Introduction to Algebraic Topology (Part-I)
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Lecture 4
Path Homotopy

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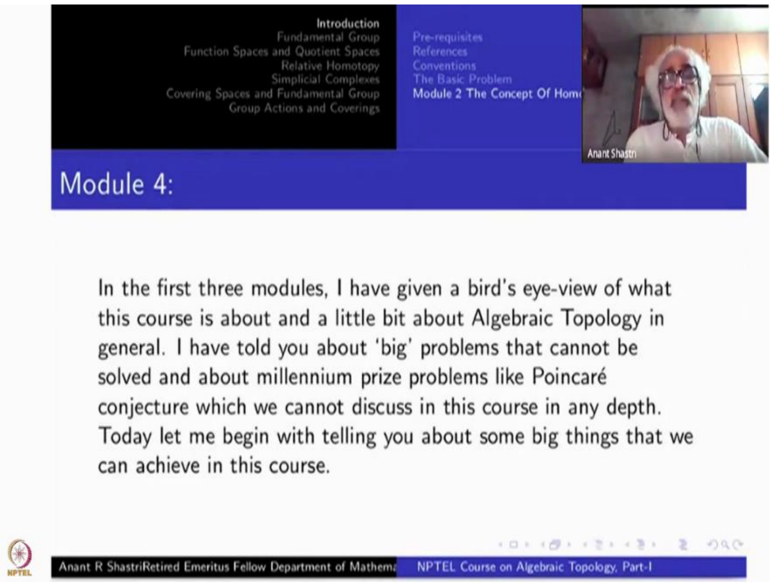

Hello. So, let us begin the fourth module today. In the first three modules, I have given a bird's eye-view of what this course is about and a little bit about algebraic topology in general. I have told you about certain big problems that cannot be solved and about certain millennium prize problems like Poincare conjecture which we cannot discuss in this course in any depth. So, today let me begin with telling you about some other big things that we can achieve in this course, on the positive side.

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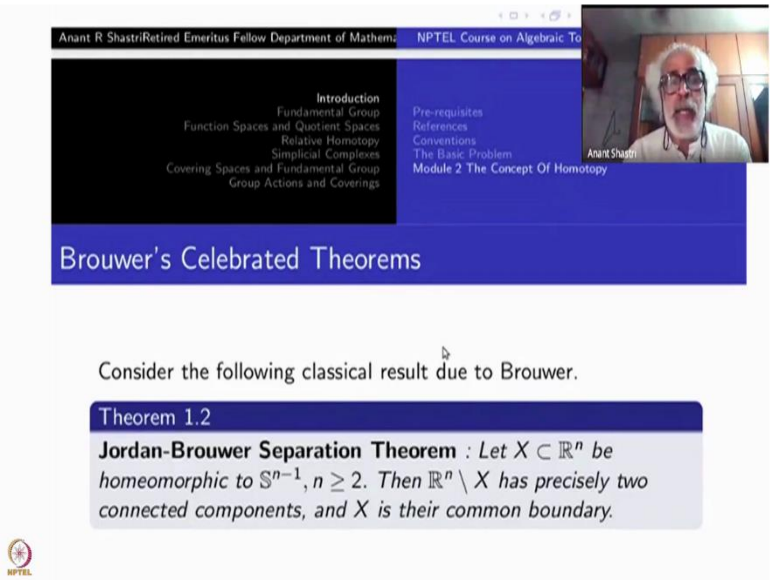

So, this is called Brouwer's Celebrated Theorems. There are two of them here. One is the Jordan-Brouwer Separation Theorem. Jordan comes here for n = 2. For higher things it is Brouwer. That is why it is Jordan-Brouwer Separation theorem.

Take a copy of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ for $n \geq 2$. Then, the complement of this $\mathbb{S}^{n-1}$, let us denote the copy of $\mathbb{S}^{n-1}$ by X, the complement of X has precisely two connected components, and X happens to be the common boundary, common boundary. So, in the case of n = 2, a copy of $\mathbb{S}^1$; one calls it a Jordan Curve, or a Jordan Loop.

So, a Jordan Loop separates the plane into exactly 2 components. One is inside, another is outside. So, the inside region is called, inside is what?--- the bounded region. That is the meaning of inside region. There is only one bounded region and only one unbounded region and the loop happens to be the common boundary of both of them. This has been completely generalized by Brouwer for all n. This theorem, we will be able to prove in this course. Maybe it will take some time but it will be proved, that is the whole idea.

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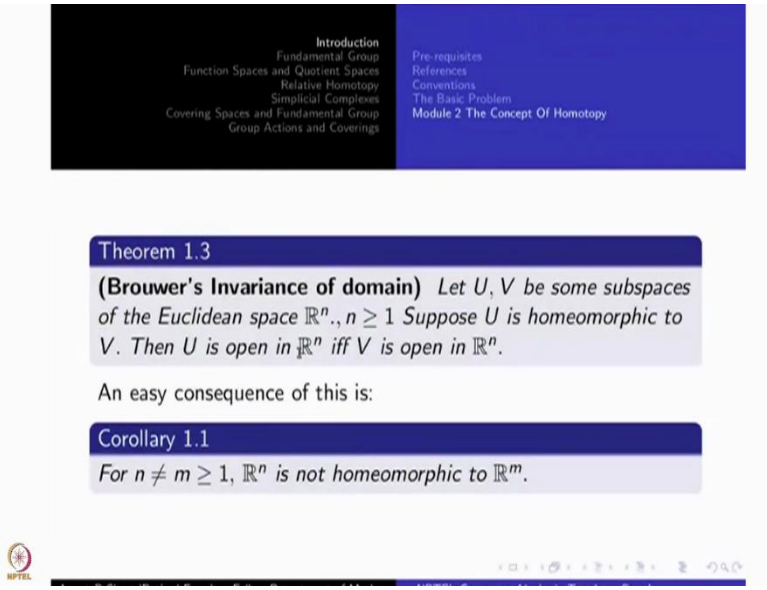

The next thing is Brouwer's Invariance of domain. Invariance of domain means ----you know what is the meaning of domain in calculus or complex analysis. It is an open and connected subset. Open and connected subsets of $\mathbb{R}^n$ are called domains. So, if something is a domain in some $\mathbb{R}^n$ that n is invariant. That is the whole thing. That is the whole idea of Brouwer's Invariance of domain.

Suppose, you have U and V, some subspaces of $\mathbb{R}^n$ and they are homeomorphic. If one of them is a domain that is one of them is open, then the other one is also open. That is like saying that invariance of domain. If something is a domain, then homeomorphic copies of that inside the same $\mathbb{R}^n$, they are all domains.

As an easy consequence of this, if you change the dimension, then they are not all domains --- can also be seen; it can also be observed, namely, for n not equal to m, $\mathbb{R}^n$ will never be homeomorphic to $\mathbb{R}^m$. So, this corollary is an easy corollary to theorem 1.3. I will let you think about it. Finally, we will solve this one. This is not, this corollary is not difficult from theorem 1.3.

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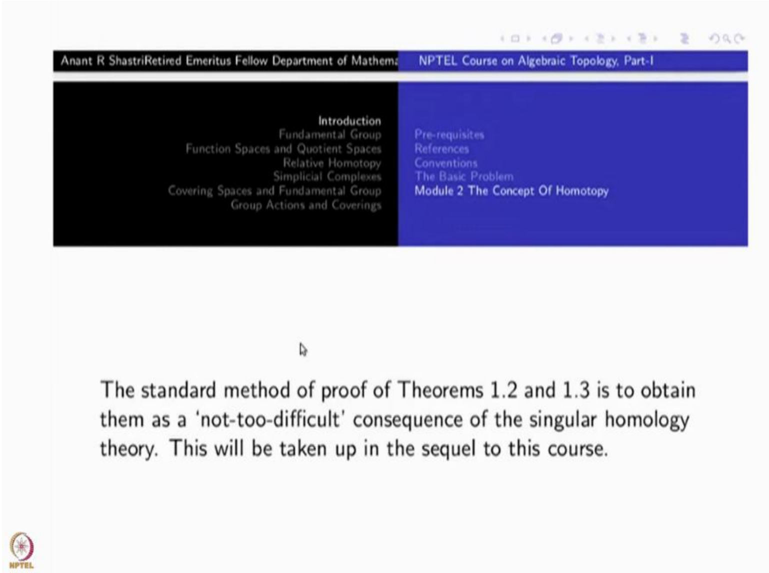

Standard method of proof of these two theorems is to obtain them as "not-too-difficult" consequences of singular homology theory. The singular homology theory will be taken up in a sequel to this course.

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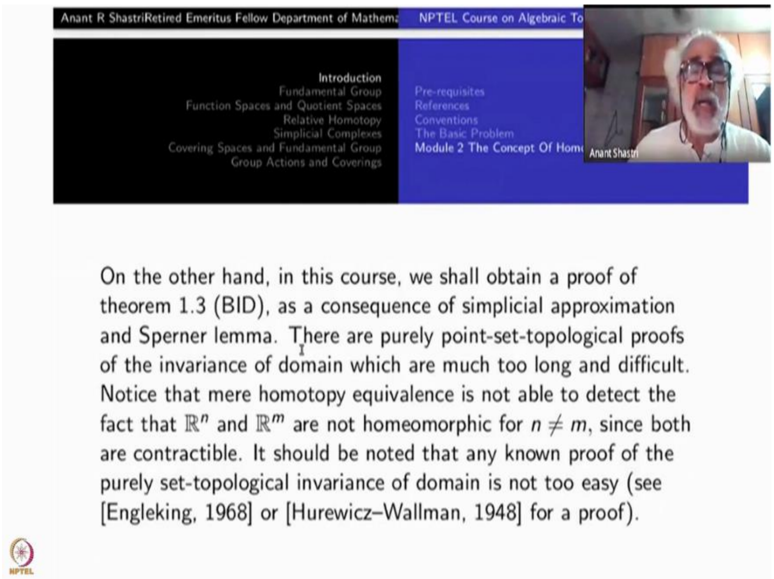

On the other hand in this course, what shall we do? We shall obtain a proof of the Brouwer's invariance of domain as a consequence of simplicial approximation and some combinatorial result called Sperner lemma. There are of course, purely point-set-topological proofs of this invariance of domain which are much too long and difficult. So called dimension theory books have been written on that.

Notice that mere homotopy equivalence is not able to detect the fact that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for n not equal to m, because both of them are contractible and therefore, they are homotopy equivalent to each other. So, how does homotopy help here? That is a strange thing no. It does. It should be noted that any known proof of purely point-set-topological invariance of domain is not too easy at all. All proofs are somewhat quite involved and lengthy. But you can look into Engleking's book and Hurewicz-Wallman's books and so on.

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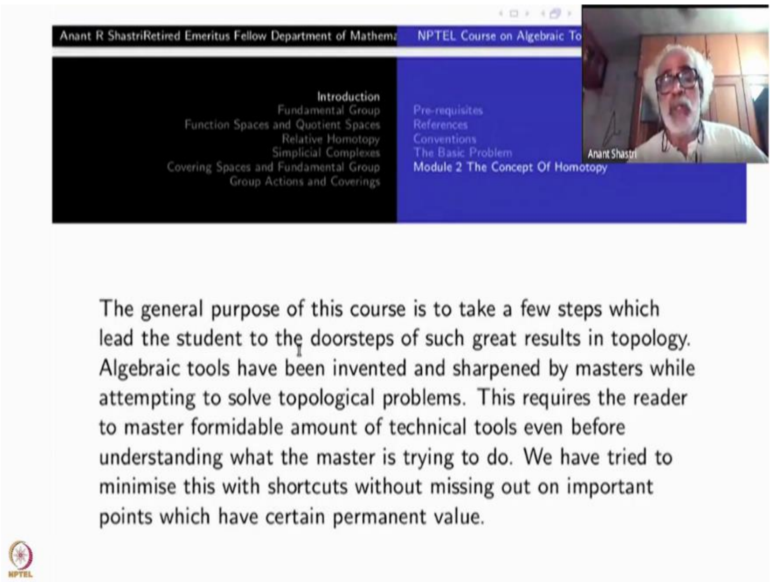

The general purpose of this course is to take a few steps which leads the students to the doorsteps of such great results in topology. We may not be able to see much of them, but once you have a couple of courses like this, you will be able to access all these results. Algebraic topology tools have been invented and sharpened by masters while attempting to solve topological problems.

This requires the reader to master a formidable amount of technical tools even before understanding what the master is trying to do; master is trying to work out. We have tried to minimize this with shortcuts without missing out on important points which can have a certain permanent value. So, this is what we are trying to do in this course.

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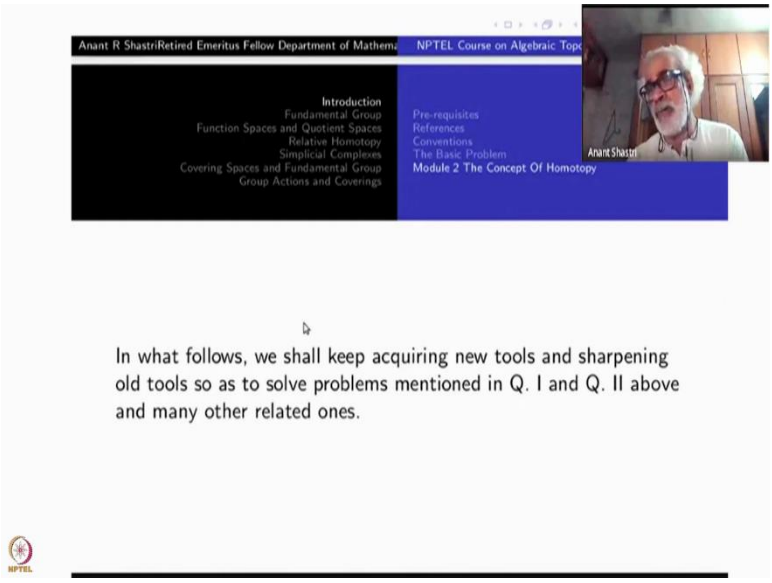

So, in what follows we shall keep acquiring new tools and sharpening the old tools, so as to solve problems mentioned in question number 1 and question number 2 above and many other related problems. So, this is the summary of whatever we want to do. So, we will now start doing them one by one.

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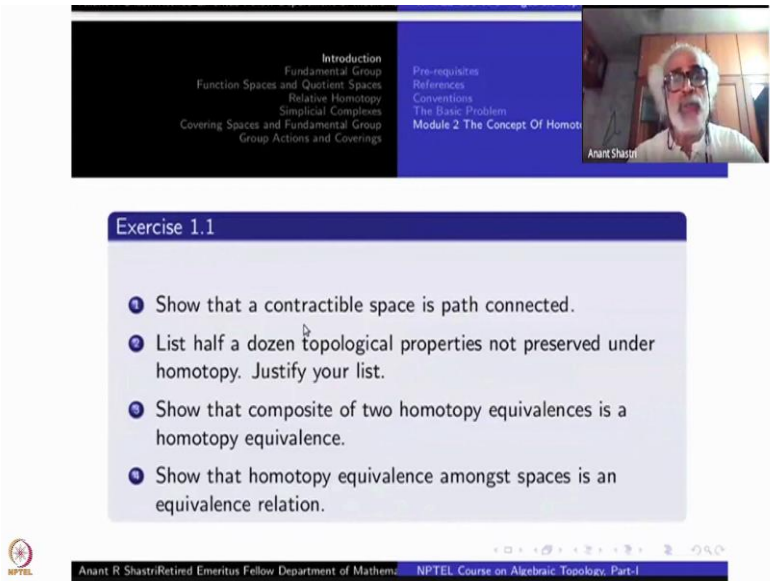

So, since we have already some technical definitions and so on, here is a set of exercises which you should try to solve them on your own and submit and the tutors will check them and you know later on we can even discuss it in one of the open sessions, live sessions. But before that you have to submit and you have to participate.

So, let me go through these exercises. First one is, (these are all simple exercises,) to show that a contractible space is always path connected. Second one is, I have told you that there are lots of topological properties which are not homotopy invariants, which are not preserved under homotopy. So, give a list of this, say a dozen topological properties--- No, half a dozen.

Show that composite of two homotopy equivalence, is homotopy equivalence. Show that homotopy equivalence amongst spaces is an equivalence relation. I have already told you how these things are but now you have to write down full details of these exercises. These Exercises are only for practise. There is nothing very hidden in them.

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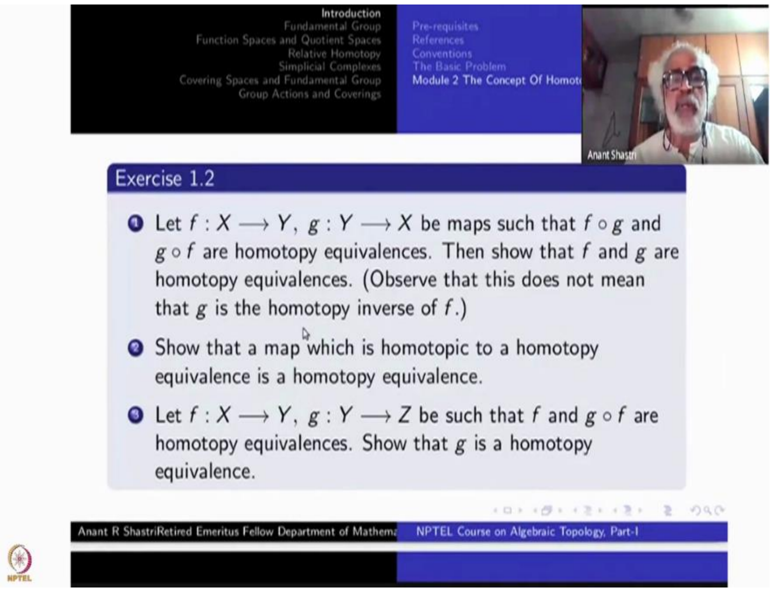

Now come to a few more exercises. First you have f: X → Y and g:Y → X, such that f ○ g and g○ f are homotopy equivalences. I am not saying that f and g are homotopy inverses of each other. The composite, I am not saying that the composite f ○ g is homotopic to the identity of Y nor g ○ f is homotopic to the identity of X. But they are themselves homotopy equivalences. Then show that f and g are homotopy equivalences.

So, I caution you. I do not mean that g is homotopic inverse of f, it may not be, it may be. It does not matter. So, that is not the question here. Here you have to think a little bit. Keep thinking. When I want to use one of these results in the exercise, given the exercises, by that time I will give you the solutions. But till then, you keep thinking about it. So, whenever you get a solution, you can submit it. The tutors and I will check them.

So similarly, the next problem here: f : X →Y; g : Y → Z be such that f and g ○ f are both homotopy equivalences. Show that g is a homotopy equivalence. It is like cancelling one of them. If f is invertible, g ○ f is invertible, then g is invertible. So, this is algebra of homotopy equivalences.

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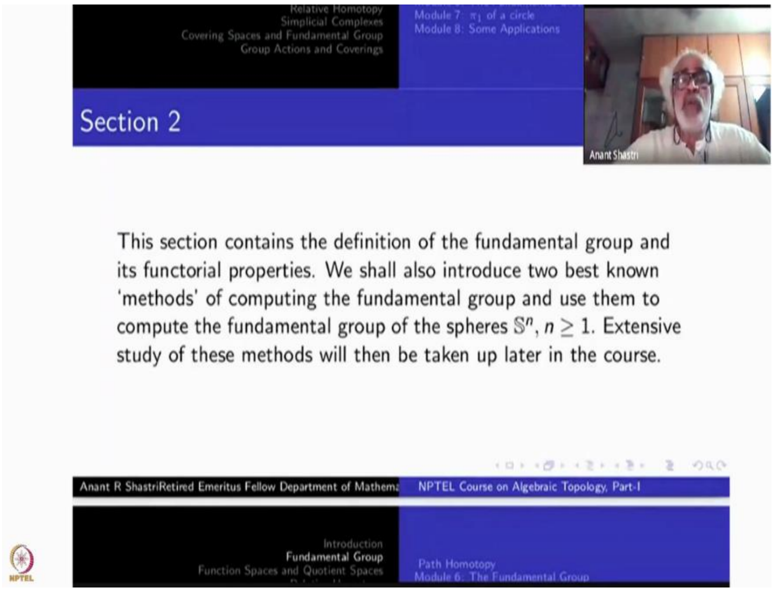

Let us begin with the brass-tags. We want to do whatever we want to do. We have to start doing them. So, this section will contain a definition of the fundamental group and its fundamental functorial properties. We shall also introduce two best 'methods' of computing fundamental groups and use them to compute the fundamental group of spheres; the neat objects. Once you have $\mathbb{R}^n$, they are simplest one; they are contractible, they do not have much homotopy properties.

The next objects are the spheres in them, unit spheres in the Euclidean spaces. Extensive study of these matters will be taken up later on. This is just now a trailer again to give you a flavour of what kind of things are coming up. So, that is what this section is about. But it will already be introduced to you, slightly deeper into the subject.

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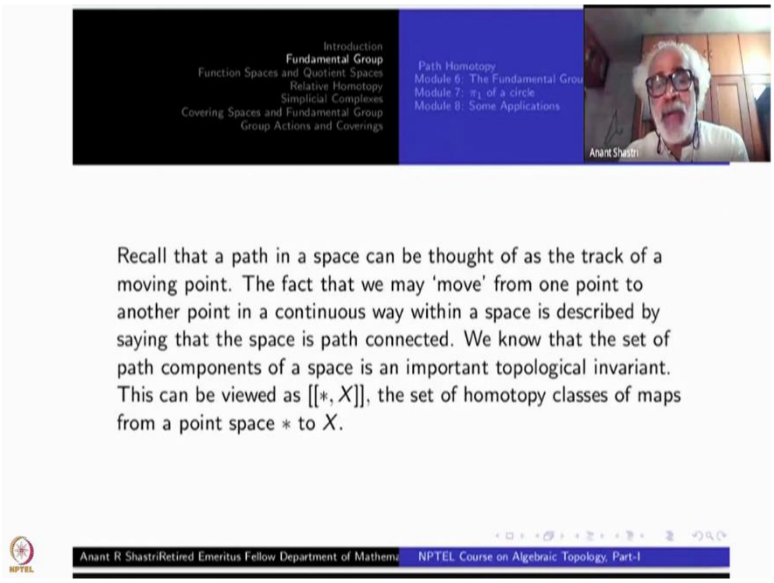

Recall that a path in a space can be thought of as the track of a moving point. The fact that we may move from one point to another point. It means what? Moving means what? In a continuous way within a space. That is described by saying that the space is path connected. You can go from one point to another point, if it is path connected. Path connectivity is a very, very old concept and which is very fundamental in all topological aspects.

We know that the set of path components of a space is an important topological invariant. We have introduced it as homotopy classes of maps from single point into X--- the set of homotopy classes of maps from a single point into X.

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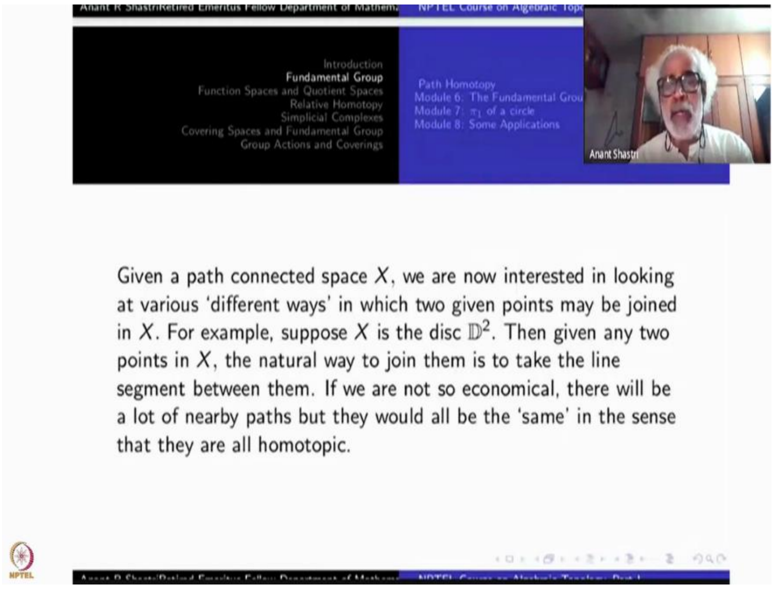

Now, given a path connected space, that is, suppose X is path connected. We are now interested in looking at various different ways in which two given points may be joined. For example, suppose X is a two-dimensional disk, -say the unit disk. Then given any two points, the natural way to join them is to take the line segment.

If we are not so economical there will be a lot of nearby paths, but they will all be in some sense the same even if you go a little bit away from the straight line. And straight lines are not always possible. You know paths are not always made up of straight lines, except perhaps in deserts. But we keep the direction the same. So, it is more or less the same in some sense. So, that is the meaning of being the same. Slightly they are away but you know like diversions in a given route when there is some road construction going on. They are homotopic paths.

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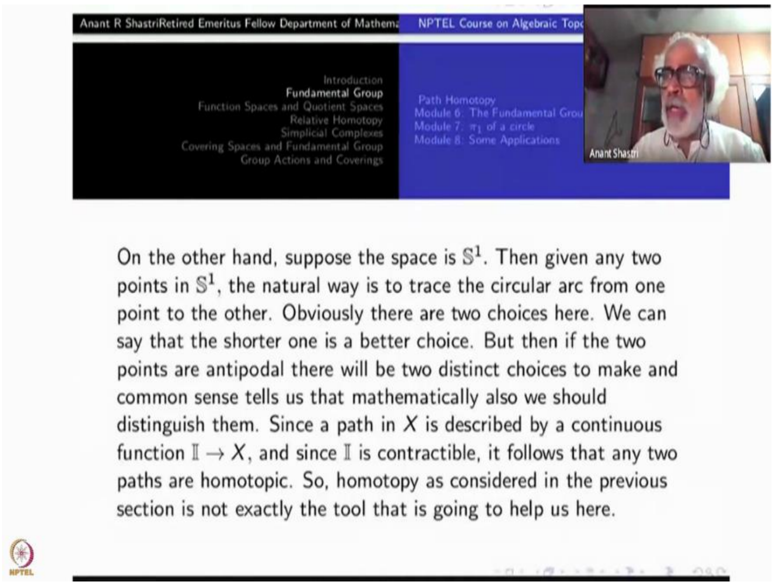

But, let us look at the picture $\mathbb{S}^1$. In $\mathbb{S}^1$, let us say, take any two points. Then I want to say that there is no straight line now, but there are two different arcs from one point to another point. So, these arcs, you know, you cannot change from one arc to another arc continuously. So, how to make this idea rigorous? That is the task we now have.

A path in X is described by a continuous function from a closed interval, which you have standardized as $\mathbb{I}$, the closed interval [0,1]. The closed interval [0,1] itself is contractible. Therefore, we know that any path, namely, a function from $\mathbb{I}$ to X must be null homotopic. We have seen that once you have a contractible space, any continuous function from a contractible space into any other space is null homotopic.

So, the homotopy that we have introduced is not very effective in distinguishing the two arcs that are there in which we want to distinguish. So, we need to sharpen the tool here. Then we want to study the paths as such. That is the meaning of sharpening a tool. We have the homotopy concept, but we want to modify it as per our requirement.

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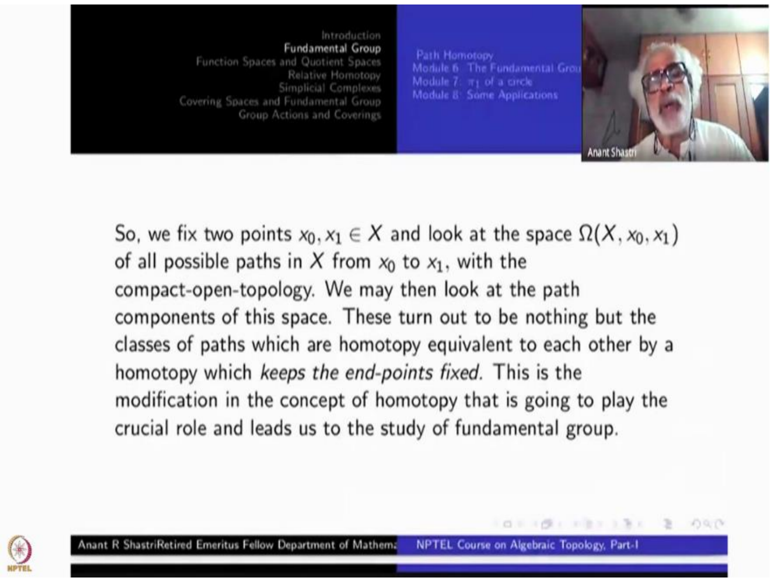

So, we fix two points x_0 and x_1 belonging to any space X. You can assume that X is path connected. There is no other way and look at the space of all paths from x_0 to x_1 inside X. So, I have denoted it by $\Omega(X, x_0, x_1)$ all paths. So, this is the collection of all paths. They are starting at x_0 and ending at x_1.

All of them are in X. So, such a space can be given a neat topology what we call compact-open topology. What is the meaning of compact-open topology? I will tell you later on. There is some topology. We may then look at paths in this space and path component of this space. This turns out to be nothing but classes of homotopy in this space namely I have to change from, I have to change the given path to another path but all the time we are in this space means the end points x_0 and x_1 remain the same. So, homotopy keeps the end points the same.

So, this is the extra condition on homotopy that we are going to introduce, a modified homotopy that we are going to introduce. So, once we see what we are trying to do, then we can do that. We have to understand what we want to do first of all. So, this leads to the concept of fundamental group of the space X.

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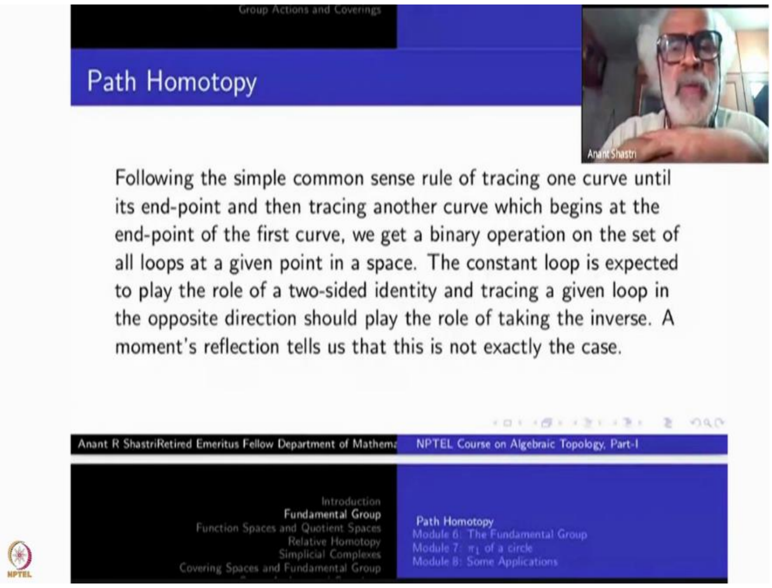

Well, let us introduce this concept of path homotopy. Following the simple common-sense rule of tracing one curve until its end-point and then tracing another curve which begins at that point, we get a binary operation on the set of all loops at a given point in space. To take a point and then look at a loop at that point. Means end point and the starting point are the same: x_1 is equal to x_0.

Take that special case. Then take a loop, take another loop. You can compose them by this method. It is called concatenation of the loops, which is just an extension of homotopy that we have already done. The constant loop, you know, you see any constant loop; it is a funny thing. Geometrically, you would like a loop as a continuous function from an interval into the space X with the endpoints the same.

But if the endpoint, not only endpoint, all the points are the same? that is also a loop by our definition. Why do we allow this one? This is a very nice thing to be allowed, this one, the constant loop. I would like to say that it will act as a two-sided identity for this operation. Because, after you trace a curve and come back and then you do not do anything. You stay there all the time; it is just like you have traced that curve-- that is all. So, that is the meaning of this constant loop being the identity element for this operation. If the right side is identity, the left side is also identity.

But there are problems, we are just now making a demand; making, anticipating something. How to do is one thing-- trying to do is another. We want to sharpen our definition of homotopy, how to make these things work and finally it should work. So, another thing is, if you trace a path in the opposite direction, it should be treated as the inverse of the path. You have gone along this path but finally you have come back the same way. So, it is as if you have done nothing. So, this kind of thing one has to do.

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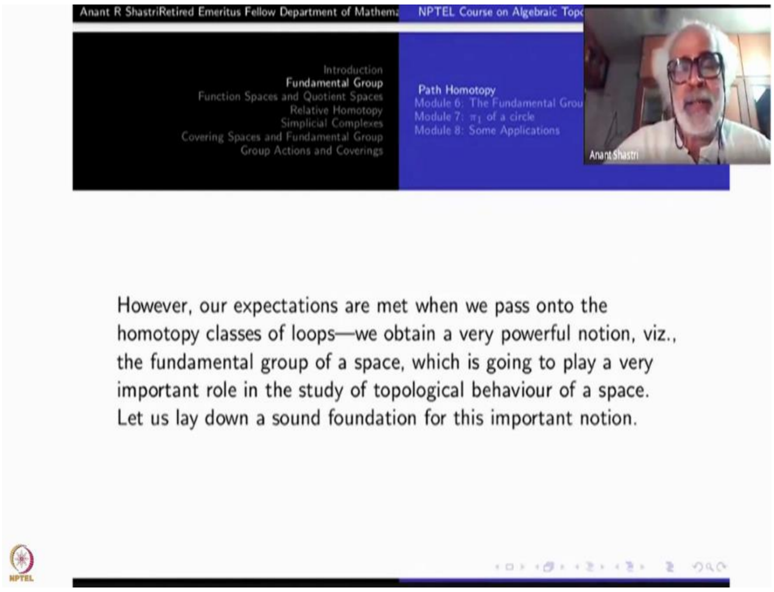

However, our expectations are met only when we pass on to the homotopy classes of loops. Otherwise as functions they are never the same. So, this is what we want to emphasize. We obtain a powerful notion namely fundamental group only when we go to homotopy classes of loops. So, this is going to play a very important role in the topological behaviour of a space.

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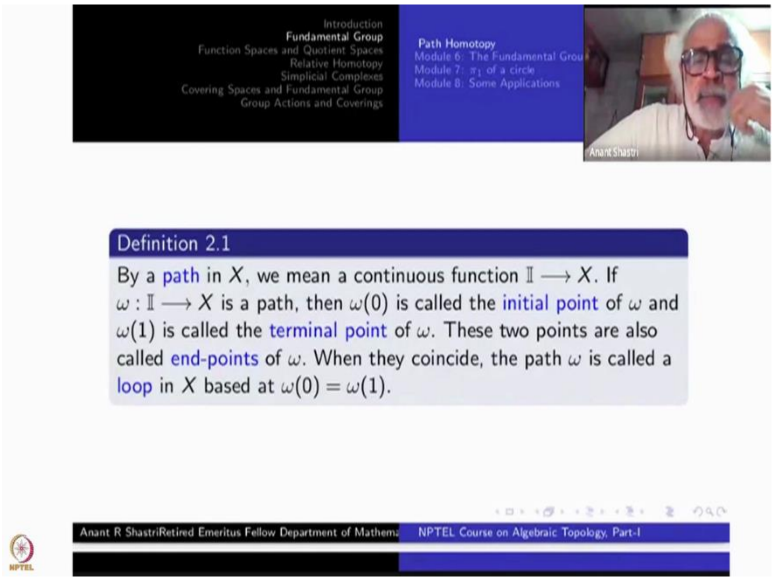

So, let us make a formal definition of path homotopy. Before that let me make a formal definition of a path also now, so that we have no confusion later. A path is just a continuous function from a closed interval from the closed interval 0,1 to X. All the time, we have fixed the domain to be the interval 0,1 closed interval 0 to 1.

If ω is a path, ω(0) is called initial point. ω(1) will be called the terminal point. Both of them together can be called end-points. When the end-points coincide, such a path is called a loop and what is the base point? The base point, namely, ω(0) which is same thing as ω(1). So, these are some basic terms. I have defined what is the path, initial point, terminal point, end point and a loop.

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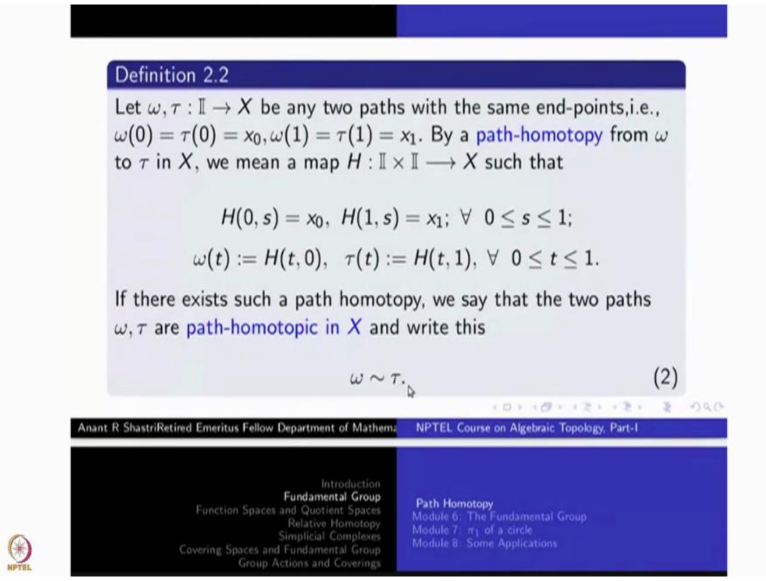

So, let us now make a path homotopy, definition of path homotopy. Take two paths, with the same end-points. ω(0) is equal to $\tau(0)$ equal to x_0. Let us call, ω(1) equal $\tau(1)$ equal to x_1. Then a path homotopy from ω to τ is first of all a homotopy- homotopy of these maps; remember if your map is from $X \to Y$, then homotopy was taken $X \times \mathbb{I} \to Y$.

Now, the maps are from $I \to X$, so homotopy will be from $\mathbb{I} \times \mathbb{I} \to X$. So, H is a continuous function from $\mathbb{I} \times \mathbb{I} \to X$ such that when you take H (0,s) for all points s-- the starting thing, is x_0. H(1,s) for all points s is x_1. So, these two points do not move at all. The second coordinate showing that it is moving. They do not move at all. For every point $0 < s < 1$.

H(t,0); it is the first part, that is ω(t). H(t,1): it is the last path; it is τ(t). So, if this happens then we call ω is path homotopic to τ. Alright? And we use a simple notation $\omega \sim \tau$. A general notation for homotopy was there is a twiddle and an arrow. Remember that. Here; this is a different notation. So, this is a different equivalence; this is a different symbol.

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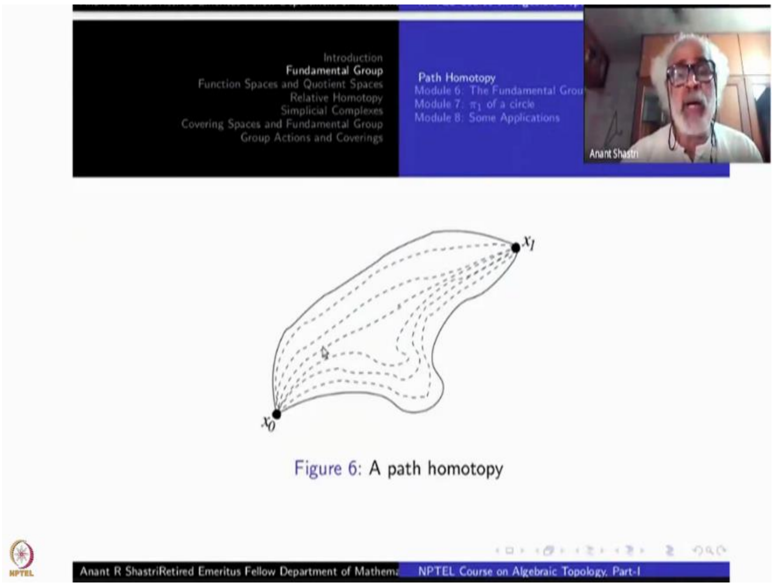

Here is a picture, starting with ω here ending in τ here. End-points are fixed here. So, this is you know, for t equal t_1, t_2, t_3, various stages shown by dotted lines. This is how homotopy is supposed to look like.

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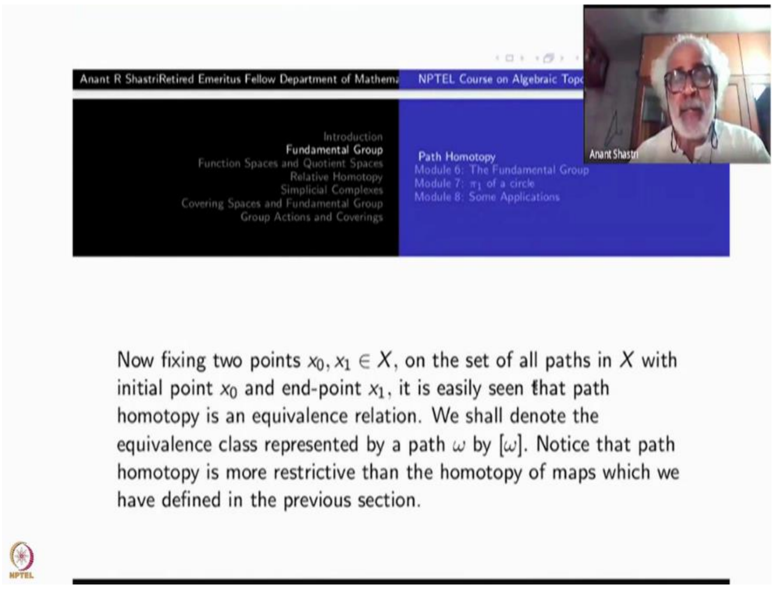

Fixing two-points x_0 and x_1, on the set of all paths in X with initial point x_0 and end point x_1, it is easily seen that path homotopy is an equivalence relation. The proof is exactly the same as the proof equivalence of homotopy of functions. We have now taken end-point being fixed at same thing.

So, every time it will fix the same thing. Transitivity, reflexivity and symmetry, all you can verify the same way. So, path homotopy is an equivalence relation amongst the class of paths which have same end points. That is what is important. Notice that path homotopy is more restrictive than the homotopy of maps which you have defined in the previous section.

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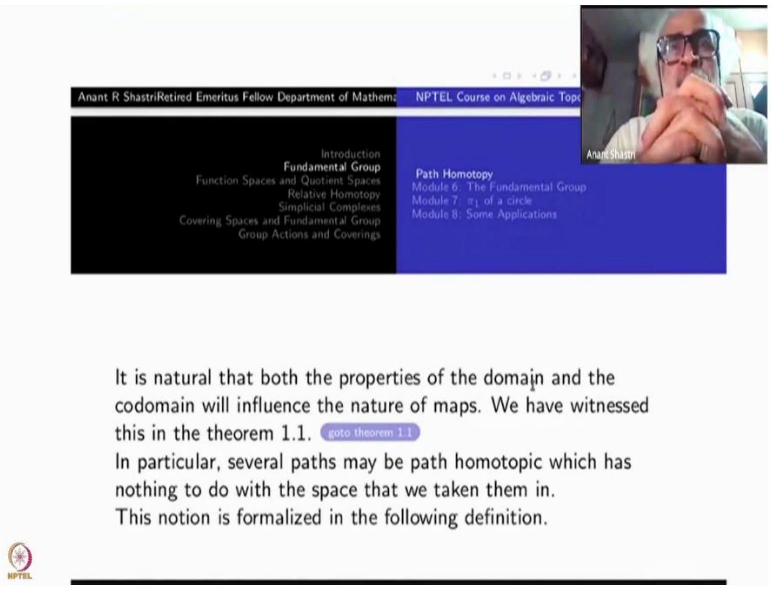

So, it is natural that both the properties of the domain and the codomain will influence the nature of maps between them. We have witnessed this in theorem 1.1 namely if the domain is contractible, then the function is null homotopic. Similarly, codomain is contractible, any function to that is null homotopic. Right?

So, even for paths and path homotopy, there must be some such thing happening. Irrespective of where I am taking $x \in X$, X is the space, I am taking the paths inside that. Let us first understand, what are these essential homotopies between paths and of course end-points must be the same. So, let us first take away this path. After that we can talk about what happens inside X. Right now, irrespective of what happens to x, where x is, this path homotopy must have certain properties. Let us understand that.

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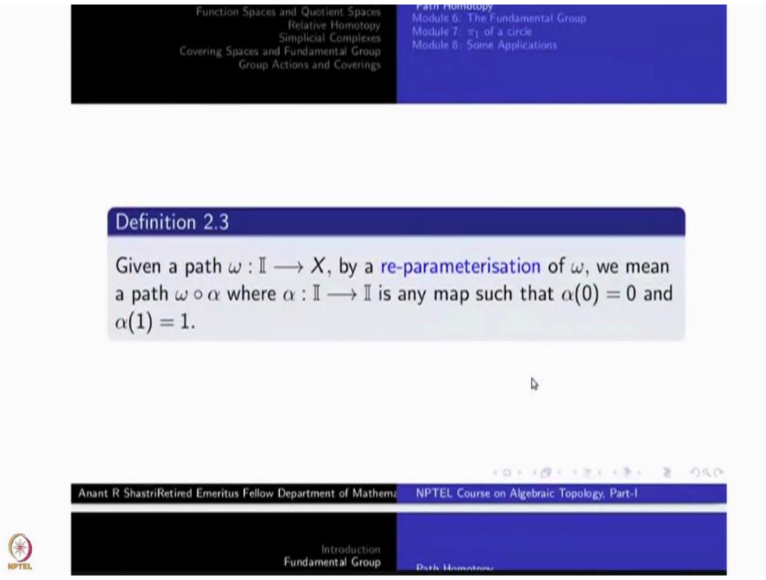

So, this leads us towards what is called re-parameterisation. Take a path ω: I → X. A re-parameterisation of ω we mean a path $\omega \circ \alpha$ where α itself is another map from I to I such that 0 goes to 0 and 1 goes to 1 under α. Any path and then you change it namely ω, instead of ω(t) you take ω (α(t)). That will be called re-parameterisation of the path ω.

One of the simplest things is the image of ω and image of $\omega \circ \alpha$ does not change. It is the same thing. So, from a layman's point of view both the paths are the same, but from a mathematician's point of view, they may not be the same. But in fact, they are not the same if α is not identity map. But the layman's point of view should be respected and what happens is these two paths will always be path homotopic to each other. So, weaker equivalence is there. Any re-parameterisation will not produce any new paths in that sense. They will all be path homotopic to the original omega. Let us see how?

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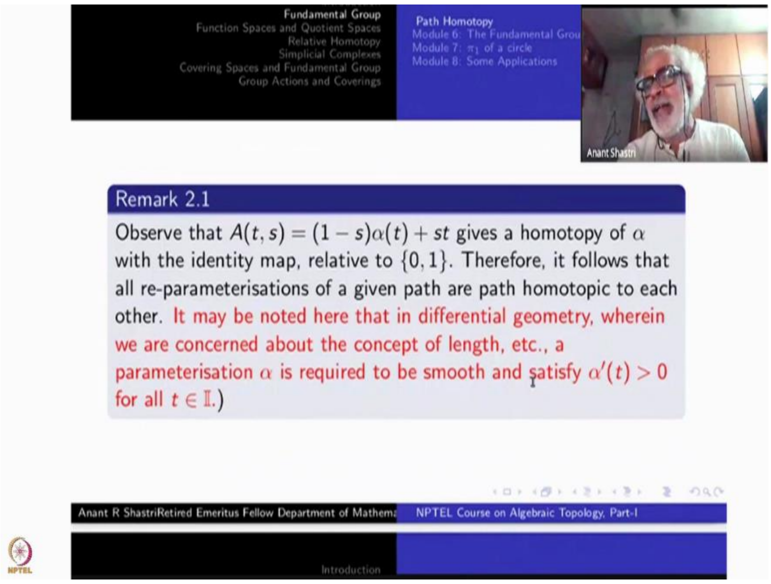

So, this is how. All that I have to do is, look at this homotopy A(t,s). A(t,s) equal to

(1- s)α(t) + s t. So, I am joining t and α(t). Where are they? They are inside the closed interval [0,1]. Therefore, line segment makes sense 1- t times this plus s times that one which is again inside the closed interval. Therefore, this gives you a homotopy. When you put s = 0, it is α(t). When you put s = 1, it is the identity map t going to t. So, α is homotopic to identity map.

Relative to the end-point 0 and 1. No matter what s is when you put t = 0, what do we get? α(0) is also 0; t is 0. So, A(0,s) is 0 for all s. Similarly, t =1, α(1) is 1 and t is

1. (1- s)1+ s is equal to 1. So, this homotopy is a homotopy of the identity map with alpha, keeping the end-points fixed.

Therefore, when you apply ω to it, what you get? You will get the homotopy of ω ○α with ω composite identity which is ω. No matter what ω is or no matter what α is, re-parameterisations of all paths are path homotopic to the original one. This is the concept.

Now, I want to warn you, you might have studied in differential geometry or even in complex analysis and so on, when there is a re-parameterisation, first of all those maps are not just maps, they are smooth maps or piecewise smooth maps. Similarly, the re-parameterisations must be smooth maps with an extra condition namely the derivative at every point must be positive. So, this is the standard condition in differential geometry. Also, in integration theory and so on in complex analysis.

But, in algebraic topology, we do not need those conditions. We are taking all continuous functions and we do not require smoothness, only end-points are the same is enough for us. If these conditions are all satisfied, there is no problem, of course. We do not need to bother about them because our spaces are arbitrary spaces. The derivative may not make sense there. So, I will stop here and we will resume from this point onwards in the next module. Thank you.