X} \to X$, where X is now locally path connected. Then something nice happens, namely, the map p is a covering projection if and only on each component C of X, component means path component, the restriction map p $p : p^{-1}(C) \to C$ is a covering projection.

Next, if p is a covering projection, then for each component $\bar{C}$ of $\bar{X}$, now see again, I am taking path components, the restriction map $p : \bar{C} \to p(\bar{C})$ is a covering projection. So, under restriction map the covering property is not disturbed if $\bar{C}$ is a path component. And in that case, $p(\bar{C})$ will be a path component of X.

The point is that you have to take a component, if you restrict just to a path connected space it is not, it is not true. Just like the other way something is path connected by removing a point it may still be path connected then it will not be a covering projection. But if you remove a point, it would not be a path component in the original thing. You have removed something. It would not be a path component. So that kind of counter example is not there. But this is a theorem and actually we have to prove this.

Namely, the second part is very important, the first part is easy because you are coming from bottom. You are taking its component here and then you are taking full inverse image. So, this is not so difficult. The second part is something which you have to doubt and so you have to pay attention to the proof of this one. So, here is the proof.

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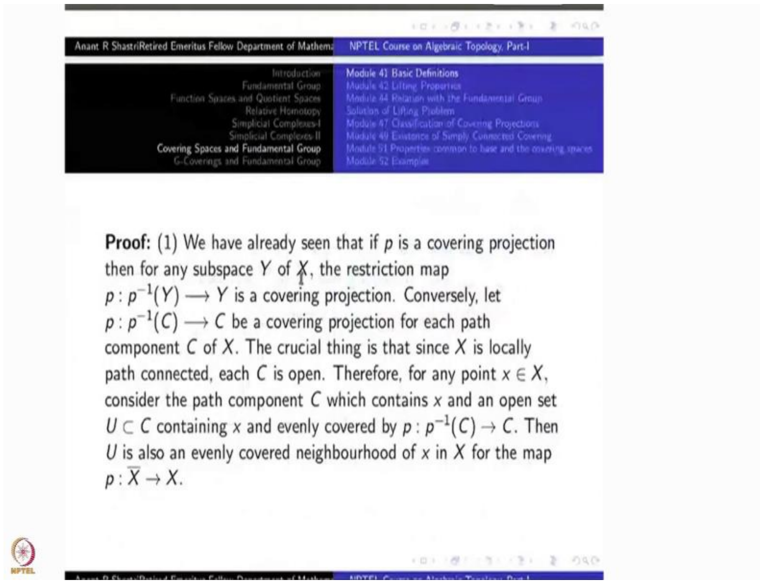

So, first part (1): if p is a covering projection for any subspace Y of X, the restriction $p : p^{-1}(Y) \to Y$ is a covering projection. So, one part is easy. Conversely, suppose for each component C of X, $p : p^{-1}(C) \to C$ is a covering projection, let us say. The crucial thing here is that this X is locally path connected and hence each C is open in X. Therefore, for any point in X consider the path component C_x, which contains x. Every point is inside a path component and that is an open set.

So, if U is open in C_x, U will be open in X also. But this is a covering projection, so I can choose U to be evenly covered by the restriction. Then since this $p^{-1}(C)$ is a full inverse image it will contain $p^{-1}(U)$. So, the same U will be evenly covered neighborhood for the full $p : \bar{X} \to X$. Since every point is covered by an evenly covered neighborhood it follows that $p : \bar{X} \to X$ is a covering projection. So, this proves part (1) here. That is a easy part.

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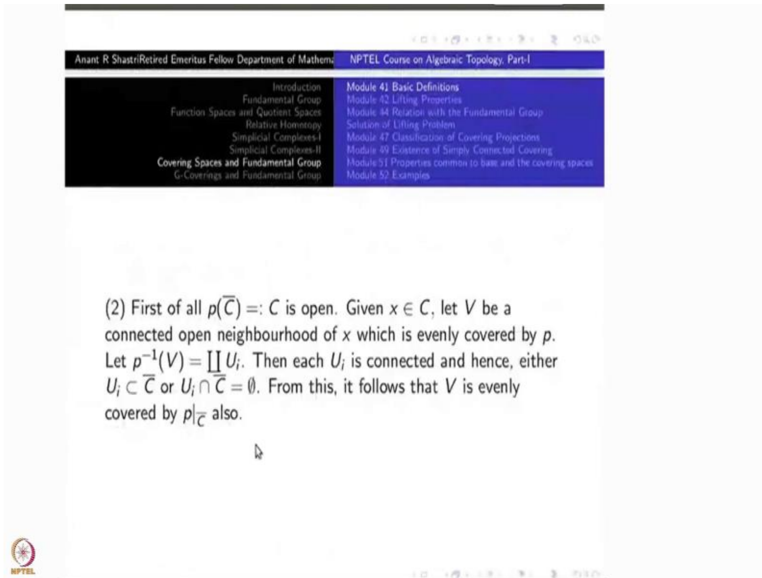

The second part (2), first of all look at $p(\bar{C})$, it is an open set in X. Why? Because, X is locally path connected, so, $\bar{X}$ is also locally path connected. Path connected components of locally path connected space are open. And p is an open mapping. So, $C := p(\bar{C})$ is an open set. Given, $x \in C$, let V be a path connected open neighborhood of x which is evenly covered by p. $p^{-1}(V) = \coprod_{i \in \Lambda} U_i$, such that $p : U_i \to V$ is a homeomorphism for each i. Then each U_i is path connected.

And hence either $U_i \subset \bar{C}$ or $U_i \cap \bar{C} = \emptyset$. Therefore, if Λ' is the set of those $i \in \Lambda$ for which $U_i \subset \bar{C}$, it follows that when you take restriction map $q := p : \bar{C} \to C$, then $q^{-1}(V) = \coprod_{i \in \Lambda'} U_i$. Some of these Ui's will go away, the rest of them will be remaining fully inside. And then they are contained in, not no part, no part of Ui will be there, either it is full or it is none. So, V is evenly covered by q also.

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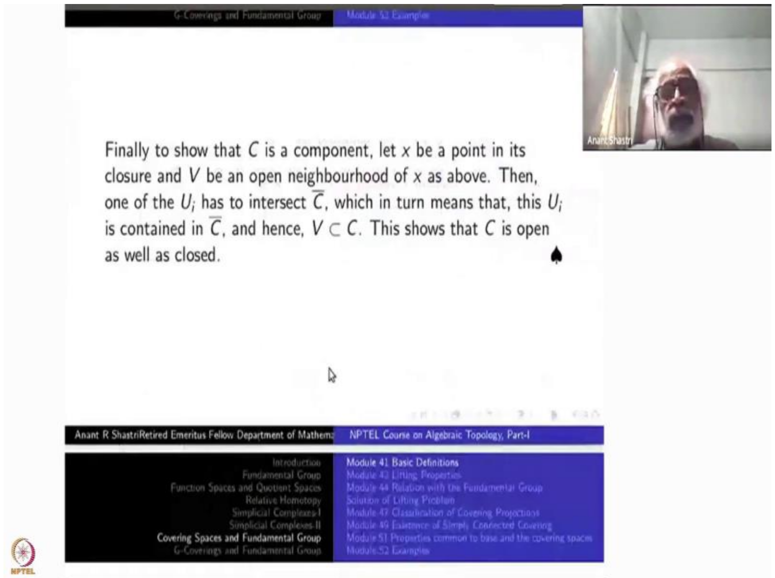

Finally, the converse of this one. sorry, one more thing we have to show, namely that C is a component. $\bar{C}$ is path connected. Therefore $C = p(\bar{C})$ is also connected. It is a component I have to show. So, let x be a point in the closure of C in X and V be a path connected evenly covered open neighborhood of x as above. Then one of the U_i has to intersect $\bar{C}$ because, $V \cap C \neq \emptyset, p^{-1}(V) \cap \bar{C} \neq \emptyset$. The whole thing intersects therefore but at least one of them has to intersect that is the point. But then that in turn means that, that particular U_i is contained inside $\bar{C}$, because, $\bar{C}$ is a component. So, these components etc., I am working inside a larger spaces $\bar{X}, X$ etc. and V is an open subset in larger space. And hence, $V = p(U_i) \subset p(\bar{C}) = C.$ So, this intersects this one of them. So, it is inside C bar. So, V is inside C. So, if V is inside C, it shows that C itself is open as well.

Usually, a component, you know it is closed. Now I have shown that every point in the closure of C has neighborhood which contained in C. So, it must be closed as well as open. Actually, I started with the closure point then show that the whole thing is inside in a whole neighborhood inside C. So, C must be open as well. If it is open and closed and connected it must be a component. So, that completes the proof of the theorem. I think we will stop here today; it will take you some time to understand this one, that go more things next time. Thank you.