Introduction to Algebraic Topology (Part 1)
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Lecture 40
Homotopical Aspects

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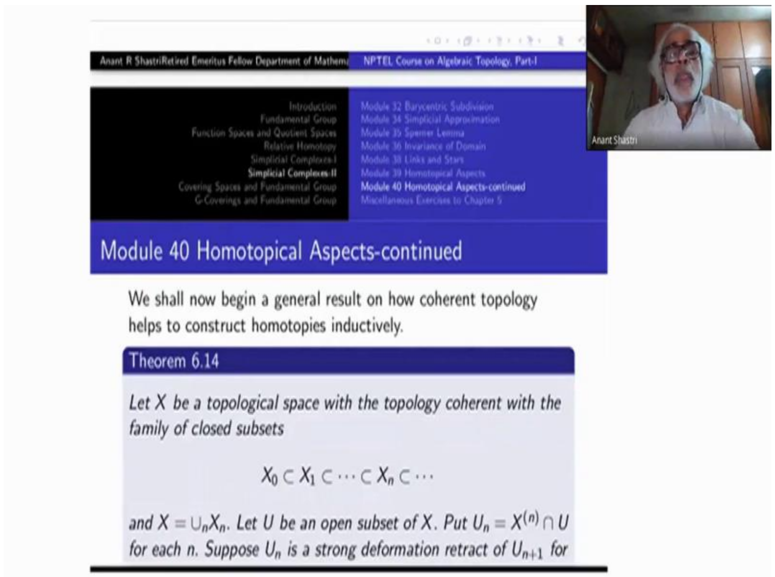

Last time, we considered somewhat easy type of homotopic properties, easier results. They are all important of course. So, today we will take a little deeper look into these aspects. So, now we have appealed to the coherent topology more clearly, in defining the homotopies also inductively. So, I would like to present a result which will be applicable to any space which has this coherent topology and then we will use it for simplicial complexes.

Namely, the topology on simplicial complex is coherent with the skeletons, n-skeletons. Also, it is coherent with each closed simplices and so on, we can use that. So, here is a general result of getting the homotopies on coherent topologies. Start with a topological space with the topology coherent with a family of closed subsets, an increasing family $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ They are all closed subsets of X and $\cup_n X_n = X$, the union of Xn is X. A set is closed inside X if and only if intersection with each X_n is closed inside X_n. That is the meaning of a coherent topology.

Now, take U to be any open subset of X, put $U_n = U \cap X_n$. Suppose, each U_n is a strong deformation retract of the U_{n+1}, the next one. U_n is subset of U_{n+1}. Then U_0 is a strong deformation retract of the entire U. So, what we are saying is similar to the previous result--- if we do not want strong deformations, just take retracts. So we had this kind of thing coming in the proof of the previous result on retracts. But now we want a strong deformation retract, means we are referring to the homotopies here.

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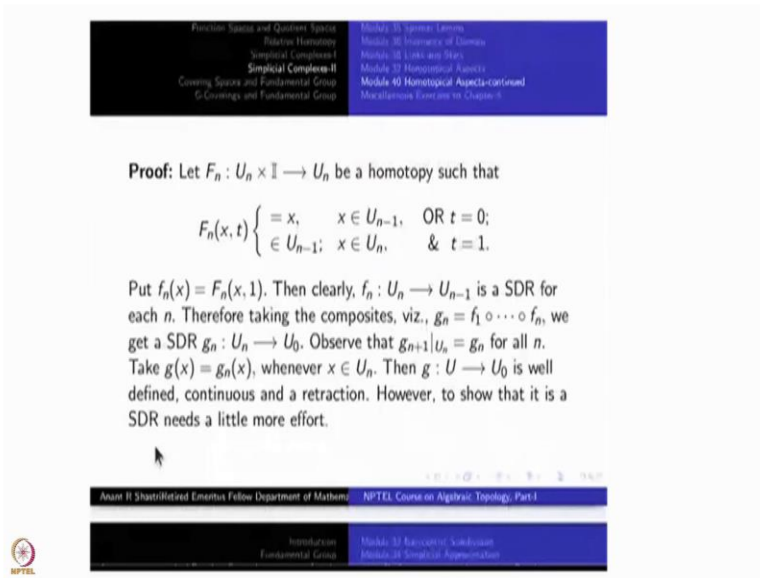

So, start with a sequence of strong deformation retractions, namely, $F_n : U_n \times \mathbb{I} \to U_n$, $F_n(x,0) = x$ is the identity on Un, when t is 0. Also $F_n(x,t) = x, x \in U_{n-1}$, it is identity for all t, on U_{n-1}. And the last thing namely, $F_n(x,1) \in U_{n-1}, x \in U_n$. So, this is a strong deformation transform of U_n into U_{n-1}. So, this is the hypothesis. Let us put $f_n(x) = F_n(x,1)$, this is just a short notation. So, f_n we know is homotopic to identity keeping U_{n-1} fixed. And $f_n : U_n \to U_{n-1}$ itself is a retraction. That is why it is a strong deformation retraction.

Let us put $g_n = f_1 \circ f_2 \circ \cdots \circ f_n$. This brings U_n to U_{n-1}, U_{n-1} to U_{n-2} so on and finally, f_1 will bring the entire thing into U_0. So, gn is a retraction of U_n to U_0. Being a composite of retractions, it is a retraction. So, each of them is a strong deformation retraction, so I can take the composites of these homotopies, composites each time so it should be a strong deformation retract to Un to U0, there is no problem.

Moreover, $g_{n+1} = g_n \circ f_{n+1}$ and since f_{n+1} is identity on U_n, so, g_{n+1} restricted U_n is g_n. Only the first n of them will operate non trivially on U_n. It follows that if $m > n$, then g_m restricted to U_n equal to g_n. Therefore, I can define $g : U \to U$ by putting $g(x) = g_n(x)$ whenever $x \in U_n$. So, this g will be defined on the entire of U. Also $g(U) \subset U_0$ because each $g_n(x) \in U_0$. So, continuity of g also follows because restricted to each X_n, i.e., on $U \cap X_n = U_n,\ g = g_n$.

So, g is continuous, it is a retraction from U to U0. The only problem here is that where is the homotopy? What is the meaning of concatenating infinite sequence of homotopies? See when you, when you take one, one homotopy here, another homotopy here the composition of homotopies is not composition of functions. It is concatenating the homotopies. That is why this requires us to pay more attention. It has to be done properly. How, to show that, g it is a strong deformation retract, needs a little more effort. So, here is what we have to do, by modifying the sequence of homotopies $\{F_n\}_n$.

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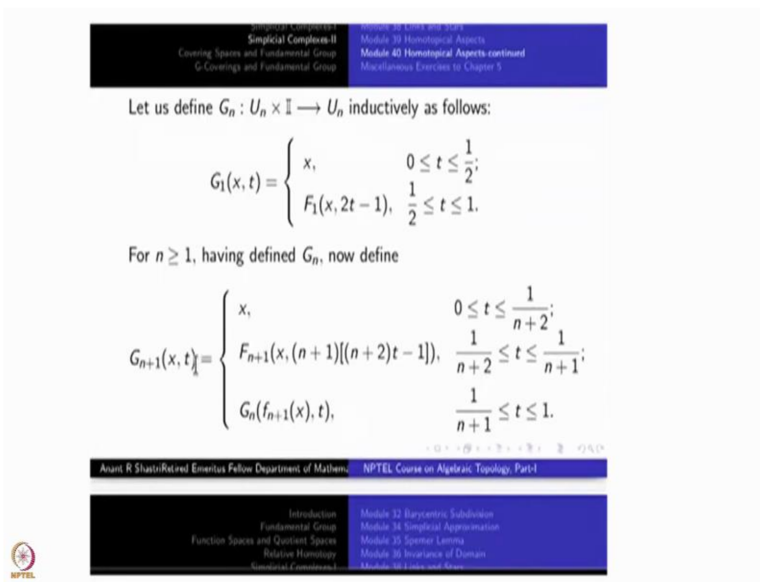

So, we start defining a sequence of homotopies $G_n : U_n \times \mathbb{I} \to U_n$. Let us define this one inductively as follows. The first one, G_1, in the first half of the interval, it is x. In the second half of the interval it is F_1, it is the `composition'. Composition means a concatenation like a composition of two paths, for each x, as t varies, it is the path composition, it is that. So, in first half of the interval, $0 \le t \le 1/2$, it is x identity map and in the second half, $1/2 \le t \le 1$, it is equal $F_1(x, 2t-1)$, we know we know what F1 is. So, this is the definition of G_1. Having defined G_n, we are now now defining G_{n+1}.

G_2 onwards, there is a slight difference in the definition. That is why I have to define separately, namely, by this formula n plus 1 of x t will be defined three different intervals here. So, the entire interval $[0, 1]$ is divided into three portions depending on $n \geq 2$. The first portion is $[0, 1/(n+2)]$. The second portion is $[1/(n+2), 1/(n+1)]$. and the third portion is the remaining all-big chunk $[1/(n+1), 1]$.

So, let us look at this last part first. It is $G_n(f_n(x), t)$, on this biggest rectangle. Gn of fn of x plus t. When $t = 1/(n+1)$, it should coincide with this one. Because these are two different formulas. So, $t = 1/(n+1)$ what happens? $(n+1)[(n+2)t - 1] = (n+2) - (n+1) = 1$. Therefore the second formula $F_{n+1}(x, (n+1)[(n+2)t-1]) = F_{n+1}(x, 1)$. On the other hand, the third formula becomes $G_n(f_{n+1}(x), 1/n+1) = f_{n+1}(x)$. This is because $G_n(x, t) = x, 0 \leq t \leq \dfrac{1}{n+1}$. Therefore these two coincide. Similarly, when you check that the first and second formulas coincide for $t = \dfrac{1}{n+2}$. We have $f_{n+1}(x, (n+1)[(n+2)t - 1] = f_{n+1}(x, 0) = x$.

So, it is identity. So, the entire definition of G_{n+1} makes sense and is a continuous function. Also, it is identity, in this small rectangular portion near 0. Remember, as n keeps increasing this will never hit 0 you know there will be some interval of positive length, so, it will never hit the line $0 \times \mathbb{I}$. Every G_n is identity on the line. So, this line this one this interval keeps coming closer and closer to to 0. But, for all n it is still some interval will be there. So, that is the nature of it.

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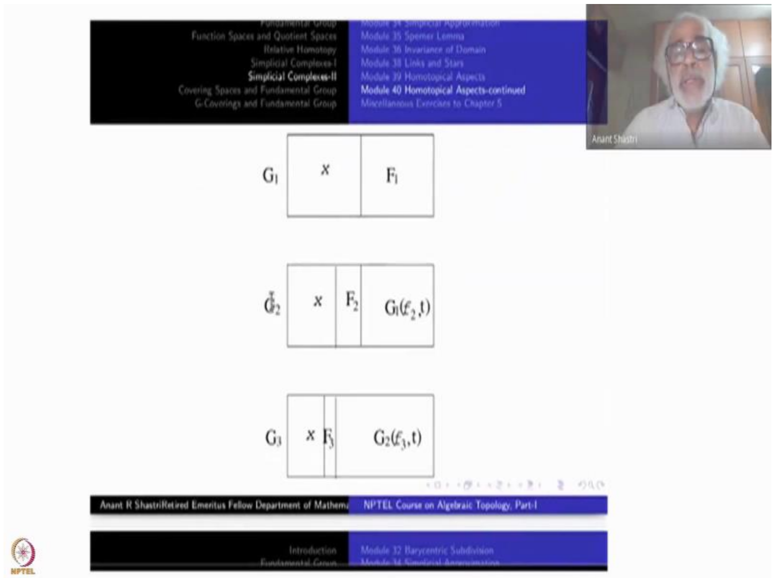

So, here is the graphical representation of G_1. On first half of the rectangle it is x, on the other half it is F_1. For G_2, in the last half, it is G1 of f2, here it is F2, and here it is x. This is 1 by 2 and this is 1 by 3. The next time again three divisions but this line will be what? Of the entire thing here I am making three divisions; in the first one it is always identity. Keep going, going keep doing that, next time you delete this one put a line on the left of this one-fifth and so on. So, keep going, define this way so this is G1, G2, G3 and on.

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Each Gn plus 1 is well defined and is a homotopy of the identity map to the last gn plus 1. Gn plus 1 is look at this definition here Gn little Gn of this already composite, composite, composite G Gn minus G last one fn plus 1, so this will become Gn of into composition fn plus 1 so it is gn plus 1. So, starting with x here the last thing is when t is equal to 1 this is Gn plus 1, that is clear. So, all Gn is homotopy of the identity map gn plus 1. Each Gn plus 1 is a strong deformation retract of Un plus 1 into U0.

Moreover, restricted to $U_n \times \mathbb{I}$, it is G_n. Therefore, therefore you can define a map $G : U \times \mathbb{I} \to U$ by the formula $G(x,t) = G_n(x,t), x \in U_n$. This is valid, because, for $m > n$, we have $G_m(x,t) = G_n(x,t), x \in U_n$.

Now, if V is an open subset of U, then $G^{-1}(V) \cap X_n = G^{-1}(V) \cap U_n$. Because the domain of G is U. But then this is equal to $G_n^{-1}(V)$ which is an open subset of U_n because G_n is continuous. This means $G^{-1}(V)$ in U. So, I am just showing that G is continuous as a map from $U \times \mathbb{I}$ into U but finally into U_0. Therefore, G is continuous. Therefore, G is a strong deformation retract of U into U_0. So, you had to work much harder to get a inductive homotopy.

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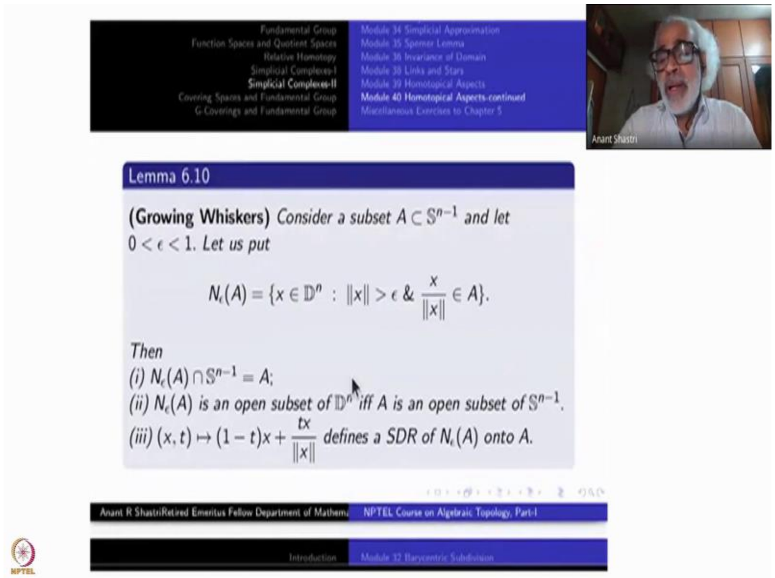

Now, we will come to a standard lemma and again inside $\mathbb{D}^n$. So, we have to get familiar with Euclidean spaces, you can do whatever you like and then you want to extend them to the simplicial complexes. This is the game we are playing. So, `growing whiskers' is a lemma, I have named it so, Growing Whiskers. You take any subset $A \subset \mathbb{S}^{n-1}$, the boundary of $\mathbb{D}^n$, and choose any epsilon between 0 and 1. Setup $N_\epsilon(A)$ equal to the set of all those $x \in \mathbb{D}^n$ which are at a distance from the boundary less than 1 minus epsilon; i.e., $\|x\| > 1 - \epsilon$, and $\frac{x}{\|x\|} \in A.$ Norm of x is bigger than one minus epsilon. But the modulus divided the modulus the unit vector must be inside A.

So, it just means that take a point inside A and then draw a small line segment of length less than epsilon inside the disk. So, that is the Whiskers, so it is Growing Whiskers then. So, $N_\epsilon(A)$ is such a thing. Clearly, this is an open subset of $\mathbb{D}^n$ provided A is open in $\mathbb{S}^{n-1}$. Because the function $x \mapsto \frac{x}{\|x\|}$ is continuous $\mathbb{D}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ and $x \mapsto \|x\|$ is continuous $\mathbb{D}^n \to \mathbb{I}$.

In any case $N_\epsilon(A) \cap \mathbb{S}^{n-1} = A$. That is property (i). The second property we already added. This is an open subset of $\mathbb{D}^n$ if and only if A is an open subset of $\mathbb{S}^{n-1}$. Because, intersection is this one. The third thing is that you can push the whole thing back to A namely, $(x,t) \mapsto (1-t)x + t\frac{x}{\|x\|}$. This is a strong deformation retraction $N_\epsilon(A)$ onto A.

Note that epsilon can be chosen as small, or as large as you pleased provide $0 < \epsilon < 1$. Depending upon your requirement this is going to give you arbitrary small open neighborhoods of this A inside $\mathbb{D}^n$ which strong deformation retracts onto A So, this is the germ of the idea, the starting point, which is happening in the Euclidean space. From this, we want to capture it and take it to all the simplicial complexes.

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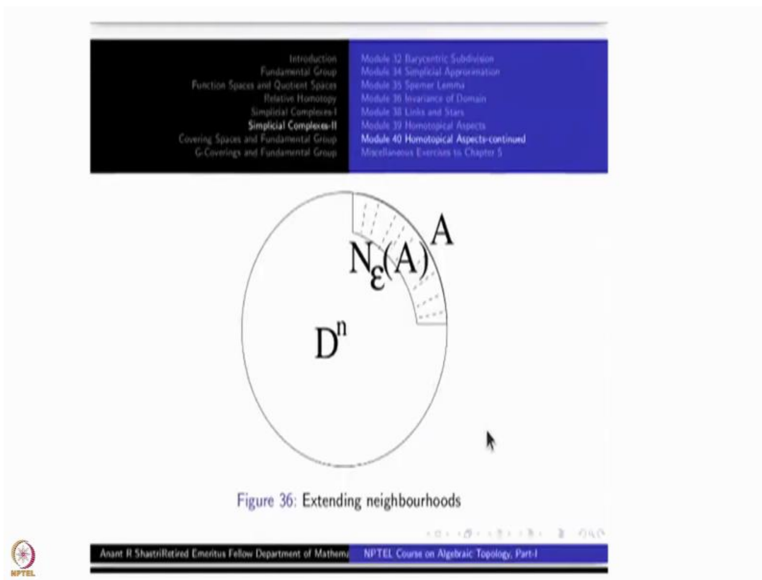

So, this is the picture of $\mathbb{D}^2$, A is some subset of the boundary circle. I have shown it as an arc here. This maybe open arc or a closed arc you do not know, may be not even connected. If it closed arc these end point will be there. If this is an open arc even this end point should not be there but I cannot show it in the picture. This $N_\epsilon(A)$ will an open subset if $A \subset \mathbb{S}^1$ is open.

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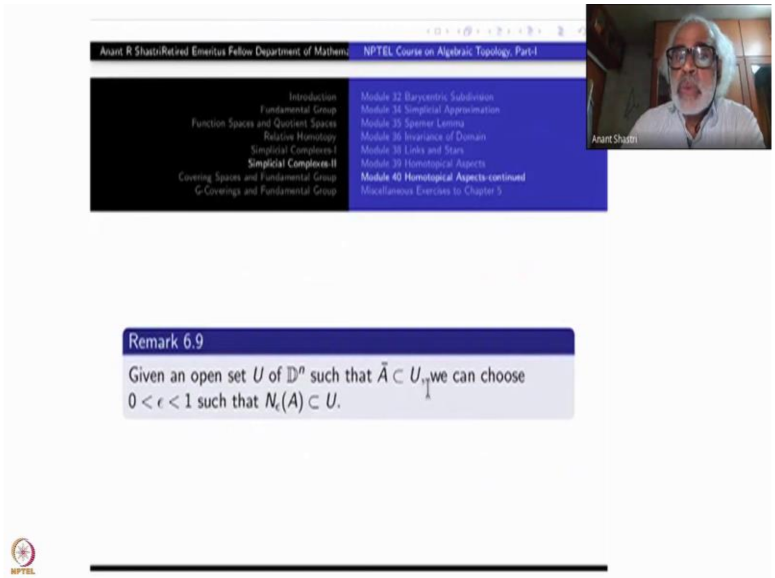

So, this remark is very important. Start with an open subset U of $\mathbb{D}^n$ such that $\bar{A} \subset U$. $\bar{A}$ is inside U allows you, $\bar{A}$ being compact, to choose $0 < \epsilon < 1$ such that this $N_\epsilon(A)$ is contained in U. So, that is because this $\bar{A}$ becomes compact. So, I think this much topology you know.

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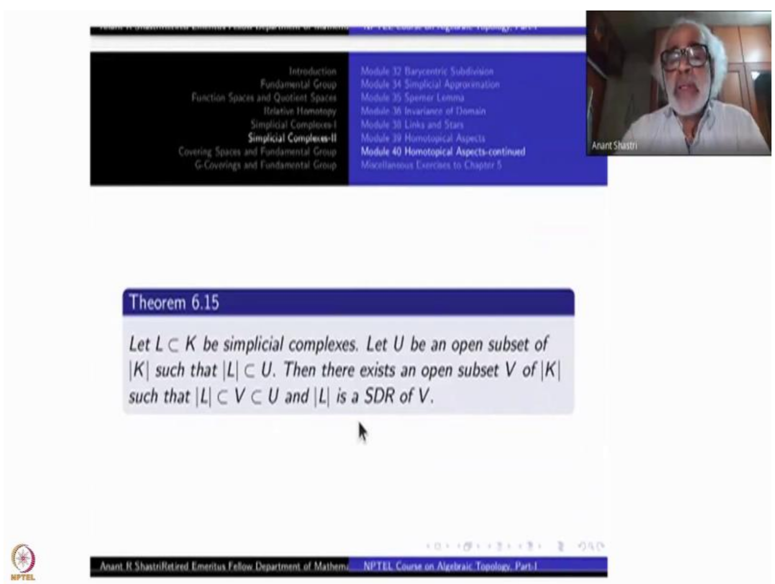

Now, we have a big theorem. Let L be any sub complex of a simplicial complex K. Let U be an open subset of $|K|$ which contains $|L|$. Then there exists an open subset V of $|K|$ contained inside U and containing $|L|$ such that this $|L|$ is a strong deformation retract of V. In particular we know that is this will imply that the inclusion map $|L|$ to V is a cofibration. It is being used in many other results. First, let us workout how we got this result.

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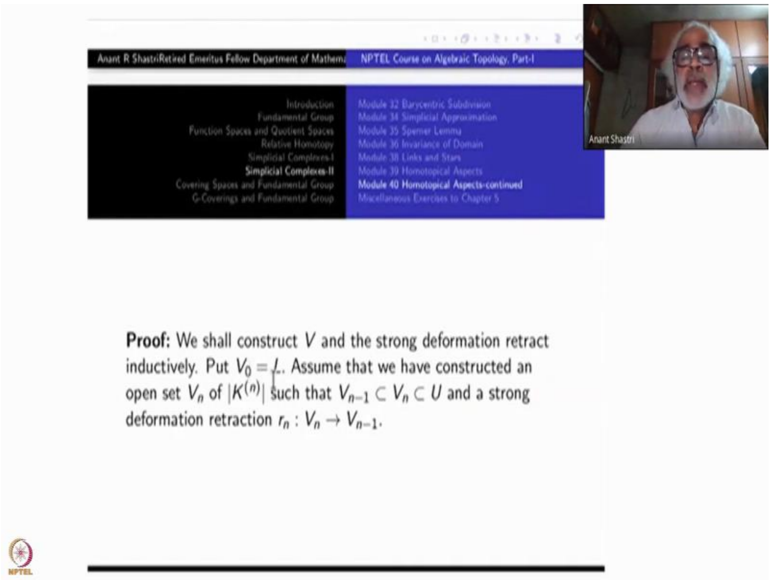

We shall construct this V, the open set as well as the strong deformation retract inductively. We start with V_0 equal to $|L|$. In the lemma, in the general lemma sorry theorem 6.14 it is this one, this theorem you see U was a given open subset of X with certain properties. We did not have to construct this one. Here we are constructing that a subset V in place of U there. As before, put $K_n = |L \cup K^{(n)}|$. Put $V_0 = |L|$. Note that V_0 is an open subset of K_0 and is contained in $U_0 = U \cap K_0$. Assume that we have constructed an open set $V_n \subset |K^{(n)}|$ such that $V_{n-1} \subset V_n \subset U_n = U \cap |K^{(n)}| \subset U$. We want a strong deformation retraction $r_n : V_n \to V_{n-1}$. So, this is the inductive step we want to do. We have not yet done it.

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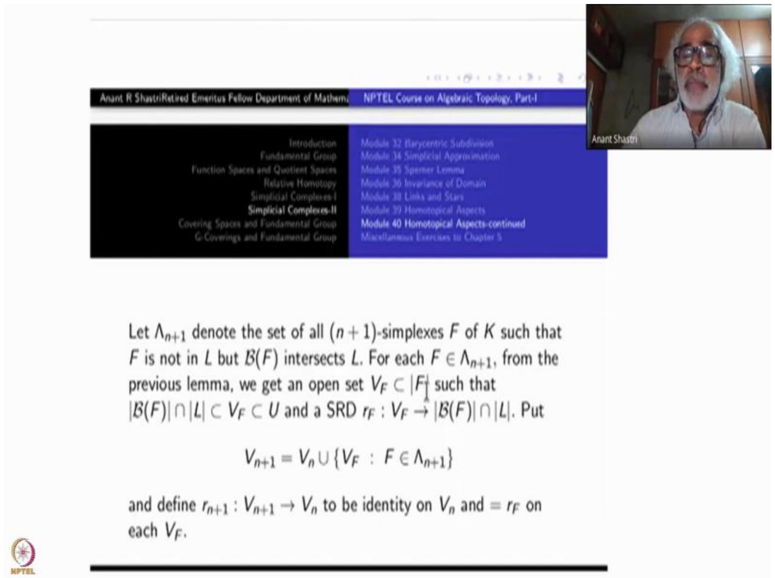

So, look at the collection Λ_{n+1} of all (n+1)- simplices F of K such that F is not in L (because, part of L we do not want to trouble), but $\mathcal{B}(F)$ intersects L. These $F's$ are near enough L, they are not far away. If $\mathcal{B}(F)$ does not intersect L you ignore them. You do not have to worry. Also, only those F which are not inside L. For each such F, in previous lemma, we got a open set $V_F \subset |F|$, such that $|\mathcal{B}(F)| \cap |L| \subset V_F \subset U \cap |F|$, and a strong deformation retraction $r_F : V_F \to |\mathcal{B}(F)| \cap |L|$. So, how did you get this one? This is by Growing Whiskers lemma. On each of this, this is just Growing Whiskers.

Now put $V_{n+1} = \cup\{V_F \ : \ F \in \Lambda_{n+1}\} \cup V_n$. Now, r_{n+1} will be defined by patching up all these $r'_F s$. So, take $r_{n+1} = r_F$ on each $F \in \Lambda_{n+1}$. So, this will give you a retraction of $V_{n+1} \to V_n$, a strong deformation retraction.

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Now, put $V = \cup_n V_n$. This V is open in $|K|$. None of the V_n may be open in $|K|$. but they are opened inside K_n. That is the point here. So, now I have just appealed to this big theorem that we proved. To say that once we have retractions like this you can patch them up. So, V is open in $|K|$, $|L| \subset V \subset U$, because each $V_n \subset U_n$ is open. You can now appeal to this theorem to obtain strong deformation retraction $r : V \to |L|$.

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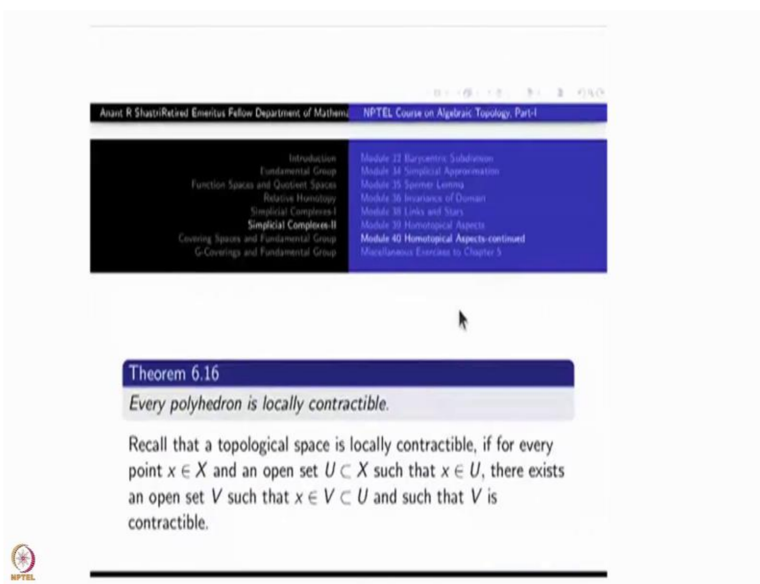

The conclusion is that every polyhedron is locally contractible. This is the stronger conclusion we have come up now which we have could not do earlier, we did only semi locally contractibility. Now, it is every polyhedron is locally contractible. How do you do that? So, take any point $x \in |K|$. Make it into a vertex by choosing an appropriate subdivision. Once it is a vertex singleton $\{v\}$, it will be a subcomplex. So, for a subcomplex you have done this one.

So, given an open set U around x, there neighborhood V of $\{x\} = |L|$, V strongly deformation retracts onto x. That means V is contractible. This time the strong deformation does not use U, but is completely inside V, $r : V \times \mathbb{I} \to V$. So, this is just a corollary to this strong result, namely, every subcomplex of a simplicial complex has this property. Namely, it is locally NDR NDR pair, there is a neighborhood which retracts strongly to $|L|$. So, let us stop here.