Introduction to Algebraic Topology
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Lecture – 39
Homotopical Aspects of Simplicial Complexes

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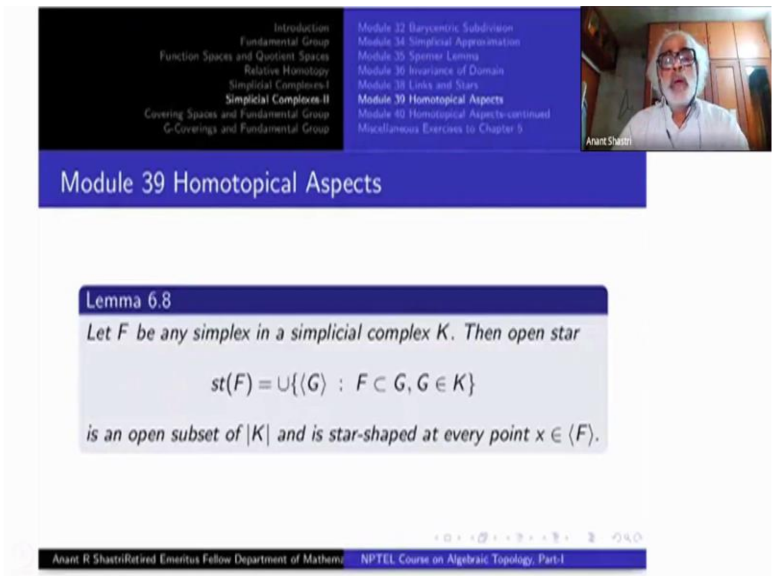

Homotopical aspects of simplicial complexes. By which we mean how to construct homotopies and what are certain basic properties not topological in nature but homotopical in nature. Like local contractibility, cofibration and so on. So, that is the topic for discussion for this module as well as the next module.

Last time we did the combinatorials of links and stars and so on. All of them were what are called as closed subspaces, closed links, closed stars and so on. But, today we have a slightly different version here. Start with any simplex F in a simplicial complex K. Then consider the open star of that , which I am denoting with the small stF.

The capital StF, remember was for the closed star. But, this is open star of F this is an open subset of $|K|$. It is not defined as a combinatorial object not a simplicial complex or something . It is an open subset of the simplicial complex StF, consisting of the union of all open simplex i$\langle G\rangle$, remember this notation is used for the open simplex of G, namely, all those alpha such that support of alpha is actually equal to G; So, take $\langle G\rangle$ for all those where G is a simplex containing F. In particular, G can be equal to F also here. But this containment sign I am using for subset equal to also.

So, claim is that this is an open set. Just because you called it open star it does not mean it is open. So claim is that it is an open subset of $|K|$. Also it is a star shaped at every point of the open simplex $\langle F \rangle$. Notice that, when you take G equal to F, this open simplex F comes here so open simplex $\langle F \rangle$ is a subset of $st\ F$. Now, you take any point there, the entire set will be star shaped at that point. That is the claim. These are all easy not difficult. But, you have to just verify and be sure of it that is all. Let us do them one by one.

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To show that any subset is open in $|K|$, what we have to show is that its intersection with each closed simplex $|H|$ is open in $|H|$, but this should happen for every simplex $H \in K$. Of course, if H is a singleton, then this is automatic. Beyond that you have to verify this. Clearly, if $F \not\subset H$ then $st\ F \cap |H| = \emptyset$. For, $\alpha \in st\ F$ implies $F \subset supp\ \alpha$.

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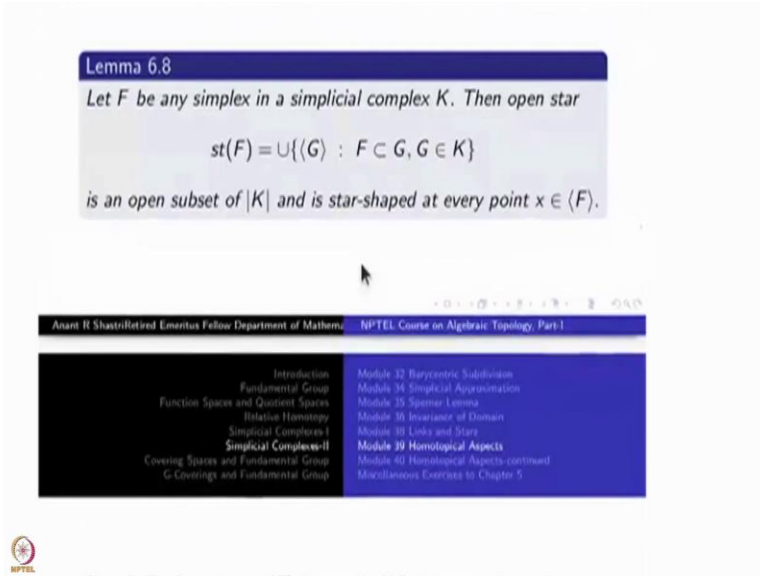

Remember the whole space $|K|$ is union now such open simplexes. This was the starting point of our discussion of topological properties. That we have seen. So, in stF, we are taking only those $\langle G\rangle$ where G contain F.

Therefore, given G, either this entire open simplex $\langle G\rangle$ is contained stF or we have stF will not intersect $\langle G\rangle$ at all.

So, let us take the case wherein $F \subset H$. Then I want to show that this intersection $st\ F \cap |H|$ is open in $|H|$. Actually, what happens now is: $|H| = |F * F'| = |F| * |F'| = (\langle F\rangle \coprod |\mathcal{B}(F)|) * |F'| = (\langle F\rangle * |F'|) \coprod (|\mathcal{B}(F)| * |F'|).$ Now we notice that $st\ F \cap |H| = \langle F\rangle * |F'|$ and $|\mathcal{B}(F)| * |F'| = |\mathcal{B}(F) * F'|$ is a closed subset. Therefore its complement $st\ F \cap |H|$ is open in $|H|$.

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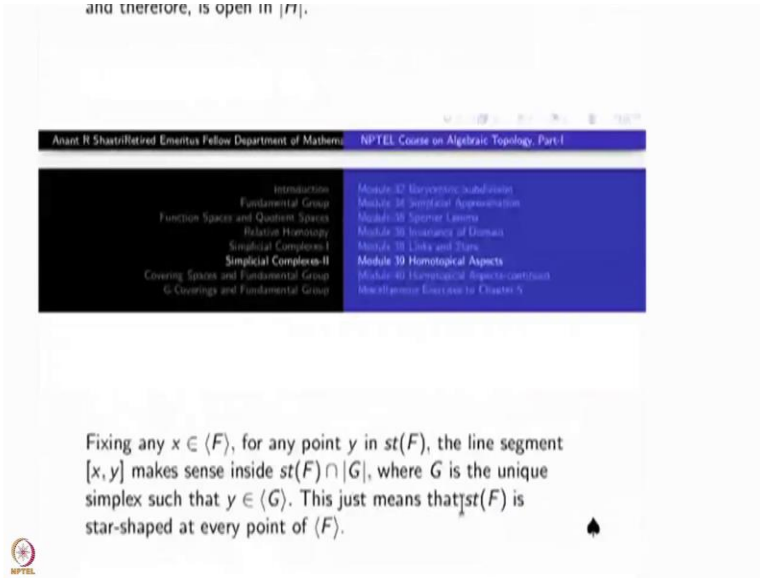

Now, take any x belonging to open simplex F. I want to show that stF is star shaped at this point. What is the meaning of that? Take any point y in stF, then the line joining x and y should be completely contained inside the stF. The line joining x and y makes sense inside both star F as well as mod G. Where G is the unique simplex for which y belongs to open simplex G.

Remember, every y will be inside some unique open simplex $\langle G \rangle$, for some G which contains F. Then x is also inside $|G|$. Also y is inside $|G|$. Therefore this line segments makes sense inside $|G|$. In fact $(x, y]$ is actually contained in $\langle G \rangle$ which is contained in $st\ F$. Since x already in $st\ F$, $[x, y] \subset st\ F$. This just means that star F is star shaped that is all. Every line segment for each point inside a star F every line segment x y must be inside.

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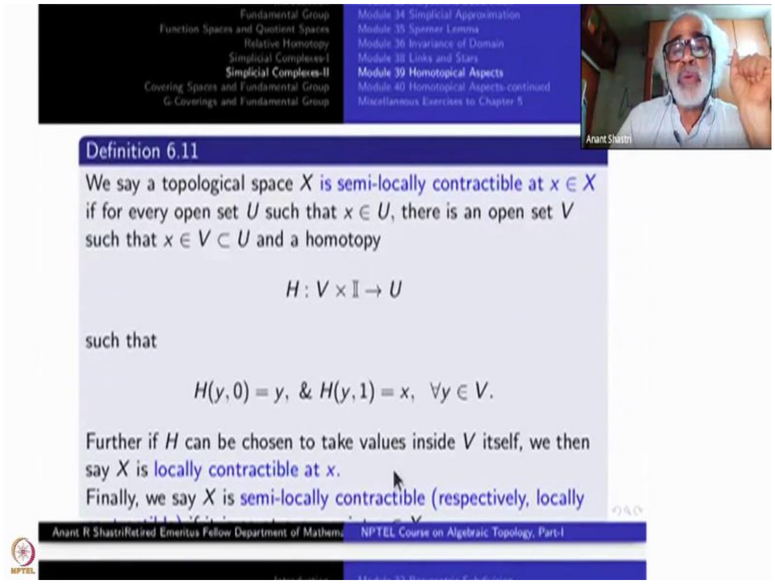

Now, we will make a definition which we might have made earlier. But now it is more relevant anyway. We say a topological space X is semi-locally contractible at a point x belonging to X if for every open set U such that x is inside U there is an open set V such that x belongs to V contained inside U and a homotopy $H : V \times \mathbb{I} \to U$ such that $H(y, 0) = y$ which means a homotopy is a deformation, starting point is identity then the end function is a constant $H(y, 1) = x$. What constant? Precisely, the point we are interested in. The only thing here is unlike contractibility of V, the homotopy is taking place U which is larger open set. If you replace U by V this is nothing but saying that V is contractible to the point x.

It is stronger than saying that V is just contractible that contractible to point x is important for this. But the whole thing is taking place inside U. The homotopy can go out of V but it has this property. That is why we call it semi-locally contractible. However, if you replace U by V itself in this definition, namely, H is taking values inside V then we actually say that this capital X is locally contractible at x.

The same condition the only thing that you have to do is replace this U by V and then you delete `semi' So, that is the meaning of locally contractible. If H can be chosen to take values inside V itself. So, locally contractibility is stronger than semi locally contractibility.

Now suppose, this happens at every point of x just like a function is continuous if it is continuous at every point, if this is happens at every point then we say X itself is semi locally contractible or in the later case X itself is locally contractible. So, that is the meaning of these two definitions.

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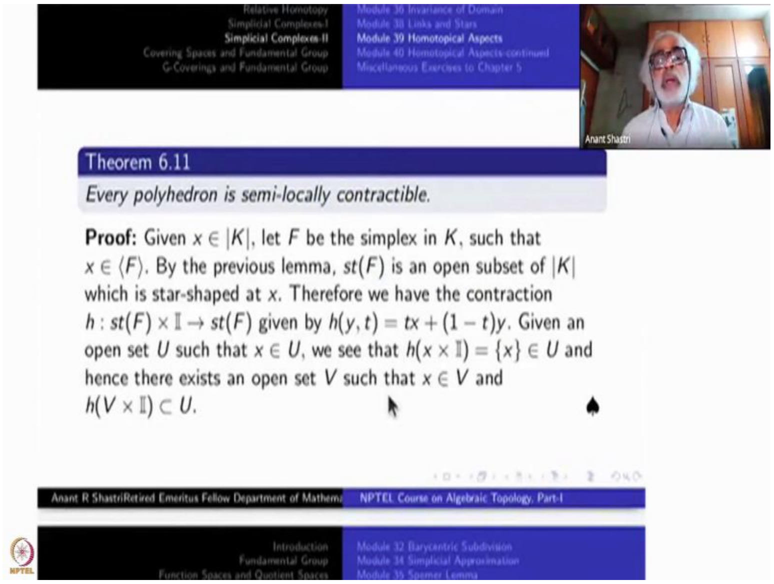

So now, the next theorem is that every polyhedron is semi locally contractible. Take a simplicial complex K, take $|K|$. I have to show that at every point $x \in |K|$, given any neighborhood U of x there is a smaller neighborhood V of that point such that V contracts to the point x inside U. This is what I have to show.

So, let us go through the proof as you see which is not difficult at all. Starting with $x \in |K|$, choose F (unique) such that $x \in \langle F \rangle$. Remember every point is inside a unique open simplex because mod K is partitioned into open simplexes.

So, that is used again and again. The previous lemma says that stF is an open subset of $|K|$ which is star shaped at x. So, this itself is a contractible neighborhood of the point x. But we want to do it inside the given open set so that is the catch here. So, how we go about it. This is how. First of all we have a contraction $st\ F \times \mathbb{I} \to st\ F$ which is just given by the formula $h(y,t) = tx + (1-t)y$.

This x is the apex of the star shaped thing so any y can join to x . This is the line segment. This whole thing is inside star F. So, this is a continuous function put t is equal to 0 this will be y. Put t equal to 1 this will be x t equal to 0 it is identity, t equal to 1 is a constant function taking the value x.

Now, start with an open set U such that x is inside U. Then look at this $h(\{x\} \times \mathbb{I}) = \{x\} \in U$. h of x cross I x is the given point h of x cross I if y is equal to x what is this map what is this one. It is just x it is singleton x. That is inside U, U is an open set. So, this whole closed interval is mapped inside an open set. The closed interval is a compact set.

Therefore, by standard methods in point set topology like the tube lemma or otherwise, there exists an open set V inside $st\ F$ such that $x \in V$ and $h(V \times \mathbb{I}) \subset U.$ One single open set V cross I, instead of $V_t \times (t - \epsilon, t + \epsilon)$ etc.

Now, you look at H which is same h restricted to $V \times \mathbb{I}$. What we get is this condition. So, given any U we have found a V such that and a homotopy H as required. Here H is little h restricted. special H here. So, this proves that every polyhedron is semi locally contractible. Very easy statement, very easy proof.

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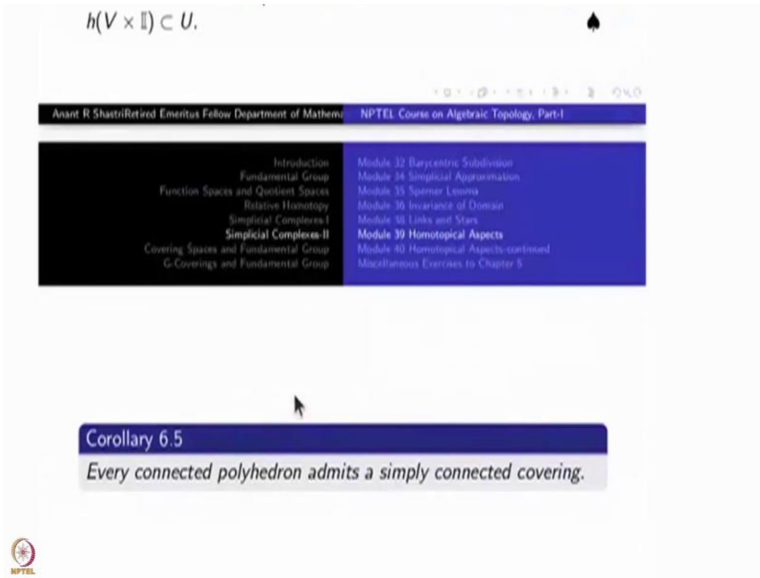

Student is questioning: Hello sir, the U that you chose the open set U around x cannot be sub divide to make the star of that point inside U? Then we can contract it?

Professor: What do you, how do you sub divide it?

Student: Yeah. I mean if we sub divide then it will be small, can it be small enough to be inside at any U?

Professor: How do you do that?

Student: We know that there is always one sub, I mean can we, we can we can take any point…

Professor: K is not a finite simplicial complex. Even at point x you have taken is not at all first of all a vertex. Even if that is the case there may be infinitely many simplices coming and meeting at x. So, it is not even locally finite. Even after that you have to make x to be a vertex etc.

Student: But cannot we have a simplicial structure that makes x a vertex?

Professor: That is the next thing which we want to do. Even then the problem is that by the barycentric subdivision that we have taken that will not do the job. That will do only if it is a finite simplicial complex. Once you have the semi local contractibility, later you will see that such spaces admit a simply connected covering later. So right now, we assume that then what we have proved is every connected polyhedron admits a simply connected covering. This is additional application of this one. Local contractibility is harder to prove.

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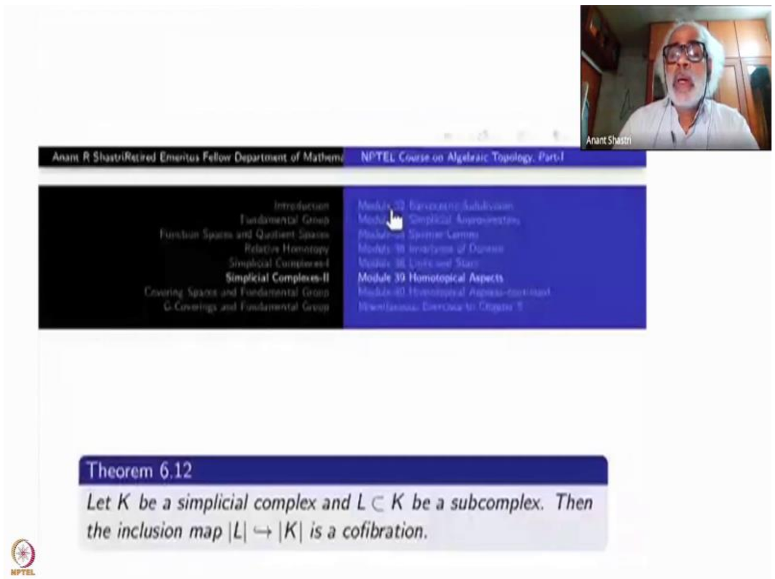

Now, let us come to the second point here, namely, a cofibration property. This theorem says that for every sub complex L, the inclusion map mod L to mod K is a cofibration. Remember that we had a proposition which gives you a method to verify cofibrations. Namely, you must produce a retraction from $|K| \times \mathbb{I} \to (|K| \times \{0\}) \cup (|L| \times \mathbb{I})$. And that is precisely what we are going to prove.

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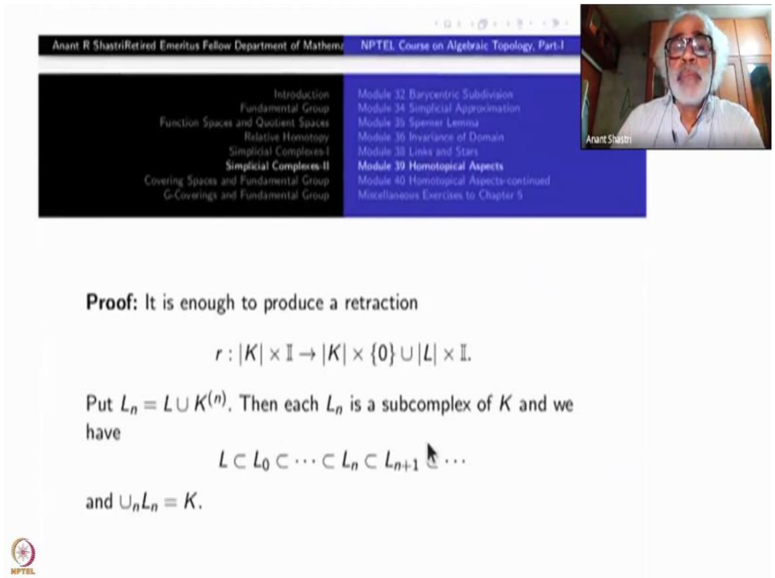

Retraction should be from mod K cross I to mod K cross 0 union mod L cross I. So, this one we are going to get such a retraction in an inductive fashion. Put $L_n = L \cup K^{(n)}$ where, Ln equal to $K^{(n)}$ denotes the nth skeleton of K. Then by the very definition, $L, K^{(n)}$ are sub complexes of K and hence so is L_n. We have the inclusion maps here. Start with $L \subset L_0$. What is L naught? It consists of all of L and all the vertices of K. Those are extra vertices coming from $K^{(0)}$ are there. Also, $L_0 \subset L_1 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots$ Finally, if you take the union of all of them that will contain all the skeletons therefore it will be the whole of K.

So, I have written K as a countable union starting with L_0. If K is a finite simplicial complex there are many other methods to do this. There is no need to go through this. This is precisely to take care of infinite simplicial complexes.

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We have seen this one several times namely, $\mathbb{D}^n \times \mathbb{I}$ deformation retract onto $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times \mathbb{I}$. Just topologically, there is no need for going to simplicial complexes. I can write this in terms of simplicial complexes viz., $|\Delta_n| \times \mathbb{I}$ deformation retracts onto $|\Delta_n| \times \{0\} \cup |\mathcal{B}(\Delta_n)| \times \mathbb{I}$.

So, there are these standard retractions for each n. I am going to use this one now. For each n greater than equal to 0 we have such a thing. So now, label the set of all (n+1) simplexes of K which are not in L. Do not worry about things which are in L. Because they are going to be here this part. So, let us index them, let us denote them $\{F_s\}_s$. For each s, there is a retraction $\rho_s : |F_s| \times \mathbb{I} \to |F_s| \times \{0\} \cup |\mathcal{B}(F_s)| \times \mathbb{I}$. It is the same rho a copy of rho but I am denoting rho s for each s. Then what do I do?

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We start constructing a sequence of slower retractions $\{r_n\}$. I am going to define these retractions

$r_0, r_1, \ldots, r_{n+1}, \cdots r_{n+1} : |L_{n+1}| \times \mathbb{I} \to |L_{n+1}| \times \{0\} \cup |L_n| \times \mathbb{I}$.

But there is slight different definition for $r_0 : |L_0| \times \mathbb{I} \to |L_0| \times \{0\} \cup |L| \times I$. Here, either x is in $|L|$ or t is 0 then take $r_0(x,t) = (x,t)$. It is identity. Then there are points in a disjoint union of vertices cross I, which corresponds to all the vertices which are not in L. Each of those vertices x, there is the line segment x cross I, you push all of it to just the base point x cross 0. Instead of x cross t all of them pushed to x cross 0. So, this is obviously a retraction of L naught cross I onto L naught cross 0 union L cross I.

That is the starting point. So, in this inductive process, for n=0, we have got a retraction corresponding to this L naught so corresponding to L. Is not a retraction to L but its L naught cross I to L naught cross 0 union L cross I. So, that is the kind of thing we are going to do here all the time. So, once you have this one you can assume that you have rewind this one for up to N and you can go for N plus 1 stage. So, what I am going to do is for.

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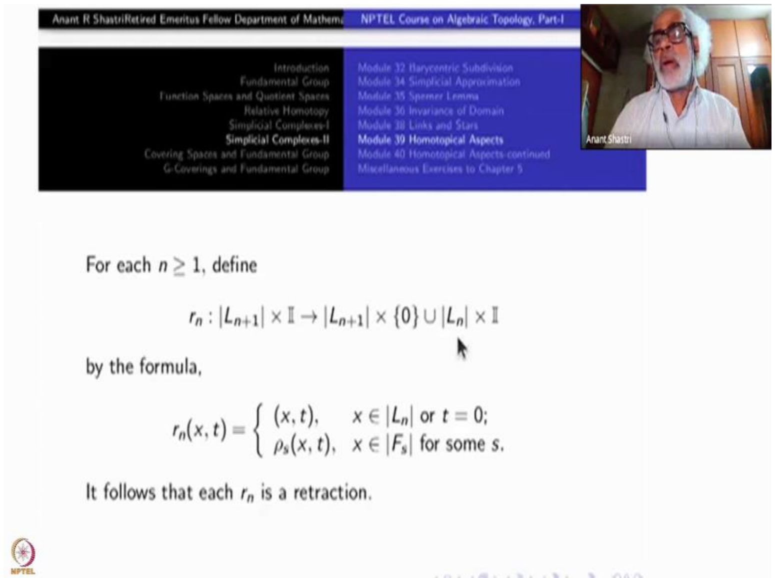

Now, for each $n \geq 0$, I am defining $r_{n+1} : |L_{n+1}| \times \mathbb{I} \to |L_{n+1}| \times \{0\} \cup |L_n| \times \mathbb{I}$ as follows: On all of $|L_{n+1}| \times \{0\} \cup |L_n| \times \mathbb{I}$, it is identity, $r_{n+1}(x,t) = (x,t)$. The second part, when they are inside $|F_s| \times \mathbb{I}$, there I take this rho s. Because, they are all indexed by s; $r_{n+1}(x,t) = \rho_s(x,t)$.

ρ_s will push $|F_s| \times \mathbb{I}$ into $|F_s| \times \{0\} \cup |\mathcal{B}(F_s)| \times \mathbb{I}$. So, if $t = 0$ or $x \in |F_s|$ for some s, $\rho_s(x,t) = (x,t)$, the identity map. So the two definitions agree. So, there is no conflict here. So, each r_{n+1} is a retraction. (In the slide there is a typo.) What is the next step then?

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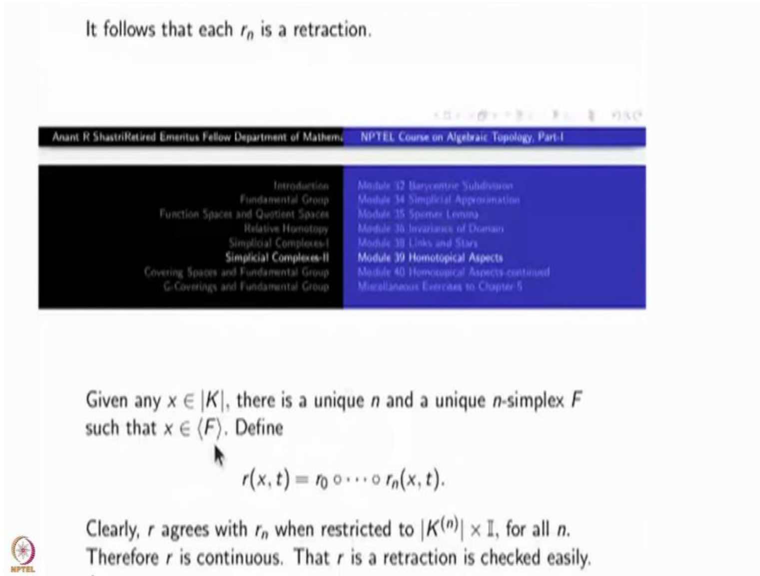

Now, take any point $x \in |K|$. This will belong to a open unique n-simplex, it is inside $\langle F \rangle$, where F is an n-simplex. This F maybe singleton also. open simplex for a vertex is a same set as a vertex. In all other cases it is different. For an edge, it will be a open interval and so on. So, x belong to open F. That means it is inside $|K^{(n)}|$ in any case. In any case $x \in |L^{(n)}|$.

So, you define r of x t as follows: start with r_n then apply r_{n-1} … and so on, come up to all the way r_0 So, that you land up inside K cross 0 union L cross I. So, that will be the final picture here. Actually, r agrees with you see once the point is inside the Kn then it is rn and then all these are r naught.

The next stage rn plus 1 will agree with this one if it is already inside rn inside Kn. Kn plus 1 from Kn plus 1 the rn plus 1 will (())(28:15). Because rn plus 1 of that part is already identity then only rn will up to r naught rn will change it. Therefore, this compatible family of this so that is why r of x t make sense.

Restricted to each Kn it is only this part. If Kn plus 1 if I have taken one more composition will come rn plus 1 we would have come that is all. But they are all well defined and when you want to check the continuity you have to check the continuity on this part. And then this is the composition of finitely many continuous function so it is continuous. The retraction part follows very easily because for each point this inside this one.

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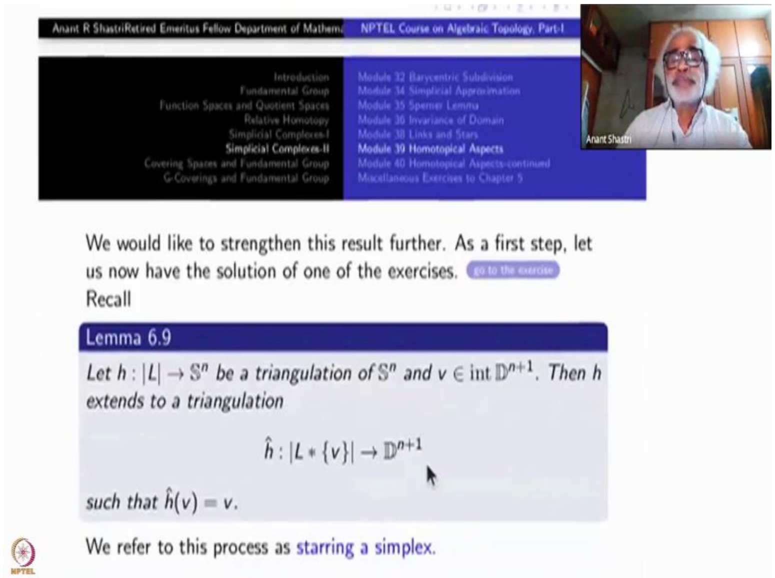

So, every subcomplex the inclusion maps are cofibration. What we want to do next? Given an arbitrary open subset V containing $|L|$, can we get a smaller open subset U such that the inclusion $|L| \subset U$ itself is a cofibration. So, these are the harder results. So, that is what we want to do next.

We would like to strengthen all this results. The first step let us have one of the exercises that we had announced earlier namely, that every point can be thought of as vertex inside a simplicial complex. The first step is again, this we have discussed several times, I will discuss it again.

Let (L, h) be a triangulating $\mathbb{S}^n$, i.e., L is a simplicial complex and $h : |L| \to \mathbb{S}^n$ is a homeomorphism. Take any point v in the interior of $\mathbb{D}^{n+1}$. $\mathbb{D}^{n+1}$ is bounded by $\mathbb{S}^n$. Then this triangulation extends to a triangulation $\hat{h} : |L * \{v\}| \to \mathbb{D}^{n+1}$, such that $\hat{h}(v) = x$.

L star v is a cone. It is the join of these two simplicial complexes L and the singleton $\{v\}$. That is just another name; cone over L with apex at v. So, take the modulus of that this h hat defines a homeomorphism of that. This follows because given any interior point $v \neq x$ in $\mathbb{D}^{n+1}$ there is unique point on the boundary, $\mathbb{S}^n$, call that u, such that $v \in [u, x]$.

Only x will be in all the line segments. That is why it becomes a cone. I am just recalling that this we have done earlier. So, this process let us call it as starring a simplex. This L is the boundary of the simplex which has been subdivided. L could be any subdivision. Not naturally delta n plus 1 boundary of delta n plus 1.

That triangulation extends the triangulation of this, starring a simplex. With whatever boundary triangulation is there that gets extended. So, this process we will call as starring a simplex. Just temporarily if do not worry about other people may call it differently and so on.

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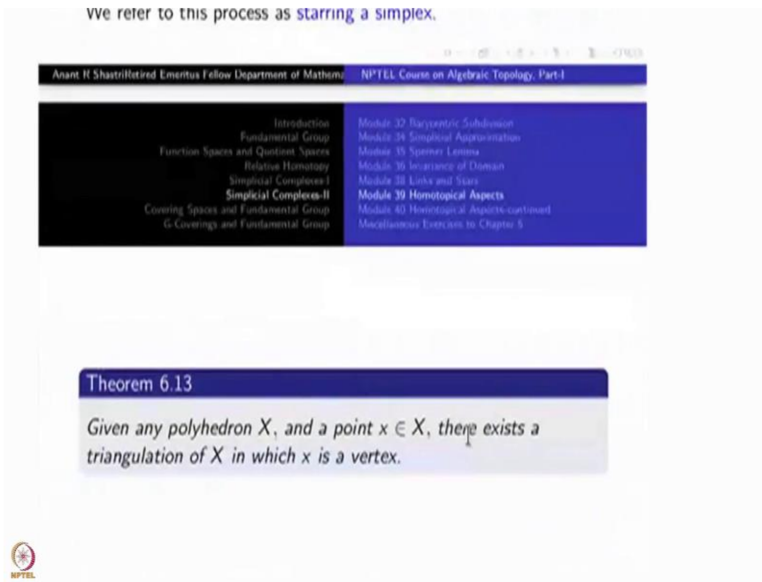

So, now we can solve this problem. Given any polyhedron $|K|$ and a point $x \in |K|$ there exists a triangulation in which x is a vertex.

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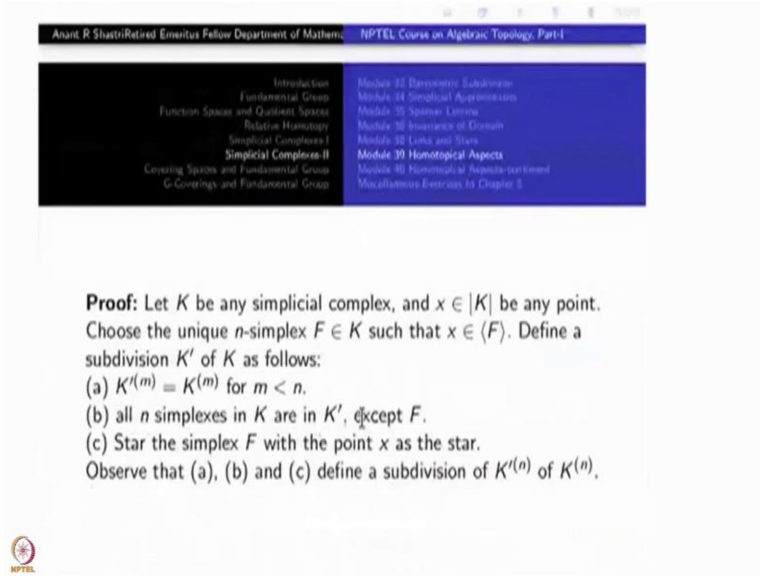

Start with a simplicial complex K. That means what? I am talking about now $X = |K|$. Let $x \in X$ be any point. Again, choose the unique n-simplex F belonging to K such that x is inside open simplex F. (If F is a singleton there is nothing more to prove.) Define a subdivision K' of K as follows.

We are going to define a subdivision and that subdivision will have this x as a vertex that is all. So, what do we do? This vertex is inside $\langle F \rangle$. So there may not be any need to disturb all other simplexes, there is no need to subdivide them. Somewhere something around F only, we have to carry out subdivision.

So, what I do first of all for all is for simplexes of dimension less than n, choose K' to be just K. Whatever it is, it means $K'^{(n-1)} = K^{(n-1)}$. When it is n you have to be bit careful. What is that? Do not divide anything other than this F. Keep them as it is. All the n-simplexes in K are in K' also except F. The simplexes are as they are that means we are not dividing that one at all. So, now you have to concentrate one simplex that is F and in the interior of that. The boundary of F that is not divided. It is whatever it is, isomorphic to $\mathcal{B}(\Delta_n)$. Now, use the previous lemma and star this simplex at the point x.

Instead of barycentre you are choosing x that is all. For example, this x becomes a vertex and all the lines joining to each vertex of the boundary will be the 1- simplexes and so on. So, that is the starring. So, this will complete the definition of subdivision $K'^{(n)}$ of the nth-skeleton $K^{(n)}$.

After this there may be many n+1-simplexes which contain this F which has been divided. Therefore, you may have to subdivide them also. So, do not worry, divide all of them by starring at the barycenters. Beyond this n dimension we need subdivide more and more simplexes. But this can be put in a general set up.

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Inductively, as soon m is bigger than n extend the subdivision $K^{(m)\prime}$ of $K^{(m)}$ to a subdivision $K^{(m+1)\prime}$ of $K^{(m+1)}$ by starring each (m+1)- simplex at its barycenter. Keep starring them put them all together get $K' = \cup_m K^{(m)}$. So, that will complete the subdivision.

Achieving that the given x as vertex it is done at the stage (c). This does not complete the subdivision. So for that you have to do the stage (d) also. This we will use next time but today we will stop here. Thank you.