Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 37
Proof of Controlled Homotopy

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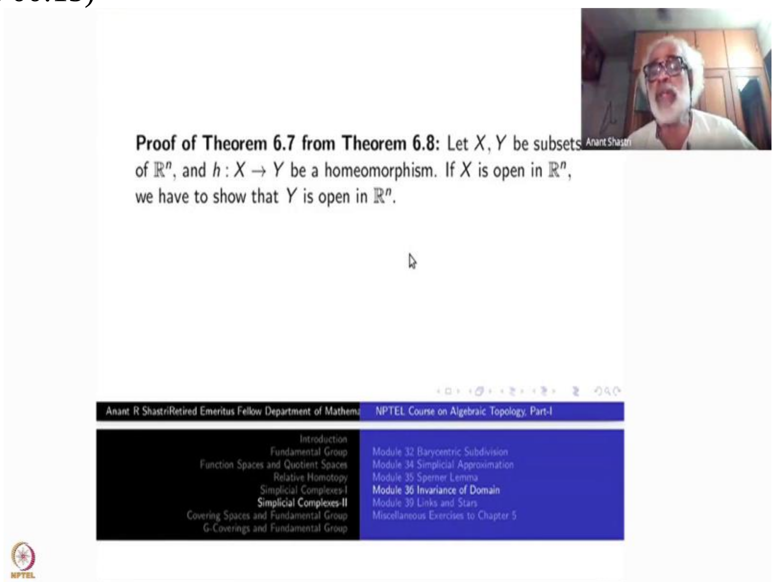

So, on the way, proving Brouwer's invariance of domain, we proved a topological criterion for a point of a subset of $\mathbb{R}^n$ to be an interior point of the subset. That was theorem 6.8, using that theorem let us complete the proof of Brouwer's invariance of domain, which is now very easy. After that, we need to complete the proof of the control homotopy lemma then, the entire proof of Brouwer's invariance of domain will be completed.

So, let us start proving, the final theorem point, 6.1, 6.7. So, for this we have, two subsets X and Y of $\mathbb{R}^n$ and a homeomorphism h from X to Y. This homeomorphism you have to understand that it is defined from X to Y, and not on the whole of $\mathbb{R}^n$ to $\mathbb{R}^n$. If there is a whole homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ which takes X to Y then the conclusion of this theorem is obvious. Then opens subset image of open subsets will be open. The crucial point is that this h is defined only from X to Y. And it may not be extendable to the whole of $\mathbb{R}^n$. But we want to prove that if X is open in $\mathbb{R}^n$ then Y is also open in $\mathbb{R}^n$.

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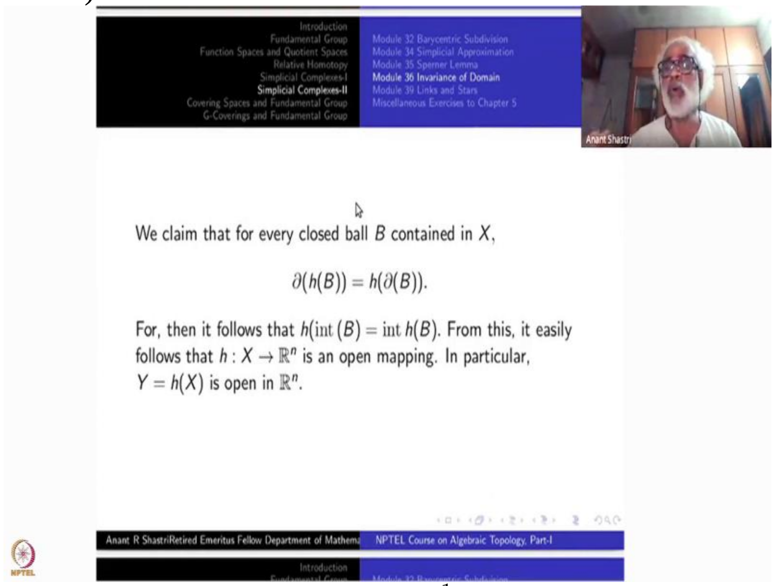

So, what is the method here? Namely, under h and h^{-1} the boundary goes to the relative boundary, both ways because the homeomorphism. Rest of the points, namely the interior will have to go to the rest of the points there namely interior. Complement goes to complement. So, if you want to say, the interior goes to interior, rest of the, this kind of thing that we want to solve. So, this we have to do for every closed ball B contained inside X. So, I have show that for every closed ball B contained inside X, boundary of h of B is equal to h of boundary of B.

Then this means h of the interior of B is interior of h of B, from this what will follow? It will follow that, h is an open mapping from X into $\mathbb{R}^n$ itself, as a map into $\mathbb{R}^n$. Of course map is from X to Y but Y is a subset of $\mathbb{R}^n$. So, this open subset in $\mathbb{R}^n$ because, take any open subset of X which will be the union of interiors of such balls contained inside X. So, union of all these is equal to union of these open sets here. These interior of h(B) s are open subsets of $\mathbb{R}^n$.

Therefore, it will follow that, this whole mapping is open mapping and in particular Y which is equal to h(X) is also open. Because X itself is open to start with. Only when you assume X is open, we will get Y is equal to hX is open. So, all these openness is now happening inside $\mathbb{R}^n$. So, we want to show this, this, this equality, h of boundary of B is equal to h of, a boundary of h of B for every closed ball B contained inside X.

Now, the closed ball is a compact set, h restricted to B to hB is a homeomorphism. Therefore, I can apply the previous theorem for compact subsets. See, the previous theorem 6.8 was for compact subsets. So, that is how this problem for arbitrary X is reduced to compact subsets. So that is a key here.

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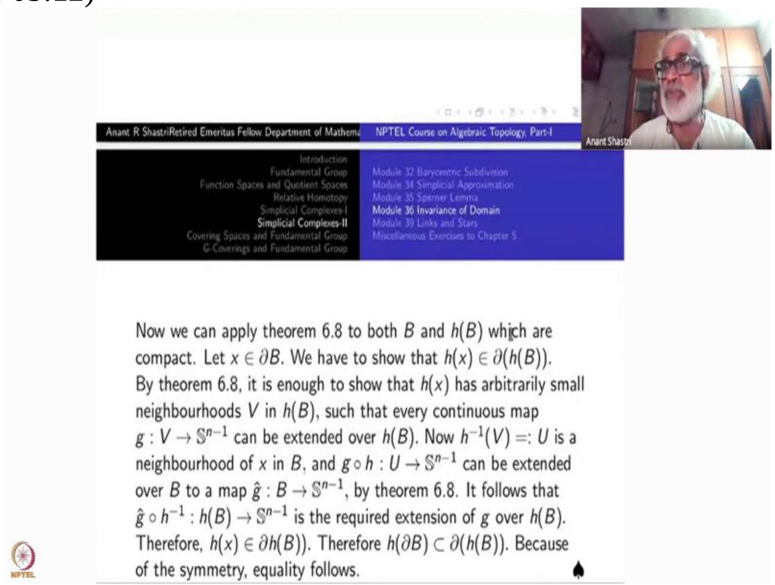

So, now we can apply 6.8 to both B and hB, which are both compact. Because B is compact, hB is also compact. Take a point in the boundary of B, we have to show that hx belongs to the boundary of hB, but the theorem 6.8, it is enough to prove that this hx has arbitrary small neighborhoods V in hB such that, every continuous map g from $f : h(B) \setminus V \to \mathbb{S}^{n-1}$V can be extended to whole of hB. This was a criterion for a boundary point.

You start with a neighborhood V, whatever neighborhood, V should be neighborhood of hx, take a function defined on $h(B) \setminus V$ into $\mathbb{S}^{n-1}$. I must show that, it can be extended over to hB. By using h, I will go back to B now, how? Take h inverse of V equal to U. That will be a neighborhood of x, because V is a neighborhood of hx. And this U is now a subset of our B.

So, via h inverse, here we have, now we have come to B, start with hx inside hB, we have come to B by h inverse of V. There is a neighborhood of x in B, if you take $g = f \circ h : B \setminus U \to \mathbb{S}^{n-1}$ this g can be extended over to B, to a map $\hat{g} : B \to \mathbb{S}^{n-1}$ It follows that, if you go back again, by taking h inverse, i.e., put $\hat{f} = \hat{g} \circ h^{-1}$, then $\hat{f} : h(B) \to \mathbb{S}^{n-1}$ will be an extension of f.

Why we have the extension $\hat{g}$ on B? Because we started X as a boundary point of, boundary, boundary point of B. Therefore, we use the criterion here of 6.8. So, it can be extended. To go back, so that problem is solved, our extension is solved in hB. Therefore, hx must be the boundary.

So, this means started with h of the boundary, is contained as a boundary of hB. But now the argument is symmetric h from B to hB but h inverse is from hB to B. Therefore, I can interchange the B and hB, so I get the equality. So, this complete is a proof of Brouwer's invariance of domain.

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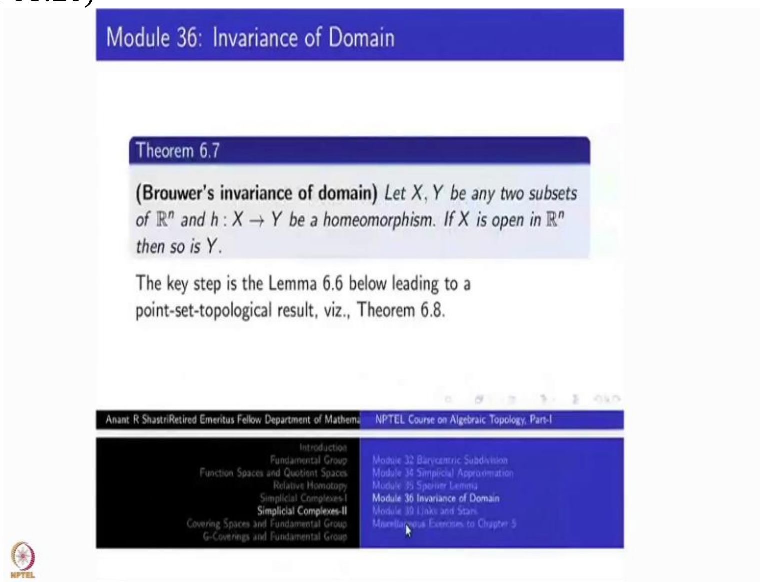

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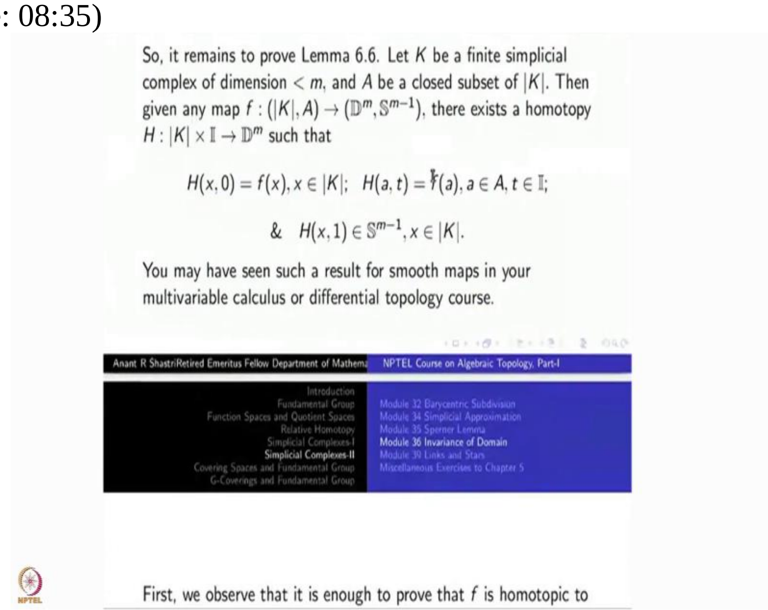

So, it remains to prove, the lemma which is purely homotopy theoretic. I will take this task now. So, we have to work harder here. We have to do, what? We have to do simplicial complexes, approximations and various things here. So, I restate this lemma here. K is a finite simplicial complex of dimension less n than m. A is a closed subset of $|K|$. A map is given $f : (|K|, A) \to (\mathbb{D}^m, \mathbb{S}^{m-1})$. We have to show, there is a homotopy $h : |K| \times \mathbb{I} \to \mathbb{D}^m$ of f

relative to A, such that the end result namely, $h(x,1)$ takes values inside $\mathbb{S}^{m-1}$, on the whole of $|K|$. So, that is what we have to show. Such a result, what I want to say is, you might have seen in a differential topology course, for simply, for smooth approximations. There are such results for smooth approximations. The proof of that and this are not much different, if you have understood that one, this will be easier. If you understand this one, you go back that will be easier for you. So, I am not assuming that you have read that one. But I say that, those things are similar.

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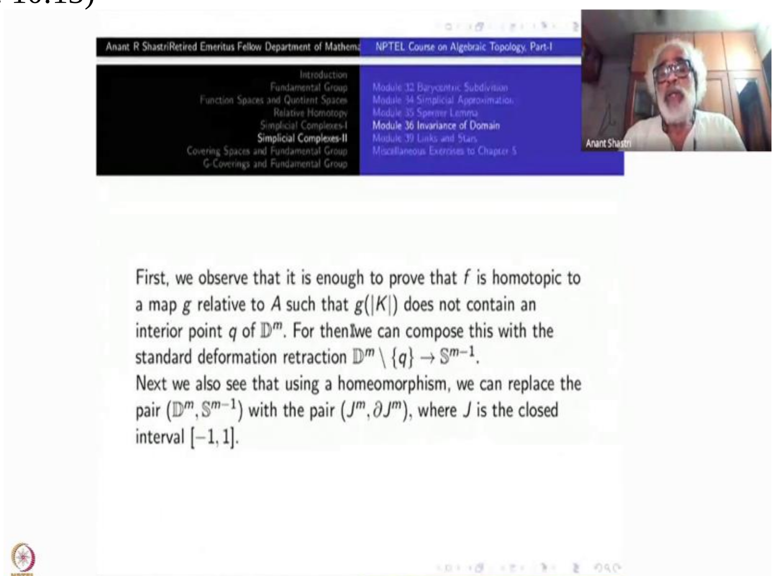

First, we observe that, it is enough to prove that f is homotopic to a map g relative to A such that, the image of g does not contain an interior point $q \in \mathbb{D}^m$. If, one point is missed here, from that point, I can take the radial projection onto the boundary, which is a deformation retract of $\mathbb{D}^m \setminus \{q\}$ onto $\mathbb{S}^{m-1}$. Then I can compose with this radial projection to get a homotopy of the original map to a map into $\mathbb{S}^{m-1}$.

So, instead of directly showing that there is a map into $\mathbb{S}^{m-1}$ which is homotopic to the given f we take do it in two stages. First all, we have to homotope f to a map g. Because everything must be controlled homotopy, i.e., A should not get disturbed, relative homotopy, such that the entire image of the new map g misses at least one point in the interior, specifically, one point in interior $\mathbb{D}^m$, not on the boundary.

Now, just to write down, that writing down the proof becomes easy, instead of the round disc, we would prefer to work with cubes: I take a homeomorphism of this pair $(\mathbb{D}^m, \mathbb{S}^{m-1}) \to (J^m, \partial(J^m))$, where $J = [-1, 1]$ is the closed interval. You could have taken $[0, 1]^m$ as well, no problem. But I am taking this one, this, this is also homeomorphic to the round disc, this we know already. And the boundary of this one will be under that homeomorphism goes to boundary of Sm minus 1. So, take a homeomorphism here, then you do the whole business, by replacing $\mathbb{D}^m$ with J^m, that is all. So, now you have to show that, the map f is relative homotopic to a map g such that $g(|K|)$ does not contain a point of interior of J^m.

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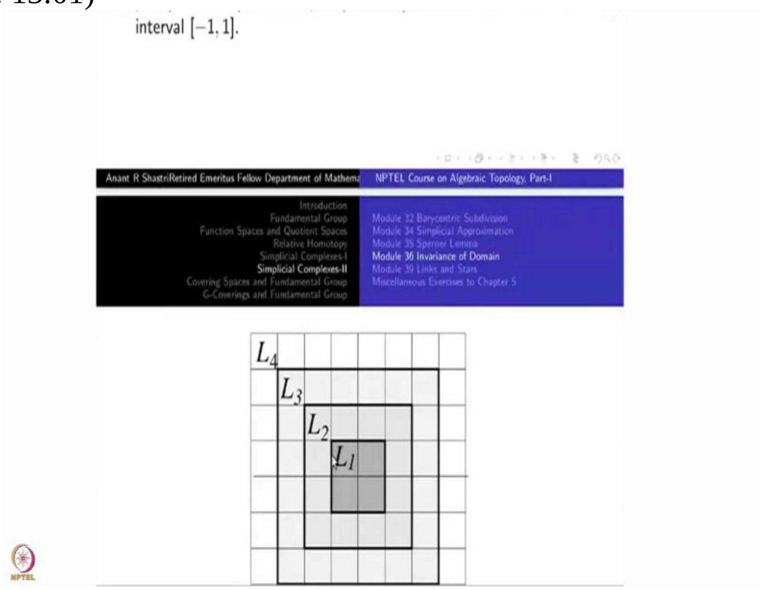

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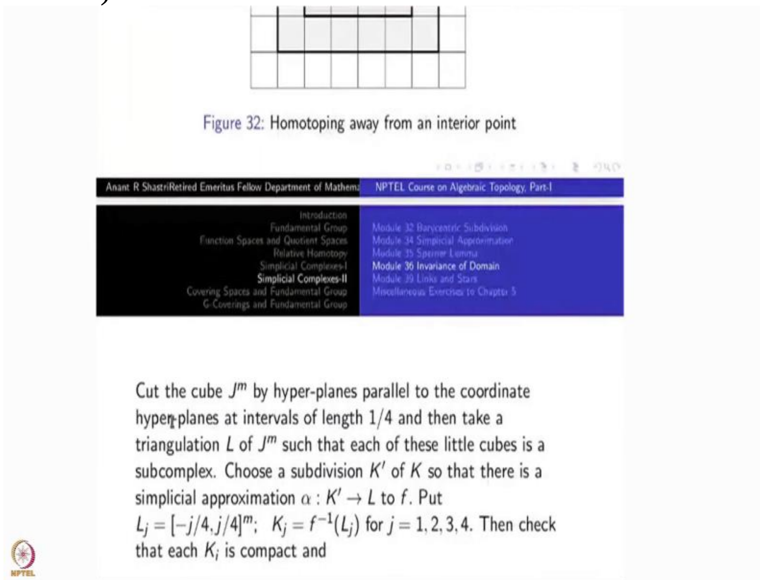

Now here is the picture of what I am going to do. So, let me first establish these notations and come back to the picture later on. Cut to the cube J^m by hyper planes parallel to the coordinate hyper planes at intervals of length ¼. So, from minus 1 to plus 1, the length is 2. So, you will be cutting the interval [-1,1], into eight equal parts like this, eight equal parts. Do it in all the coordinate directions, in all the, in all the planes. That is the meaning of cutting it into little cubes.

This picture is in $\mathbb{R}^2$. There we have only two directions, this horizontal vertical. So, take all the planes, parallel to the parallel to the, parallel to these coordinate planes in all the directions, in all the 1, 2, 3 of X0 directions and cut it into like this square, smaller square, smaller cubes, at equal enter. This is not very crucial, but you have to cut it like this. So, each will be a cube, I cross, it is like J cross, J cross J m times.

Now, you can just cut it each square further into two triangles and so on and make them a simplicial complex. The cubes are they are not simplicial complex. But we know that, any way, J^m can be given a triangulation. So, you take a consistent triangulation for the whole of this one, any triangulation L that you like. So that, the modulus of L is homeomorphic to this picture. All these lines etcetera are immaterial for me so, I have not drawn them here, only I should know that this is a simplicial complex, that is all. Under any simplicial complex just plays for the logical part here, nothing more than that.

The picture, the geometry is precise in this much. So, now what is it I am telling you. So, that triangulation, triangulated thing that simplicial complex I am denoting by L. $|L|$ will be equal to

J^m, such that each of these little cubes becomes sub a complex, the smallest, 8^m of them. Make each of them a subcomlex of L.

Next, choose a subdivision K' of K. Now what is K? K is of the map. We began with

$f : (|K|, A) \to (J^m, \partial(J^m))$. And now I am going to subdivide K, maybe several times by barycentric subdivision, so that I get a simplicial approximation $\alpha : K' \to L$ to the function f. This we know, that we can do this one.

Now, put $L_j = \left[\frac{-j}{4}, \frac{j}{4}\right]^m, j = 1, 2, 3, 4.$ So, now, you have come back to this picture,

$L_1 = \left[\frac{-1}{4}, \frac{1}{4}\right]^2, L_2 = \left[\frac{-1}{2}, \frac{1}{2}\right]^2$ etc, L4 is entire L. So, this is what the picture is L1, L2, L3, L4, L4 is the entire L. So, these will be automatically sub complexes of L contained in the other. So, this is what will help us in the controlled homotopy here.

Put $K_j = f^{-1}(L_j), j = 1, 2, 3, 4.$ Actually, we are now using the notation L_j to denote both the simplicial complex as well as its underlying topological space $|L_j|$, which should not cause any confusion.

So, Kj is f inverse of the underlined topological space. Since f is continuous these K_j are closed subsets, Kj. Since $L_j \subset L_{j+1}$, we have $K_1 \subset \text{int} K_2 \subset K_2 \subset \cdots$ K4 is entire K, whatever to begin with, or whole of K.

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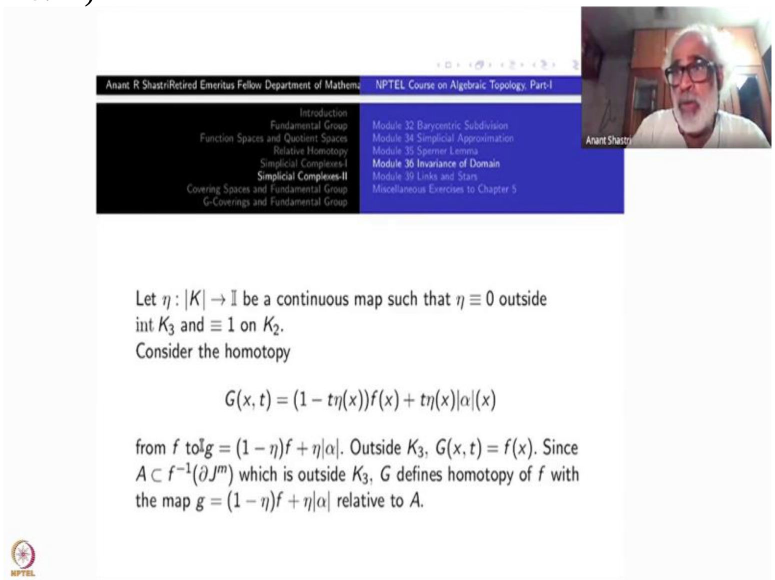

Let eta from K to I, this is not Dm, this is not correct. Let $: |K| \to \mathbb{I}$ be a continuous map such that, eta is 0 on $|K| \setminus \text{int } K_3,$ (which is a closed set) and identically 1 on $K_2.$ So, this is where I am just using that $|K|$ is normal which we have proved. On K2, it is identically 1, and outside this K3 it is 0. So, here is identity on this part, from here to here, it takes the value between 0 and 1.

Now, instead of taking t times f into 1 minus t times mod alpha, as usual, I take a modified homotopy here. $G(x,t) = (1-\eta(x)t)f(x) + \eta(x)t|\alpha|(x).$ This homotopy is, it is again not very strange. Because fx and mod alpha x are in the same simplex, there and this is going to be a convex combination. Between f and its simplicial approximation; you know that they lie inside the same simplex, therefore the convex combination always makes sense. Look here: 1 minus t times eta x is there here. And I am adding t times eta x, sum total is 1.

And eta x takes values between 0 and 1, t also takes value between 0 and 1. So, this is also taking values between 0 and 1. Therefore, this is just a convex combination makes sense and is continuous etc. So, this entire thing will be inside L. So, G is a homotopy. What happens when t is 0? t is 0 this is just 1 and t is 0 this goes away, therefore , this is fx. This is the starting function f. What happens when t equal to 1? Let it be denoted by g: $g(x) = G(x,1) = (1\eta(x))f(x) + \eta(x)|\alpha|(x).$ So, this is our function g, whatever it is, we want to say that this function, misses one of the interior points of L1, in its image. Then we are done. Now, let us analyze how it is done. This is purely pointset topology. Outside K3 what happens to eta? Eta is 0, it is this point 0, this is 0 so, $g(x) = f(x).$ But A is contained inside $f^{-1}(\partial(L)) \subset |K| \setminus K_3.$ Remember that f of A is going inside the boundary, to begin with. The function is from $(|K|, A) \to (J^m, \partial(J^m))$. Therefore, this A is contained in the complement of K3. What is, what is K3? K3 is the inverse image of L3, L3 contains the boundary of L after all. So, look at this picture. So L3, L3 is the this part. So, if something is outside this one, so, it must be coming from here, but this whole of boundary is there. So, that is all I meant. This set theoretic conclusion here.

So, so, we are here since A is outside K3, this whatever we have derived this applies here. Therefore, $G(x,t) = x$ for all x inside A. This means G is a homotopy of f relative to A. So, homotopy we have seen, it is relative to A. This is important. So, what we are left out do be done now, we have to show that this g misses a point in the interior of L1.

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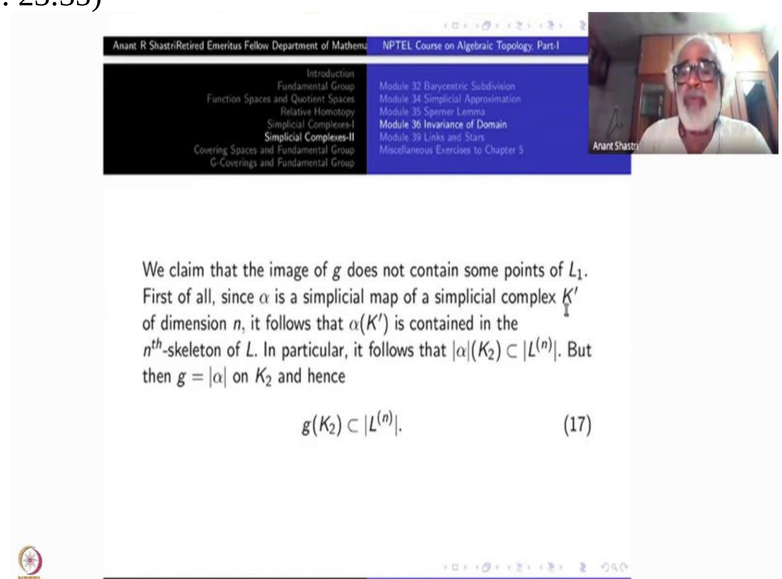

So, we claim that g does not contain some point in the interior of L1. First of all, since alpha is a simplicial map from the simplicial complex K' which is a subdivision of K. We have taken K, to be of dimension n<m . It follows that, the image of alpha is contained in the nth skeleton of L. No simplex here, will be of dimension more than n. Because the same thing applies to our K' so the image cannot be of more dimension than that. That is a simple thing about simplicial maps, because simplicial maps are after all vertex maps.

So nth skeleton of L, in particular it follows that when you pass to $|\alpha|$, $|\alpha|(K_2)$ will be contained in $|L^{(n)}| \cap L_2$. $L^{(n)}$ is the nth skeleton of L. But then what happens here? g is identically alpha on K2, go back here, on K2, eta is 1. So, this is 1 minus t times fx plus t times eta is 1, t times mod alpha x. And on K2 this is what is this one, so what I have what I am talking. Yeah, so, all that I want to say that g is equal to $|\alpha|$ on K2.

On K2, eta is identically 1. Look at this one, eta is identically 1. So, this 1 minus f, this eta is 1. This is 0, this alpha, eta is 1, g is actually mod alpha on K2. Since $|\alpha|(K_2 \subset |L^{(n)}|$, this means $g(K_2) \subset |L^{(n)}| \cap L_2.$ Finally, we wanted to conclude something $g(K)$. So, this is one part. So, I marked it as (17).

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The next part is, suppose x is not in K2, x is in K2 we have taken. Also A intersection K2 is empty. This we have seen, because A is in the complement of K3 . If x in not in K2, f(x) is not in L_2. Therefore both $f(x), |\alpha|(x)$ will belong to $|\sigma|$, where σ is simplex of L and σ does not intersect L_1. I want to say, whatever x is. Where is the picture? Whatever x is, if it is not inside this K2 ,f of x is not inside this one, we mean fx is either here or here.

As soon as fx is here, mod alpha fx will be also inside the corresponding simplex, maybe this simplex is here or simplex is here. It is in one side of simplex which is outside this part, that is all I want. Both mod alpha x and fx will be in this part, in one simplex here, it may be one simplex here, may be one simplex here, maybe some same thing here and so on, one of them, one of the simplex is in particular it will be one of these cubes, that is all I wanted to know.

So, geometry is very much used here. So, fx and alpha fx are contained in a single simplex of L which does not intersect L1 at all, no problem. Therefore, the line segment $[f(x), |\alpha|(x)]$, this entire thing does not intersect L1. It is contained inside some $|\sigma|$ which does not intersect L1. This whole simplex is inside, this whole line is inside this simplex and this does not intersect L1.

Therefore, this entire thing does not intersect L1. Which just means that, $g(x)$ is one of the points here in the line segment. because g is the endpoint of this. gx belongs to this line segment, it follows that gx is not in L1. As soon as x is not in K2, gx is not in L1. Therefore, $g(|K| \setminus K_2) \cap L_1 = \emptyset.$ So, this is, is purely set theory, is it clear?

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Now put (17) and (18) together, what do I get? Any x in K is K2 or is not in K2, there are the two cases. Accordingly, g(x) will belong to $|L^{(n)}| \cap L_2$ or to $L \setminus L_1$. $|L^{(n)}|$ will not cover all of L1, So, not all of L1 will be contained inside the union of these two sets. Therefore, L1 is not contained inside g (K), which just means that the image of g misses at least some point in L1. And all of L1 is in the interior L.

So, this completes the proof of 6.8, hence theorem 6.7 is also completed. So, the, the proofs here are much more educative than the, the statement of theorems. If you read it carefully, so you should learn many, many subtle points here. So, thank you.