Introduction to Algebraic Topology (Part 1)
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Lecture 38
Links and Stars

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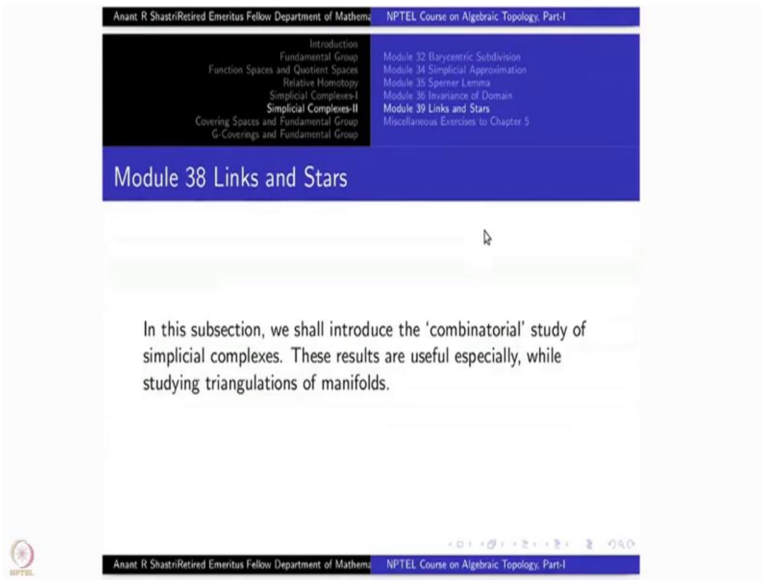

Today, we will do some combinatorial study of simplicial complexes. Of course, there is some topology there, but mostly it is a relation between where something is incident and neighborhoods and such things which are all studied by a combinatorial relation. A combinatorial relation decides which face is where, that kind of information.

These things are useful while doing many more combinatorial problems and also triangulation of manifolds and various things polyhedron topology and so on. Within this course we are not going to use this one. So, there is less material based on this one, that is one reason why I have put it at the end of the study of simplicial complexes. Just because we are not going to study much of it does not mean that this is less important, though.

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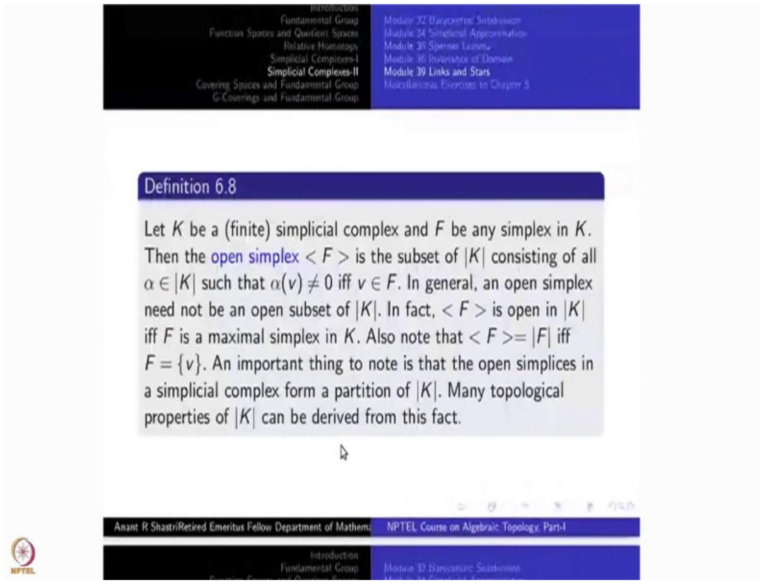

So, for this study to begin with you can just concentrate on finite simplicial complexes, to get a hold on what is going on. But, all the definitions and all the results except when stated otherwise, they are valid for any simplicial complex. But, to get your ideas clear you just work on some finite simplicial complexes, what is happening. So, take a finite simplicial complex K and F be any simplex in K.

Recall we already defined this open simplex $\langle F \rangle$ to be a subset of |K|, which consist of all α in $|K|$ with $\alpha(v) \neq 0$ for every point v inside F and conversely, i.e., $\alpha(v) \neq 0$ iff $v \in F$. So, $\langle F \rangle \subset |F|$, where $|F|$ consists of all $\alpha \in |K|$ with $supp\ \alpha \subset F$. So, this set is actually the interior of that in some sense. It may not be interior of $|F|$ inside $|K|$; this will be an open set in $|K|$ and the interior of $|F|$ in $|K|$ provided F is a maximal simplex in K. So, all this we have seen.

In general, this open simplex need not be an open subset of $|K|$. In fact, this is open if and only if F is a maximal simplex. Note that $|F| = \langle F \rangle$ if and only if F is a singleton, all these things we have seen but I am just driving a point, driving your attention to these things. An important thing to note is that the set of all open simplices in any simplicial complex forms a partition of $|K|$. This is the fact we have used also. Now, let us do a little more.

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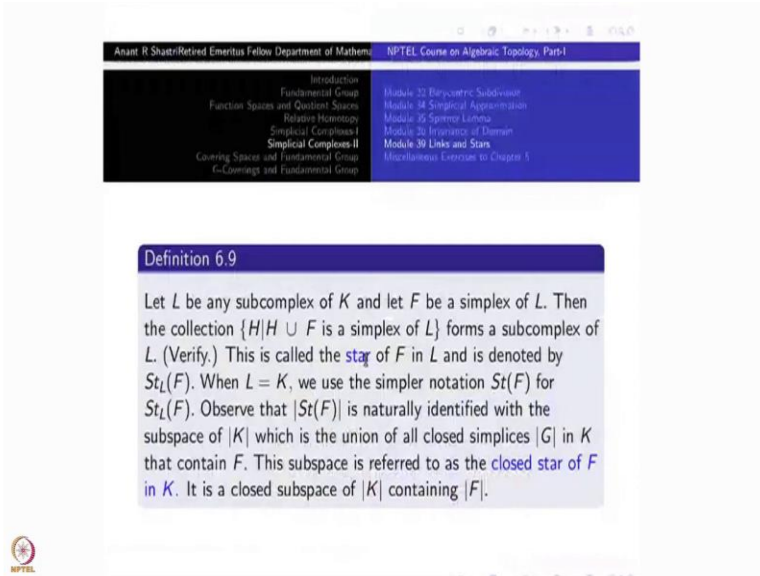

Now, take L to be a subcomplex of K and F be inside L. Then look at $St_L(F) = \{H \in K \ : \ H \cup F \in L\}$, the set of all simplices H of Ksuch that $H \cup F$ is still a simplex of L. It must be simplex of L; not just in K but inside L. $H \cup F$ should be a simplex. Then you put H in this one, in this collection. In particular $F = F \cup F$ is also there, clearly, all subsets of F will be also there. Once a simplex H is there, all subset of H will be also there. So, a subcomplex of L. Therefore $St_L(F)$ is a subcomplex of L. I have just told you why if $H' \subset H$ where $H \in St_L(F)$, then $H' \in St_L(F)$ because L is a subcomplex. So, this set has a name. It is called star of F in L. The notation indicates that these things are taken inside L. In particular, if L is all of K, then we can make a simpler notation, $St(F)$=$St_K(F)$, the suffix K can just be dropped. Because the ambient simplicial complex K is fixed, it may not be any need to write it. $St_L(F)$, if you take different subcomplexes L, could be different, because then it will be only those things which are inside that particular subcomplex L, that is all.

As usual, with any subcomplex, $|St_L(K)|$, the underlying topological space is naturally identified with a subspace $|L| \subset |K|$. This is true for any subcomplex after all. And this is nothing but the union of all closed simplexes $|G|$, where G contains F, and belongs to L. If G contains F and is a simplex in L, it will be inside $St_L(F)$. That is easy to see. This subspace $|St_L(F)|$ is referred to as the closed star of F in L. It is a a closed subspace of $|K|$, and contains $|F|$. It is a quite large of course, not a small one, compared to $|F|$. Note that $|F|$ is also closed but this will be another larger closed subset called closed star of F. There will be an open star also we will define that soon.

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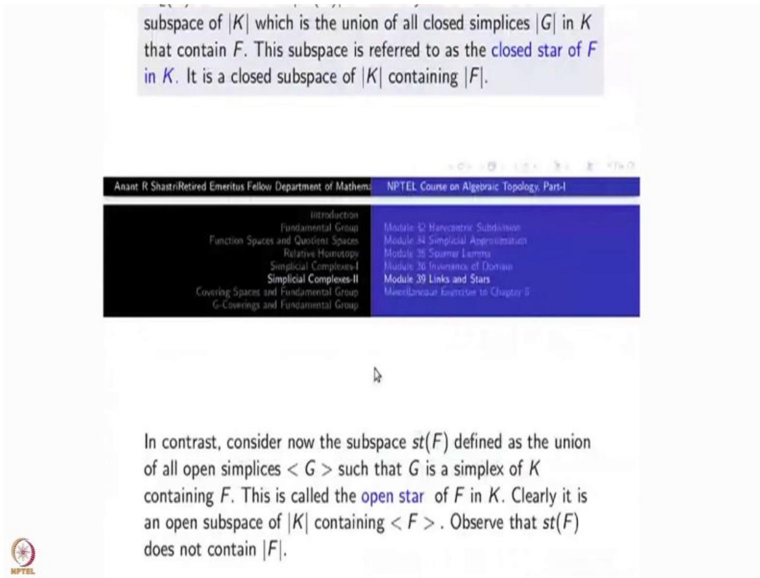

In contrast, see that this is capital St here small st. $st(F) = \cup\{\langle G\rangle \ : \ G \in K, F \subset G\}$. This mall st of F defined as the union of all open simplices, G open. Such that G is a simplex of K containing F. This is called the open star of F in K. Clearly this is an open subspace of mod K because you have taken all simplexes larger than F, so it will be open subset of $|K|$. And it contains this open F. You have to be very careful with it.

For example, this open $st(F)$ may not even contain all points of $|F|$. It may not contain it actually does not contain $|F|$ in general. Indeed, this will contain $|F|$ only if F is a singleton.

For example, take an empty triangle only the three sides are there, $K = \mathcal{B}(\Delta_2)$; take one of the sides as F. then, what is open $st(F)$? $st(F) = \langle F\rangle$, is the open interval, the two points are gone. Next take the full triangle $K = \Delta_2$. What is $st(F)$ now? The only simplexes that contain F are F and the full triangle G. Therefore, $st(F)$ is equal to the union of the open edge $\langle F\rangle$ and the open triangle $\langle G\rangle$. So, this example whatever I have given must make it clear what is happening here. Inside the closed triangle a full triangle this open star will be an open subset.

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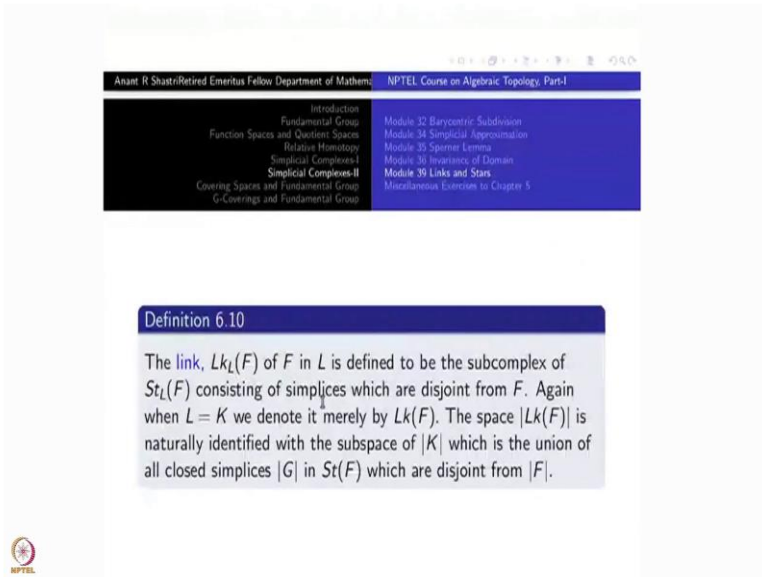

Now, I will define another concept called a link. This topic is called links and stars. So, another concept called link. $L$is subcomplex of K, and $K \in L$ is some simplex. Then then link of F in L is defined to be the subcomplex of star of F consisting of only those simplexes which are disjoint from F. Remember that star of F contains all G, such that G union F is a simplex inside L. We look at that but only take those consisting of simplices which are disjoint from F. Say some G is there, G intersection F is empty but G union F is inside L. Because it must be in star L. If it is far away then that will not be taken. The collection of all such things will be called link of L link of F inside L; $Lk_L(F) = \{G \in St_L(F) \;:\; G \cap F = \emptyset\}$

Once again if L is all of K, we just drop this suffix L and simply write $Lk(F)$. The underlying topological space naturally is a subspace of $|St(F)|$ and of course the subspace of mod K also. And $|St(F)|$ is a union of all closed simplexes $|G|$, such that $|G| \cap |F| = \emptyset$ and $G \in St(F)$. Now I will give you some examples of links.

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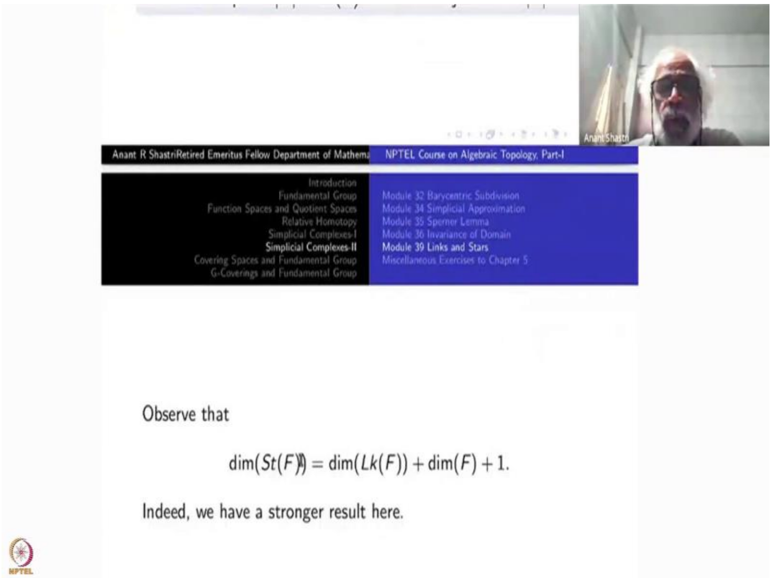

First, purely combinatorically, observe what happens to the dimension of $St(F)$. Dimension of star F is actually equal to dimension of link F plus dimension of F plus 1. Why? How does any simplex here look like. It is G union F inside that one which is a simplex. And what is simplex here? It is G union F but G intersection F must be empty. So, starting with a simplex here you write it as G union F where G intersection F is empty. You can write any union as disjoint union because F is a subset. Then that G is disjoint from F will be as will be an element here. What is a dimension of G disjoint union F? Number of elements in G plus number of elements in F minus 1. Dimension of each of them is number of elements minus 1. Therefore, you get this plus dimension of F plus 1. This F is kept fixed, the element which comes here is disjoint from F such that union is here. So, this formula is obvious. So, that is what I mean by what we are doing if purely combinatorial, no topology here as such. But this is a only combinatorial but now I will tell you a stronger result which topology here or geometry that is the result here.

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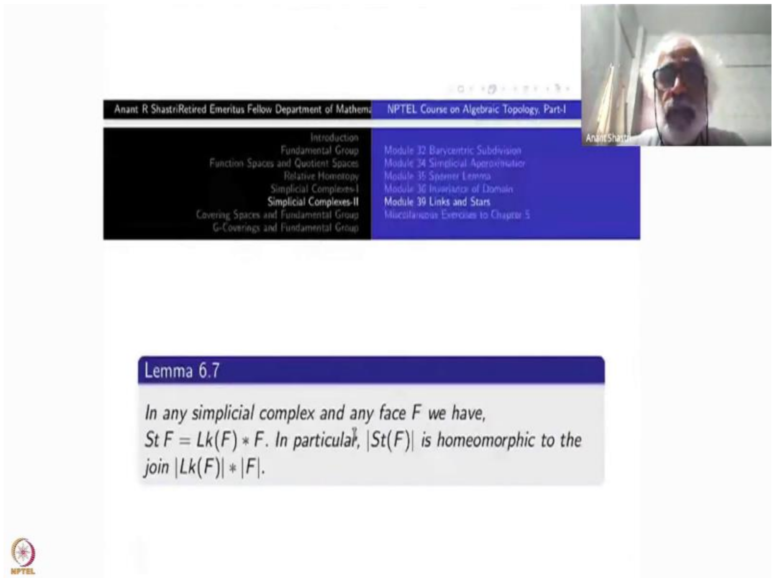

Take any simplicial complex, any face F we have the star of F. (I am not writing the sub subcomplex here L but if you write that that will be also true. Because this is true for every simplicial complex. I can put an L here then it will be all true.) Star of F is equal to the join of link of F with F. What is this F here. F is a simplicial complex on its own. It is isomorphic Δ_n, where n is the dimension of F; all subsets of that are also taken here that is the meaning of F as a simplicial complex.

So, we will take the join of these two, you get this one. So, this is again combinatorial but what is the meaning of this when you take, when you pass through the geometric realisation. Geometric realisation of star F is homeomorphic to the join of the geometric realization of link of F with the geometric realization of F. This part we have already seen in more generality: $|K_1 * K_2| \equiv |K_1| * |K_2|$. K 1 and K 2 you first take the join, then take the underlying topology mod K, is same thing as first take the mods and then take the join.

So, only this part we have to check is the following. Every simplex in $St(F)$ can be written as a disjoint union of a simplex in $Lk(F)$ and a subset of ; $G = A \coprod B$, where $A \in Lk(F)$ and $B \subset F$. It may happen that B is not the full F it maybe some part of it. So, that is precisely the definition of the join of $Lk(F)$ with F That what we have to be careful about this.

So, suppose, I take a subset here, by very definition all subsets of that will have to be also there because this is a simplicial complex subcomplex. So, after taking this one you can take the subset also there. Now, here is, in particular in particular I have solved this one already but here I have given the explanation here. So, this is a standard straight forward. You have seen it several times.

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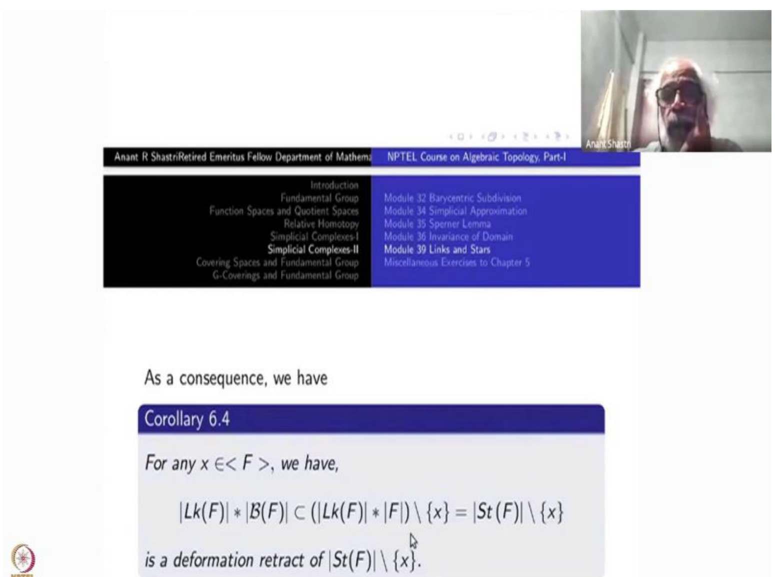

Now, as a consequence in particular take any point x inside the open simplex, then look at the mod of link of F starred with the boundary complex of F instead of taking the full thing. That is contained in the mod of link of F star with F the minus point x. x is a single point here in $\langle F \rangle$. So, we throw away that, that point is not there anyway in the whole thing also. Because, we have taken the boundary here.

But the RHS is equal to $|St(F)| \setminus \{x\}$. This much is obvious. But what happens is this LHS is a deformation retract of $|St(F)| \setminus \{x\}$. The subspace here is a deformation retract, what is the meaning? That there is a homotopy of the identity map of this one with the end function being a retract it is a map on to this part which is identity on the boundary on this part. So, here is the proof.

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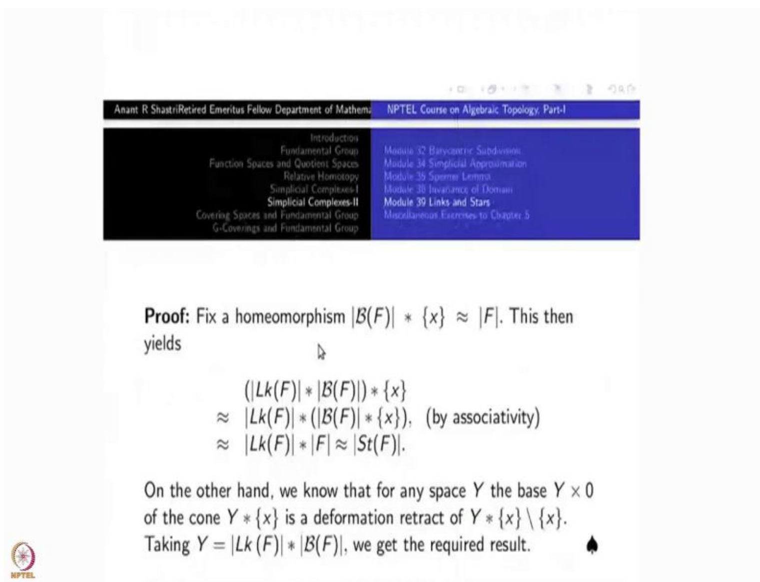

Fix a homeomorphism from $|\mathcal{B}(F) * \{x\} \to |F|$, from this is a cone to this one. Then look at $|Lk(F)| * (|\mathcal{B}(F)| * \{x\})$. So, this I already know I am taking again the joint with this one, link of F. This is homeomorphic to $(|Lk(F) * \mathcal{B}(F)|) * \{x\}$ by associativity of the join operation. I am putting bracket here that is all by associativity. First bracket was here, now the bracket is here. But once this is F I can put F here. Link of F star F we have seen that it is star of F. So, this is one way of looking at it.

On the other hand, we know that for any space Y, the base Y cross 0 of the cone $CY = Y * \{x\}$ is a deformation retract of $CY \setminus \{x\}$. I use that, take $Y = |Lk(F) * \mathcal{B}(F)|$. This is the base and that is the cone. So, conclusion is that this star of F $|St(F)| \setminus \{x\}$ will deform retract on to this part. So, we have developed these things earlier in a abstract, so I am applying it in a special case here.

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Now, let K be any simplicial complex and F and G be any two disjoint faces of a face F union G. I will start with a face and then write it as the disjoint union as above that is all. Now I am taking links and star in K. Look at the link of F in star of G. First look at st G, it contains F, G union F is a simplex in K. F is a some simplex of the subcomplex stG. So, inside this one, link of F is equal to star of G in the link of F.

Now, you can see why the the title of this section was links and stars. Link inside a star is star inside the link. Of course, F and G get interchanged, $Lk_{St\ G}(F) = St_{Lk(F}(G)$. Purely combinatorial result there is no topology here you see, it is just combinatorial result. You have to check left hand side is equal to right hand side take a simplex here, show that it is here and vice versa.

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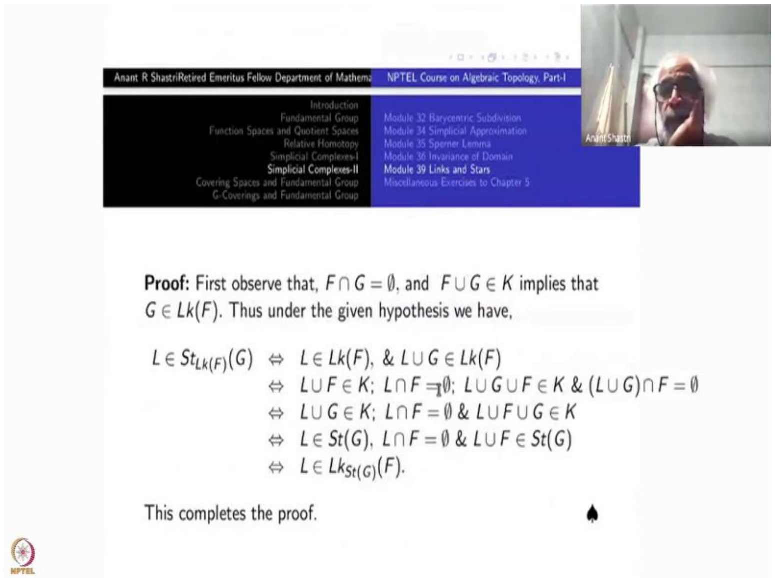

First observe that, F intersection G is empty, this is the starting assumption. And the union is in K. Therefore this G is in the link of F. And this F is in the star of G. Therefore, under the given hypothesis that F and G are disjoint, what we have is: L is a face of star of G in the link of F. I am starting from here. This just means and it is if and only if L is first of all in the link of F and L union G must be also in the link of F. That is a definition of star.

But that is the same thing as saying L union F is a simplex in K, L intersection F must be empty because it is in the link and L union G union F must be inside K. And L union G intersetion F must be empty. So, this is same thing as saying L union G inside K, L intersection F is empty, L union G union K inside K. I am just rewriting this one so nothing is lost here. But this is same thing as L is inside star of G , L intersection F is empty, L union F is in star of G. That is the same thing as saying that L is inside link of F inside star of G.

So, notice that to prove something inside link of F and link of G I am going all the way to K. Because these things were taken inside K. Link of F is inside K similarly star of G is also inside K. So, that is why I am going inside K. So, this just means that L itself is in the link of star of G. This is in star of G and it is intersection with F is empty. Therefore, this is inside the link of this. And this is all reversible, all in it all implications are reversible that is what we are saying.

This result was used by Munkres in deriving result, in which was algebraist were interested in. I will not be able to state it here in this particular case, This is called the Reinser's condition for a topological space to be a homology sphere. So, Munkres proved that such a condition is true under the relevant hypothesis. And that became key result for Reisner to prove his result in combinatorics which was, in turn used by Stanley to prove a big theorem viz., the Upper Bound Conjecture.

So, there are history here so, that is one of the idea why I just spent some time on this links and stars. The only thing is unless you are familiar with then a little more, even the definition you may forget. So, to get familiar you can start we by solving a few exercises here which I have mentioned.

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So, for example show that star of G inside star of F is star of F intersection star of G both taken inside the K. The only thing you have to assume that is F union G is a simplex. Then such a thing happens. You can just prove or disprove it then you will know you would have worked out you would have known whether you have understood these things correctly. So, that is all for today. Thank you.