

**Introduction to Algebraic Topology**  
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**Department of Mathematics**  
**Indian Institute of Technology, Bombay**  
**Lecture – 39**  
**Homotopical Aspects of Simplicial Complexes**

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**Module 39 Homotopical Aspects**

**Lemma 6.8**  
 Let  $F$  be any simplex in a simplicial complex  $K$ . Then open star  

$$st(F) = \cup\{\langle G \rangle : F \subset G, G \in K\}$$
 is an open subset of  $|K|$  and is star-shaped at every point  $x \in \langle F \rangle$ .

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Homotopical aspects of simplicial complexes. By which we mean how to construct homotopies and what are certain basic properties not topological in nature but homotopical in nature. Like local contractibility, cofibration and so on. So, that is the topic for discussion for this module as well as the next module.

Last time we did the combinatorials of links and stars and so on. All of them were what are called as closed subspaces, closed links, closed stars and so on. But, today we have a slightly different version here. Start with any simplex  $F$  in a simplicial complex  $K$ . Then consider the open star of that, which I am denoting with the small  $stF$ .

The capital  $StF$ , remember was for the closed star. But, this is open star of  $F$  this is an open subset of  $|K|$ . It is not defined as a combinatorial object not a simplicial complex or something. It is an open subset of the simplicial complex  $StF$ , consisting of the union of all open simplex  $\langle G \rangle$ , remember this notation is used for the open simplex of  $G$ , namely, all those  $\alpha$  such that support of  $\alpha$  is actually equal to  $G$ ; So, take  $\langle G \rangle$  for all those where  $G$  is a simplex containing  $F$ . In

particular,  $G$  can be equal to  $F$  also here. But this containment sign I am using for subset equal to also.

So, claim is that this is an open set. Just because you called it open star it does not mean it is open. So claim is that it is an open subset of  $|K|$ . Also it is a star shaped at every point of the open simplex  $\langle F \rangle$ . Notice that, when you take  $G$  equal to  $F$ , this open simplex  $F$  comes here so open simplex  $\langle F \rangle$  is a subset of  $st F$ . Now, you take any point there, the entire set will be star shaped at that point. That is the claim. These are all easy not difficult. But, you have to just verify and be sure of it that is all. Let us do them one by one.

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<ul style="list-style-type: none"> <li>Introduction</li> <li>Fundamental Group</li> <li>Function Spaces and Quotient Spaces</li> <li>Relative Homotopy</li> <li>Simplicial Complexes-I</li> <li><b>Simplicial Complexes-II</b></li> <li>Covering Spaces and Fundamental Group</li> <li>G-Coverings and Fundamental Group</li> </ul>	<ul style="list-style-type: none"> <li>Module 32 Barycentric Subdivision</li> <li>Module 34 Simplicial Approximation</li> <li>Module 35 Sperner Lemma</li> <li>Module 36 Invariance of Domain</li> <li>Module 38 Links and Stars</li> <li><b>Module 39 Homotopical Aspects</b></li> <li>Module 40 Homotopical Aspects-continued</li> <li>Miscellaneous Exercises to Chapter 5</li> </ul>
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**Proof:** To show that  $st(F)$  is open in  $|K|$ , it is enough to show that  $st(F) \cap |H|$  is open in  $|H|$  for every simplex  $H$  of  $K$ . Clearly, if  $F \not\subset H$ , then  $st(F) \cap H = \emptyset$ . So let  $F \subset H$ . Write  $H = F \sqcup F'$ . Then it follows easily that

$$st(F) \cap |H| = |H| \setminus |B(F) * F'|$$

and therefore, is open in  $|H|$ .

To show that any subset is open in  $|K|$ , what we have to show is that its intersection with each closed simplex  $|H|$  is open in  $|H|$ , but this should happen for every simplex  $H \in K$ . Of course, if  $H$  is a singleton, then this is automatic. Beyond that you have to verify this. Clearly, if  $F \not\subset H$  then  $st F \cap |H| = \emptyset$ . For,  $\alpha \in st F$  implies  $F \subset \text{supp } \alpha$ .

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**Lemma 6.8**

Let  $F$  be any simplex in a simplicial complex  $K$ . Then open star

$$st(F) = \cup\{\langle G \rangle : F \subset G, G \in K\}$$

is an open subset of  $|K|$  and is star-shaped at every point  $x \in \langle F \rangle$ .

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Introduction	Module 32 Barycentric Subdivision
Fundamental Group	Module 34 Simplicial Approximation
Function Spaces and Quotient Spaces	Module 35 Spanner Lemma
Relative Homotopy	Module 36 Invariance of Domain
Simplicial Complexes-I	Module 38 Links and Stars
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Covering Spaces and Fundamental Group	Module 40 Homotopical Aspects-continued
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Remember the whole space  $|K|$  is union now such open simplexes. This was the starting point of our discussion of topological properties. That we have seen. So, in  $stF$ , we are taking only those  $\langle G \rangle$  where  $G$  contain  $F$ .

Therefore, given  $G$ , either this entire open simplex  $\langle G \rangle$  is contained  $stF$  or we have  $stF$  will not intersect  $\langle G \rangle$  at all.

So, let us take the case wherein  $F \subset H$ . Then I want to show that this intersection  $st F \cap |H|$  is open in  $|H|$ . Actually, what happens now is:  $|H| = |F * F'| = |F| * |F'| = (\langle F \rangle \coprod |\mathcal{B}(F)|) * |F'| = (\langle F \rangle * |F'|) \coprod (|\mathcal{B}(F)| * |F'|)$ . Now we notice that  $st F \cap |H| = \langle F \rangle * |F'|$  and  $|\mathcal{B}(F)| * |F'| = |\mathcal{B}(F) * F'|$  is a closed subset. Therefore its complement  $st F \cap |H|$  is open in  $|H|$ .

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and therefore, is open in  $|F|$ .

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<ul style="list-style-type: none"> <li>Introduction</li> <li>Fundamental Group</li> <li>Function Spaces and Quotient Spaces</li> <li>Relative Homology</li> <li>Simplicial Complexes I</li> <li><b>Simplicial Complexes II</b></li> <li>Covering Spaces and Fundamental Group</li> <li>Coverings and Fundamental Group</li> </ul>	<ul style="list-style-type: none"> <li>Module 37: Barycentric Subdivision</li> <li>Module 38: Simplicial Approximation</li> <li>Module 39: Sperner Lemma</li> <li>Module 40: Invariance of Domain</li> <li>Module 41: Links and Stars</li> <li><b>Module 42: Homotopical Aspects</b></li> <li>Module 43: Homotopical Aspects (continued)</li> <li>Intermediate Exercises to Chapter 5</li> </ul>
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Fixing any  $x \in \langle F \rangle$ , for any point  $y \in \text{st}(F)$ , the line segment  $[x, y]$  makes sense inside  $\text{st}(F) \cap |G|$ , where  $G$  is the unique simplex such that  $y \in \langle G \rangle$ . This just means that  $\text{st}(F)$  is star-shaped at every point of  $\langle F \rangle$ .

Now, take any  $x$  belonging to open simplex  $F$ . I want to show that  $\text{st}F$  is star shaped at this point. What is the meaning of that? Take any point  $y$  in  $\text{st}F$ , then the line joining  $x$  and  $y$  should be completely contained inside the  $\text{st}F$ . The line joining  $x$  and  $y$  makes sense inside both star  $F$  as well as mod  $G$ . Where  $G$  is the unique simplex for which  $y$  belongs to open simplex  $G$ .

Remember, every  $y$  will be inside some unique open simplex  $\langle G \rangle$ , for some  $G$  which contains  $F$ . Then  $x$  is also inside  $|G|$ . Also  $y$  is inside  $|G|$ . Therefore this line segments makes sense inside  $|G|$ . In fact  $[x, y]$  is actually contained in  $\langle G \rangle$  which is contained in  $\text{st} F$ . Since  $x$  already in  $\text{st} F$ ,  $[x, y] \subset \text{st} F$ . This just means that star  $F$  is star shaped that is all. Every line segment for each point inside a star  $F$  every line segment  $x y$  must be inside.

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The screenshot shows a video lecture slide. At the top, there is a navigation menu with the following items: Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, Simplicial Complexes-I, **Simplicial Complexes-II**, Covering Spaces and Fundamental Group, G-Coverings and Fundamental Group, Module 34 Simplicial Approximation, Module 35 Spanner Lemma, Module 36 Invariance of Domain, Module 38 Links and Stars, **Module 39 Homotopical Aspects**, Module 40 Homotopical Aspects-continued, and Miscellaneous Exercises to Chapter 5. A small video window in the top right corner shows a man with glasses and a beard, identified as Anant Shastri. The main content of the slide is:

**Definition 6.11**  
 We say a topological space  $X$  is semi-locally contractible at  $x \in X$  if for every open set  $U$  such that  $x \in U$ , there is an open set  $V$  such that  $x \in V \subset U$  and a homotopy

$$H : V \times \mathbb{I} \rightarrow U$$

such that

$$H(y, 0) = y, \text{ \& } H(y, 1) = x, \forall y \in V.$$

Further if  $H$  can be chosen to take values inside  $V$  itself, we then say  $X$  is **locally contractible** at  $x$ .  
 Finally, we say  $X$  is **semi-locally contractible** (respectively, **locally contractible**) if it is semi-locally contractible (respectively, locally contractible) at every point  $x \in X$ .

At the bottom of the slide, it says: Anant R Shastri Retired Emeritus Fellow Department of Mathematics, NPTEL Course on Algebraic Topology, Part-I.

Now, we will make a definition which we might have made earlier. But now it is more relevant anyway. We say a topological space  $X$  is semi-locally contractible at a point  $x$  belonging to  $X$  if for every open set  $U$  such that  $x$  is inside  $U$  there is an open set  $V$  such that  $x$  belongs to  $V$  contained inside  $U$  and a homotopy  $H : V \times \mathbb{I} \rightarrow U$  such that  $H(y, 0) = y$  which means a homotopy is a deformation, starting point is identity then the end function is a constant  $H(y, 1) = x$ . What constant? Precisely, the point we are interested in. The only thing here is unlike contractibility of  $V$ , the homotopy is taking place  $U$  which is larger open set. If you replace  $U$  by  $V$  this is nothing but saying that  $V$  is contractible to the point  $x$ .

It is stronger than saying that  $V$  is just contractible that contractible to point  $x$  is important for this. But the whole thing is taking place inside  $U$ . The homotopy can go out of  $V$  but it has this property. That is why we call it semi-locally contractible. However, if you replace  $U$  by  $V$  itself in this definition, namely,  $H$  is taking values inside  $V$  then we actually say that this capital  $X$  is locally contractible at  $x$ .

The same condition the only thing that you have to do is replace this  $U$  by  $V$  and then you delete 'semi' So, that is the meaning of locally contractible. If  $H$  can be chosen to take values inside  $V$  itself. So, locally contractibility is stronger than semi locally contractibility.

Now suppose, this happens at every point of  $x$  just like a function is continuous if it is continuous at every point, if this is happens at every point then we say  $X$  itself is semi locally contractible or in the later case  $X$  itself is locally contractible. So, that is the meaning of these two definitions.

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Relative Homology  
Simplicial Complexes I  
**Simplicial Complexes II**  
Covering Spaces and Fundamental Group  
G-Coverings and Fundamental Group

Module 38 Instance of Domain  
Module 39 Links and Stars  
**Module 39 Homotopical Aspects**  
Module 40 Homotopical Aspects-continued  
Miscellaneous Exercises to Chapter 5

Anant Shastri

**Theorem 6.11**  
Every polyhedron is semi-locally contractible.

**Proof:** Given  $x \in |K|$ , let  $F$  be the simplex in  $K$ , such that  $x \in \langle F \rangle$ . By the previous lemma,  $st(F)$  is an open subset of  $|K|$  which is star-shaped at  $x$ . Therefore we have the contraction  $h: st(F) \times \mathbb{I} \rightarrow st(F)$  given by  $h(y, t) = tx + (1 - t)y$ . Given an open set  $U$  such that  $x \in U$ , we see that  $h(x \times \mathbb{I}) = \{x\} \in U$  and hence there exists an open set  $V$  such that  $x \in V$  and  $h(V \times \mathbb{I}) \subset U$ .

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Introduction  
Fundamental Group  
Function Spaces and Homotopy Spaces

Module 32 Barycentric Subdivision  
Module 34 Simplicial Approximation  
Module 35 Seifert Lemma

So now, the next theorem is that every polyhedron is semi locally contractible. Take a simplicial complex  $K$ , take  $|K|$ . I have to show that at every point  $x \in |K|$ , given any neighborhood  $U$  of  $x$  there is a smaller neighborhood  $V$  of that point such that  $V$  contracts to the point  $x$  inside  $U$ . This is what I have to show.

So, let us go through the proof as you see which is not difficult at all. Starting with  $x \in |K|$ , choose  $F$  (unique) such that  $x \in \langle F \rangle$ . Remember every point is inside a unique open simplex because mod  $K$  is partitioned into open simplexes.

So, that is used again and again. The previous lemma says that  $stF$  is an open subset of  $|K|$  which is star shaped at  $x$ . So, this itself is a contractible neighborhood of the point  $x$ . But we want to do it inside the given open set so that is the catch here. So, how we go about it. This is how. First of all we have a contraction  $st F \times \mathbb{I} \rightarrow st F$  which is just given by the formula  $h(y, t) = tx + (1 - t)y$ .

This  $x$  is the apex of the star shaped thing so any  $y$  can join to  $x$ . This is the line segment. This whole thing is inside star  $F$ . So, this is a continuous function put  $t$  is equal to 0 this will be  $y$ . Put  $t$  equal to 1 this will be  $x$   $t$  equal to 0 it is identity,  $t$  equal to 1 is a constant function taking the value  $x$ .

Now, start with an open set  $U$  such that  $x$  is inside  $U$ . Then look at this  $h(\{x\} \times \mathbb{I}) = \{x\} \in U$ .  $h$  of  $x$  cross  $\mathbb{I}$  is the given point  $h$  of  $x$  cross  $\mathbb{I}$  if  $y$  is equal to  $x$  what is this map what is this one. It is just  $x$  it is singleton  $x$ . That is inside  $U$ ,  $U$  is an open set. So, this whole closed interval is mapped inside an open set. The closed interval is a compact set.

Therefore, by standard methods in point set topology like the tube lemma or otherwise, there exists an open set  $V$  inside  $st F$  such that  $x \in V$  and  $h(V \times \mathbb{I}) \subset U$ . One single open set  $V$  cross  $\mathbb{I}$ , instead of  $V_t \times (t - \epsilon, t + \epsilon)$  etc.

Now, you look at  $H$  which is same  $h$  restricted to  $V \times \mathbb{I}$ . What we get is this condition. So, given any  $U$  we have found a  $V$  such that and a homotopy  $H$  as required. Here  $H$  is little  $h$  restricted. special  $H$  here. So, this proves that every polyhedron is semi locally contractible. Very easy statement, very easy proof.

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The screenshot shows a presentation slide with the following content:

At the top, the equation  $h(V \times \mathbb{I}) \subset U$  is displayed.

Below the equation is a navigation bar with the text: "Anant R Shastri Retired Emeritus Fellow Department of Mathematics" and "NPTEL Course on Algebraic Topology, Part-I".

The main content area is a table of contents with two columns:

Introduction	Module 12 Barycentric Subdivision
Fundamental Group	Module 14 Simplicial Approximation
Function Spaces and Quotient Spaces	Module 15 Spanner Lemma
Relative Homology	Module 16 Invariance of Domain
Simplicial Complexes I	Module 18 Links and Stars
<b>Simplicial Complexes II</b>	<b>Module 19 Homotopical Aspects</b>
Covering Spaces and Fundamental Group	Module 21 Homotopical Aspects-continued
C. Coverings and Fundamental Group	Miscellaneous Exercises to Chapter 5

Below the table of contents is a blue bar with the text "Corollary 6.5" and a white box containing the statement: "Every connected polyhedron admits a simply connected covering."

At the bottom left of the slide is the NPTEL logo.

Student is questioning: Hello sir, the  $U$  that you chose the open set  $U$  around  $x$  cannot be sub divide to make the star of that point inside  $U$ ? Then we can contract it?

Professor: What do you, how do you sub divide it?

Student: Yeah. I mean if we sub divide then it will be small, can it be small enough to be inside at any  $U$ ?

Professor: How do you do that?

Student: We know that there is always one sub, I mean can we, we can we can take any point...

Professor:  $K$  is not a finite simplicial complex. Even at point  $x$  you have taken is not at all first of all a vertex. Even if that is the case there may be infinitely many simplices coming and meeting at  $x$ . So, it is not even locally finite. Even after that you have to make  $x$  to be a vertex etc.

Student: But cannot we have a simplicial structure that makes  $x$  a vertex?

Professor: That is the next thing which we want to do. Even then the problem is that by the barycentric subdivision that we have taken that will not do the job. That will do only if it is a finite simplicial complex. Once you have the semi local contractibility, later you will see that such spaces admit a simply connected covering later. So right now, we assume that then what we have



proved is every connected polyhedron admits a simply connected covering. This is additional application of this one. Local contractibility is harder to prove.

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The screenshot shows a video lecture interface. In the top right corner, there is a small video window of the speaker, Anant Shastri. Below it, a navigation bar identifies the speaker as 'Anant R Shastri, Retired Emeritus Fellow, Department of Mathematics' and the course as 'NPTEL Course on Algebraic Topology, Part I'. A table of contents is displayed, listing various modules and topics. The current slide is titled 'Theorem 6.12' and contains the following text:

**Theorem 6.12**  
*Let  $K$  be a simplicial complex and  $L \subset K$  be a subcomplex. Then the inclusion map  $|L| \hookrightarrow |K|$  is a cofibration.*

Now, let us come to the second point here, namely, a cofibration property. This theorem says that for every sub complex  $L$ , the inclusion map mod  $L$  to mod  $K$  is a cofibration. Remember that we had a proposition which gives you a method to verify cofibrations. Namely, you must produce a retraction from  $|K| \times \mathbb{I} \rightarrow (|K| \times \{0\}) \cup (|L| \times \mathbb{I})$ . And that is precisely what we are going to prove.

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Anant R. Shastri, Retired Emeritus Fellow, Department of Mathematics, NPTEL Course on Algebraic Topology, Part I

Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes I <b>Simplicial Complexes II</b> Covering Spaces and Fundamental Group G-Coverings and Fundamental Group	Module 32 Barycentric Subdivision Module 33 Simplicial Approximation Module 34 Sperner Lemma Module 35 Invariance of Domain Module 36 Links and Stars <b>Module 39 Homotopical Aspects</b> Module 40 Homotopical Aspects-continued Miscellaneous Exercises to Chapter 5
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**Proof:** It is enough to produce a retraction

$$r: |K| \times \mathbb{I} \rightarrow |K| \times \{0\} \cup |L| \times \mathbb{I}.$$

Put  $L_n = L \cup K^{(n)}$ . Then each  $L_n$  is a subcomplex of  $K$  and we have

$$L \subset L_0 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots$$

and  $\cup_n L_n = K$ .

Retraction should be from mod  $K$  cross  $I$  to mod  $K$  cross  $0$  union mod  $L$  cross  $I$ . So, this one we are going to get such a retraction in an inductive fashion. Put  $L_n = L \cup K^{(n)}$  where,  $L_n$  equal to  $K^{(n)}$  denotes the  $n$ th skeleton of  $K$ . Then by the very definition,  $L, K^{(n)}$  are sub complexes of  $K$  and hence so is  $L_n$ . We have the inclusion maps here. Start with  $L \subset L_0$ . What is  $L$  naught? It consists of all of  $L$  and all the vertices of  $K$ . Those are extra vertices coming from  $K^{(0)}$  are there. Also,  $L_0 \subset L_1 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots$ . Finally, if you take the union of all of them that will contain all the skeletons therefore it will be the whole of  $K$ .

So, I have written  $K$  as a countable union starting with  $L_0$ . If  $K$  is a finite simplicial complex there are many other methods to do this. There is no need to go through this. This is precisely to take care of infinite simplicial complexes.

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<ul style="list-style-type: none"> <li>Introduction</li> <li>Fundamental Group</li> <li>Function Spaces and Quotient Spaces</li> <li>Relative Homotopy</li> <li>Simplicial Complexes-I</li> <li><b>Simplicial Complexes-II</b></li> <li>Covering Spaces and Fundamental Group</li> <li>G-Coverings and Fundamental Group</li> </ul>	<ul style="list-style-type: none"> <li>Module 32 Barycentric Subdivision</li> <li>Module 34 Simplicial Approximation</li> <li>Module 35 Seifert Lemma</li> <li>Module 36 Invariance of Domain</li> <li>Module 38 Links and Stars</li> <li><b>Module 39 Homotopical Aspects</b></li> <li>Module 40 Homotopical Aspects continued</li> <li>Magisterial Exercises to Chapter 9</li> </ul>
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Recall that we have the standard retractions

$$\rho : |\Delta_{n+1}| \times \mathbb{I} \rightarrow |\Delta_{n+1}| \times \{0\} \cup |\mathcal{B}(\Delta_{n+1})| \times \mathbb{I}$$

for each  $n \geq 0$ . Let  $\{F_s\}$  denote the family of all  $(n+1)$ -simplexes in  $L_{n+1}$  which are not in  $L$ . Let  $\rho_s$  be a copy of  $\rho$  as above for each  $s$ .

We have seen this one several times namely,  $\mathbb{D}^n \times \mathbb{I}$  deformation retract onto  $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times \mathbb{I}$ . Just topologically, there is no need for going to simplicial complexes. I can write this in terms of simplicial complexes viz.,  $|\Delta_n| \times \mathbb{I}$  deformation retracts onto  $|\Delta_n| \times \{0\} \cup |\mathcal{B}(\Delta_n)| \times \mathbb{I}$ .

So, there are these standard retractions for each n. I am going to use this one now. For each n greater than equal to 0 we have such a thing. So now, label the set of all (n+1) simplexes of K which are not in L. Do not worry about things which are in L. Because they are going to be here this part. So, let us index them, let us denote them  $\{F_s\}_s$ . For each s, there is a retraction  $\rho_s : |F_s| \times \mathbb{I} \rightarrow |F_s| \times \{0\} \cup |\mathcal{B}(F_s)| \times \mathbb{I}$ . It is the same rho a copy of rho but I am denoting rho s for each s. Then what do I do?

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Anant H. Shastry, Retired Emeritus Fellow, Department of Mathematics, NPTEL Course on Algebraic Topology, Part I

<ul style="list-style-type: none"> <li>Introduction</li> <li>Fundamental Group</li> <li>Function Spaces and Quotient Spaces</li> <li>Relative Homotopy</li> <li>Simplicial Complexes I</li> <li><b>Simplicial Complexes II</b></li> <li>Covering Spaces and Fundamental Group</li> <li>Coverings and Fundamental Group</li> </ul>	<ul style="list-style-type: none"> <li>Module 32: Barycentric Subdivision</li> <li>Module 34: Simplicial Approximation</li> <li>Module 35: Sperner's Lemma</li> <li>Module 36: Invariance of Domain</li> <li>Module 38: Links and Stars</li> <li><b>Module 39: Homotopical Aspects</b></li> <li>Module 42: Homotopical Aspects continued</li> <li>Miscellaneous Exercises to Chapter 3</li> </ul>
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Define  $r_0 : |L_0| \times \mathbb{I} \rightarrow |L_0| \times \{0\} \cup |L| \times \mathbb{I}$  as follows:

$$r_0(x, t) = \begin{cases} (x, t), & x \in |L| \text{ or } t = 0; \\ (x, 0), & x \text{ is a vertex not in } L. \end{cases}$$

Check that  $r_0$  is a retraction.

We start constructing a sequence of slower retractions  $\{r_n\}$ . I am going to define these retractions

$$r_0, r_1, \dots, r_{n+1}, \dots, r_{n+1} : |L_{n+1}| \times \mathbb{I} \rightarrow |L_{n+1}| \times \{0\} \cup |L_n| \times \mathbb{I}.$$

But there is slight different definition for  $r_0 : |L_0| \times \mathbb{I} \rightarrow |L_0| \times \{0\} \cup |L| \times \mathbb{I}$ . Here, either  $x$  is in  $|L|$  or  $t$  is 0 then take  $r_0(x, t) = (x, t)$ . It is identity. Then there are points in a disjoint union of vertices cross  $\mathbb{I}$ , which corresponds to all the vertices which are not in  $L$ . Each of those vertices  $x$ , there is the line segment  $x$  cross  $\mathbb{I}$ , you push all of it to just the base point  $x$  cross 0. Instead of  $x$  cross  $t$  all of them pushed to  $x$  cross 0. So, this is obviously a retraction of  $L$  naught cross  $\mathbb{I}$  onto  $L$  naught cross 0 union  $L$  cross  $\mathbb{I}$ .

That is the starting point. So, in this inductive process, for  $n=0$ , we have got a retraction corresponding to this  $L$  naught so corresponding to  $L$ . Is not a retraction to  $L$  but its  $L$  naught cross  $\mathbb{I}$  to  $L$  naught cross 0 union  $L$  cross  $\mathbb{I}$ . So, that is the kind of thing we are going to do here all the time. So, once you have this one you can assume that you have rewind this one for up to  $N$  and you can go for  $N$  plus 1 stage. So, what I am going to do is for.

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Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

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Anant Shastri

For each  $n \geq 1$ , define

$$r_n : |L_{n+1}| \times \mathbb{I} \rightarrow |L_{n+1}| \times \{0\} \cup |L_n| \times \mathbb{I}$$

by the formula,

$$r_n(x, t) = \begin{cases} (x, t), & x \in |L_n| \text{ or } t = 0; \\ \rho_s(x, t), & x \in |F_s| \text{ for some } s. \end{cases}$$

It follows that each  $r_n$  is a retraction.

Now, for each  $n \geq 0$ , I am defining  $r_{n+1} : |L_{n+1}| \times \mathbb{I} \rightarrow |L_{n+1}| \times \{0\} \cup |L_n| \times \mathbb{I}$  as follows: On all of  $|L_{n+1}| \times \{0\} \cup |L_n| \times \mathbb{I}$ , it is identity,  $r_{n+1}(x, t) = (x, t)$ . The second part, when they are inside  $|F_s| \times \mathbb{I}$ , there I take this rho s. Because, they are all indexed by s;  $r_{n+1}(x, t) = \rho_s(x, t)$ .  $\rho_s$  will push  $|F_s| \times \mathbb{I}$  into  $|F_s| \times \{0\} \cup |\mathcal{B}(F_s)| \times \mathbb{I}$ . So, if  $t = 0$  or  $x \in |F_s|$  for some s,  $\rho_s(x, t) = (x, t)$ , the identity map. So the two definitions agree. So, there is no conflict here. So, each  $r_{n+1}$  is a retraction. (In the slide there is a typo.) What is the next step then?

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It follows that each  $r_n$  is a retraction.

Anant R Shastri Retired Emeritus Fellow Department of Mathematics, NPTEL Course on Algebraic Topology, Part I

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Given any  $x \in |K|$ , there is a unique  $n$  and a unique  $n$ -simplex  $F$  such that  $x \in \langle F \rangle$ . Define

$$r(x, t) = r_0 \circ \dots \circ r_n(x, t).$$

Clearly,  $r$  agrees with  $r_n$  when restricted to  $|K^{(n)}| \times \mathbb{I}$ , for all  $n$ . Therefore  $r$  is continuous. That  $r$  is a retraction is checked easily.

Now, take any point  $x \in |K|$ . This will belong to a open unique  $n$ -simplex, it is inside  $\langle F \rangle$ , where  $F$  is an  $n$ -simplex. This  $F$  maybe singleton also. open simplex for a vertex is a same set as a vertex. In all other cases it is different. For an edge, it will be a open interval and so on. So,  $x$  belong to open  $F$ . That means it is inside  $|K^{(n)}|$  in any case. In any case  $x \in |L^{(n)}|$ .

So, you define  $r$  of  $x, t$  as follows: start with  $r_n$  then apply  $r_{n-1} \dots$  and so on, come up to all the way  $r_0$ . So, that you land up inside  $|K| \times 0 \cup |L| \times \mathbb{I}$ . So, that will be the final picture here. Actually,  $r$  agrees with you see once the point is inside the  $|K^{(n)}|$  then it is  $r_n$  and then all these are  $r$  naught.

The next stage  $r_{n+1}$  will agree with this one if it is already inside  $r_n$  inside  $|K^{(n)}|$ .  $r_{n+1}$  from  $|K^{(n+1)}|$  the  $r_{n+1}$  will be the identity on that part is already identity then only  $r_n$  will up to  $r$  naught  $r_n$  will change it. Therefore, this compatible family of this so that is why  $r$  of  $x, t$  make sense.

Restricted to each  $|K^{(n)}|$  it is only this part. If  $|K^{(n+1)}|$  if I have taken one more composition will come  $r_{n+1}$  we would have come that is all. But they are all well defined and when you want to check the continuity you have to check the continuity on this part. And then this is the composition of finitely many continuous function so it is continuous. The retraction part follows very easily because for each point this inside this one.

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Anant R Shashi Retired Emeritus Fellow Department of Mathematics, NPTEL Course on Algebraic Topology, Part I

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Module 36 Invariance of Domain  
Module 38 Links and Stars  
**Module 39 Homotopical Aspects**  
Module 40 Homotopical Aspects-continued  
Miscellaneous Exercises to Chapter 5

Anant Shashi

We would like to strengthen this result further. As a first step, let us now have the solution of one of the exercises. [go to the exercise](#)

Recall

**Lemma 6.9**

Let  $h : |L| \rightarrow \mathbb{S}^n$  be a triangulation of  $\mathbb{S}^n$  and  $v \in \text{int } \mathbb{D}^{n+1}$ . Then  $h$  extends to a triangulation

$$\hat{h} : |L * \{v\}| \rightarrow \mathbb{D}^{n+1}$$

such that  $\hat{h}(v) = v$ .

We refer to this process as **starring a simplex**.

So, every subcomplex the inclusion maps are cofibration. What we want to do next? Given an arbitrary open subset  $V$  containing  $|L|$ , can we get a smaller open subset  $U$  such that the inclusion  $|L| \subset U$  itself is a cofibration. So, these are the harder results. So, that is what we want to do next.

We would like to strengthen all this results. The first step let us have one of the exercises that we had announced earlier namely, that every point can be thought of as vertex inside a simplicial complex. The first step is again, this we have discussed several times, I will discuss it again.

Let  $(L, h)$  be a triangulating  $\mathbb{S}^n$ , i.e.,  $L$  is a simplicial complex and  $h : |L| \rightarrow \mathbb{S}^n$  is a homeomorphism. Take any point  $v$  in the interior of  $\mathbb{D}^{n+1}$ .  $\mathbb{D}^{n+1}$  is bounded by  $\mathbb{S}^n$ . Then this triangulation extends to a triangulation  $\hat{h} : |L * \{v\}| \rightarrow \mathbb{D}^{n+1}$ , such that  $\hat{h}(v) = x$ .

$L * v$  is a cone. It is the join of these two simplicial complexes  $L$  and the singleton  $\{v\}$ . That is just another name; cone over  $L$  with apex at  $v$ . So, take the modulus of that this  $\hat{h}$  defines a homeomorphism of that. This follows because given any interior point  $v \neq x$  in  $\mathbb{D}^{n+1}$  there is unique point on the boundary,  $\mathbb{S}^n$ , call that  $u$ , such that  $v \in [u, x]$ .

Only  $x$  will be in all the line segments. That is why it becomes a cone. I am just recalling that this we have done earlier. So, this process let us call it as starring a simplex. This  $L$  is the boundary of

the simplex which has been subdivided.  $L$  could be any subdivision. Not naturally  $\Delta^{n+1}$  boundary of  $\Delta^{n+1}$ .

That triangulation extends the triangulation of this, starring a simplex. With whatever boundary triangulation is there that gets extended. So, this process we will call as starring a simplex. Just temporarily if do not worry about other people may call it differently and so on.

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we refer to this process as starring a simplex.

The slide contains a table of contents for the course 'NPTEL Course on Algebraic Topology, Part-I' by Anant P. Shastriflated Emeritus Fellow Department of Mathum. The table lists modules 32 through 40. Below the table, a blue box highlights 'Theorem 6.13' which states: 'Given any polyhedron  $X$ , and a point  $x \in X$ , there exists a triangulation of  $X$  in which  $x$  is a vertex.'

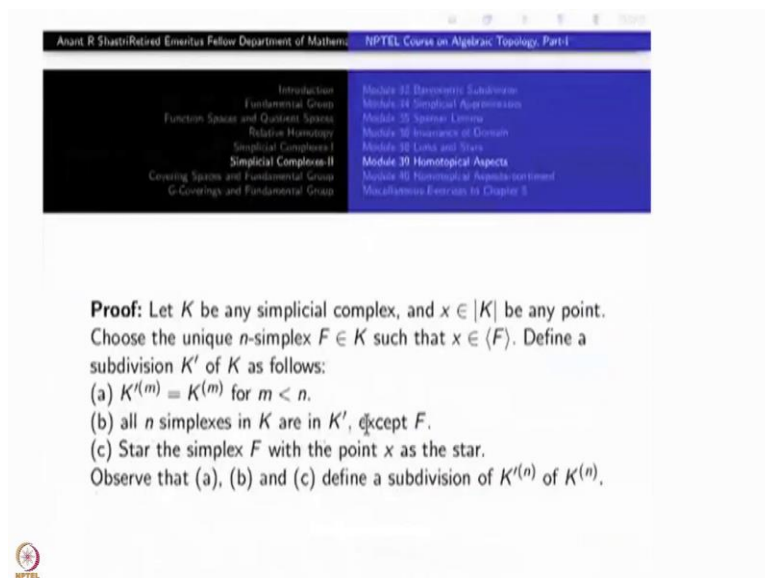
Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes-I <b>Simplicial Complexes-II</b> Covering Spaces and Fundamental Group II: Coverings and Fundamental Group	Module 32 Barycentric Subdivision Module 33 Simplicial Approximation Module 34 Sperner's Lemma Module 35 Invariance of Domain Module 36 Links and Stars <b>Module 39 Homotopical Aspects</b> Module 40 Homotopical Aspects-continued Miscellaneous Exercises to Chapter 5
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**Theorem 6.13**  
 Given any polyhedron  $X$ , and a point  $x \in X$ , there exists a triangulation of  $X$  in which  $x$  is a vertex.

So, now we can solve this problem. Given any polyhedron  $|K|$  and a point  $x \in |K|$  there exists a triangulation in which  $x$  is a vertex.



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Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes I Simplicial Complexes II Covering Spaces and Fundamental Group G-Coverings and Fundamental Group	Module 32 Barycentric Subdivision Module 34 Simplicial Approximation Module 35 Spatial Lenses Module 36 Invariance of Domain Module 38 Links and Stars Module 39 Homotopical Aspects Module 40 Topological Aspects on Homotopy Miscellaneous Exercises to Chapter 5
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**Proof:** Let  $K$  be any simplicial complex, and  $x \in |K|$  be any point. Choose the unique  $n$ -simplex  $F \in K$  such that  $x \in \langle F \rangle$ . Define a subdivision  $K'$  of  $K$  as follows:

- (a)  $K'^{(m)} = K^{(m)}$  for  $m < n$ .
- (b) all  $n$  simplexes in  $K$  are in  $K'$ , except  $F$ .
- (c) Star the simplex  $F$  with the point  $x$  as the star.

Observe that (a), (b) and (c) define a subdivision of  $K^{(n)}$  of  $K^{(n)}$ .



Start with a simplicial complex  $K$ . That means what? I am talking about now  $X = |K|$ . Let  $x \in X$  be any point. Again, choose the unique  $n$ -simplex  $F$  belonging to  $K$  such that  $x$  is inside open simplex  $F$ . (If  $F$  is a singleton there is nothing more to prove.) Define a subdivision  $K'$  of  $K$  as follows.

We are going to define a subdivision and that subdivision will have this  $x$  as a vertex that is all. So, what do we do? This vertex is inside  $\langle F \rangle$ . So there may not be any need to disturb all other simplexes, there is no need to subdivide them. Somewhere something around  $F$  only, we have to carry out subdivision.

So, what I do first of all for all is for simplexes of dimension less than  $n$ , choose  $K'$  to be just  $K$ . Whatever it is, it means  $K'^{(n-1)} = K^{(n-1)}$ . When it is  $n$  you have to be bit careful. What is that? Do not divide anything other than this  $F$ . Keep them as it is. All the  $n$ -simplexes in  $K$  are in  $K'$  also except  $F$ . The simplexes are as they are that means we are not dividing that one at all. So, now you have to concentrate one simplex that is  $F$  and in the interior of that. The boundary of  $F$  that is not divided. It is whatever it is, isomorphic to  $\mathcal{B}(\Delta_n)$ . Now, use the previous lemma and star this simplex at the point  $x$ .

Instead of barycentre you are choosing  $x$  that is all. For example, this  $x$  becomes a vertex and all the lines joining to each vertex of the boundary will be the 1- simplexes and so on. So, that is the starring. So, this will complete the definition of subdivision  $K^{(n)}$  of the  $n$ -skeleton  $K^{(n)}$ .

After this there may be many  $n+1$ -simplexes which contain this  $F$  which has been divided. Therefore, you may have to subdivide them also. So, do not worry, divide all of them by starring at the barycenters. Beyond this  $n$  dimension we need subdivide more and more simplexes. But this can be put in a general set up.

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Inductively, as soon  $m$  is bigger than  $n$  extend the subdivision  $K^{(m)}$  of  $K^{(m)}$  to a subdivision  $K^{(m+1)}$  of  $K^{(m+1)}$  by starring each  $(m+1)$ -simplex at its barycenter. Keep starring them put them all together get  $K' = \cup_m K^{(m)}$ . So, that will complete the subdivision.

Achieving that the given  $x$  as vertex it is done at the stage (c). This does not complete the subdivision. So for that you have to do the stage (d) also. This we will use next time but today we will stop here. Thank you.