Introduction to Algebraic Topology (Part-I)
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Lecture No.-36
Invariance of Domain

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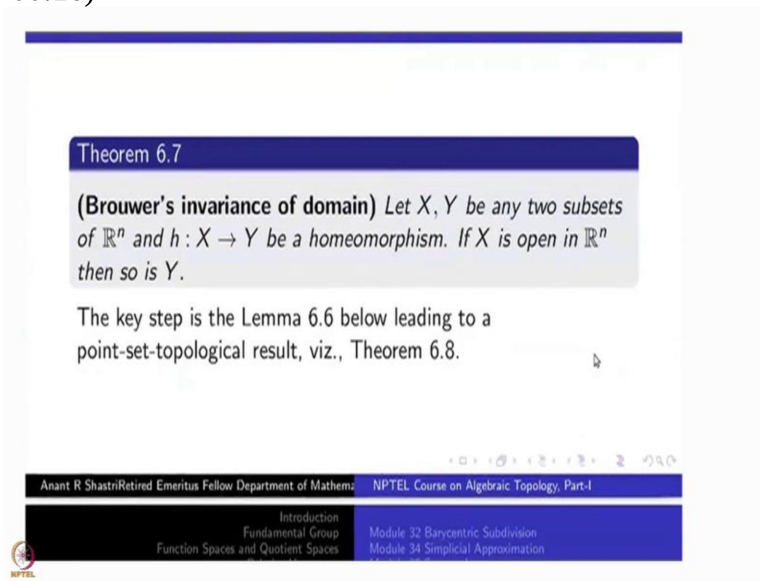

So, today's topic is the big theorem, Bruower's invariance of domain. We have already seen last time, a weaker version of this theorem, namely for n not equal to m, the Euclidian space $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$. This will be a consequence of the big theorem, that what we are going to prove today, namely, if X and Y are two subsets of $\mathbb{R}^n$, homeomorphic to each other and if one of them is open, then the other one is also open.

So, what you can do is think of $\mathbb{R}^n$ as a subspace of $\mathbb{R}^m$, for $n < m$ viz., via the coordinate inclusion $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$. Now you have two subsets, one is $\mathbb{R}^n \times 0 \times \cdots \times 0$ and another is $\mathbb{R}^m$. Both of them are subsets of $\mathbb{R}^m$. If they are homeomorphic to each other, since the whole $\mathbb{R}^m$ is open in $\mathbb{R}^m$, it should follow that $\mathbb{R}^n \times 0 \times \cdots \times 0$ should be also open in $\mathbb{R}^m$. That is very easily seen be not the case. You can easily show that coordinate inclusions are not open sets, they are closed sets if at all but definitely not open sets. It is very easy to see.

Therefore, $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic. So, that will be an easy consequence of this big theorem. However, how have we arrived at it? We arrived at it by showing that $\mathbb{S}^n, \mathbb{S}^m$ are not even homotopy equivalent to each other. This we would not have got from the big theorem, Sn and Sm are not homeomorphic we can prove from this theorem, but they are not even homotopy equivalent to each other, is stronger.

So, that step was achieved by going through Bruower's fixed point theorem and BFT in turn was proved via Sperner lemma. Sperner lemma is not implied by this big theorem. So, you must know there is this justification for proving those results separately. So, let us now prove the big theorem. This proof wil actually use BFT also. However, the key lemma is a strengthening of our homotopy simplicial approximation, namely controlled homotopy lemma. And that will give you a topological criterion for a point to be an interior point of a subset in $\mathbb{R}^n$.

I am talking about a subsets of $\mathbb{R}^n$. Take a subset X of $\mathbb{R}^n$. Whether some point $x \in X$ is an interior point of X or not, we will give you a topological criterion for this question. Topological means what? The criterion will be an invariant of the homeomorphism type of , whereas an interior point is a concept of an embedded object X as a subspace of $\mathbb{R}^n$. So, what is, what is its interior? If you take just a topological space X, just like that, interior of X is the whole X. There is no problem. But X as a subset of $\mathbb{R}^n$ may have different interior.

However, this is independent of how X is contained $\mathbb{R}^n$ is an is the gist of this thing. So, that is what we are going to achieve, which is actually much more stronger than invariance of domain itself, invariance of domain we will be deriving it as a corollary, as a consequence. So, this is the, the general idea and what is achieved in the proof of this one.

So, let us now go step by step. As I told you, the key lemma is whatever say, key step is lemma 6.6 which will lead to another theorem 6.8, I will state it separately because it seems to be much more stronger, in some sense not exactly, than the Brouwer's invariance of domain itself. So, this lemma is the following.

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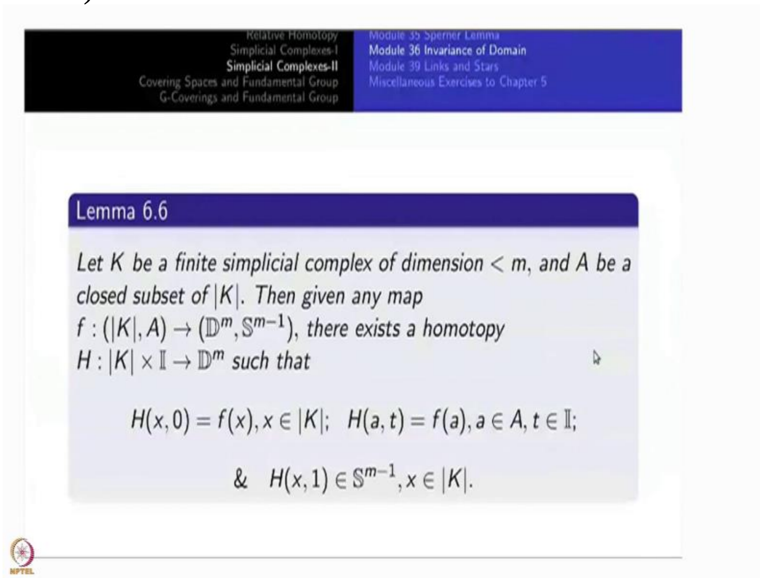

Let K finite simplicial complex, such that its dimension is less than some number m. Take a subset A which is closed inside $|K|$. Given any continuous function $f : |K| \to \mathbb{D}^m$ ($\mathbb{D}^m \subset \mathbb{R}^m$ is the unit disk) such that $f(A) \subset \mathbb{S}^{m-1}$, the lemma says that there exists a homotopy $H : |K| \times \mathbb{I} \to \mathbb{D}^m$ of this function f, namely, $H(x,0) = f(x),\ \forall x \in |K|$, this homotopy is relative to A, which means the points of A are fixed throughout the homotopy and the end result $g(x) = H(x,1)$ tkes values inside $\mathbb{S}^{m-1}$.

In short , you can say that, a function like this, when the dimension of K is smaller than m, such a function can be homotopically pushed into the boundary completely, and the homotopy is relative one , controlled one. So, that is the way to understand this one. The proof of this itself takes some time. So, that we will be postponed for the next time. However, we shall assume this result and go ahead towards proving Brouwer's invariance of domain.

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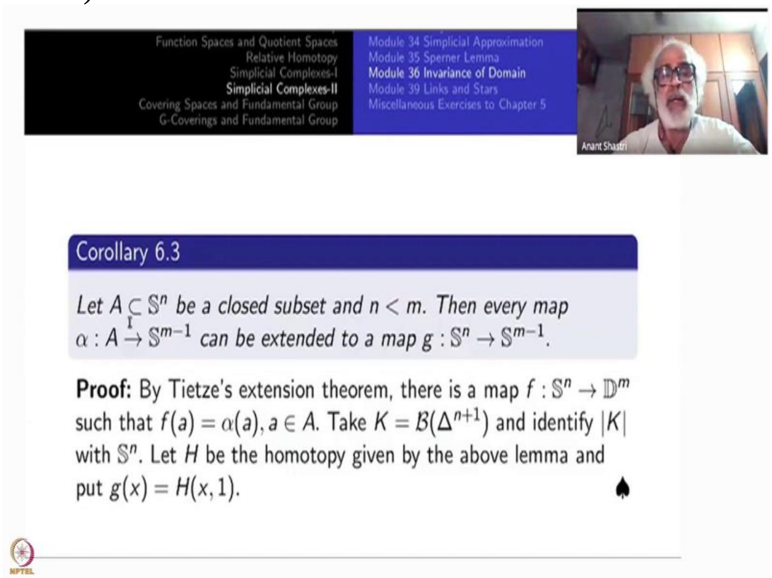

So, the first step is a ready-made homotopy theoretic corollary. Now we will take a subset, A is a close subset of $\mathbb{S}^n$. Assume that this n is less than m (so that m minus 1 is than m minus 1 that you have to understand). Every map $\alpha : A \to \mathbb{S}^{m-1}$ can be extended to a map $\hat{\alpha} : \mathbb{S}^n \to \mathbb{S}^{m-1}$.

Note that if you replace this codomain $\mathbb{S}^{m-1}$ by $\mathbb{I}^m, \mathbb{D}^m, \mathbb{R}^m$ etc. this is nothing but Tietze's extension theorem. But instead of that if you have a round sphere here, this is not an easy thing nor the result so general,viz, you require some condition on m. But this comes immediately as as a corollary to the above theorem. So, this is homotopy theory now, you see, the can be extended.

Like we told you that the homotopy theory, right in the beginning consists of this kind of things extensions and liftings. So, here is the first time you are seeing such an example very clearly. That a map defined on a closed subset, any close subset can be extended to the whole space provided the extending space, $\mathbb{S}^n$ the domain is itself is of smaller dimension than the codomain sphere. Without this condition, it is even false.

In a way, this resembles Tietze's extension theorem, Tietze's extension theorem will be used here, how? You have a map from $\alpha : A \to \mathbb{S}^{m-1}$, you can include it in $\mathbb{D}^m$. So, you have a map $f = i \circ \alpha : A \to \mathbb{D}^m$. Once we are in $\mathbb{D}^m$, we can appeal to Tietze's extension theorem, which gives you an extension to the whole of $\mathbb{S}^n$ because $\mathbb{S}^n$ is normal, that is all.

If you might have seen Tietze's Extension theorem only f for real valued functions, but then this can be, you can take this as various coordinate functions, first of all $\mathbb{D}^m$ can be replaced by $\mathbb{I}^m$.

Then there are m coordinate functions, f_i. To each of them you apply TET to get $\hat{f}_i : \mathbb{S}^n \to \mathbb{I}$. Then you put them together $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_m)$. Then you go back to $\mathbb{D}^m$. So, finally, you conclude that $i \circ \alpha : A \to \mathbb{D}^m$ can be extended to the whole of $\mathbb{S}^n$. That is the first step.

Now, you take $K = \mathcal{B}\Delta_{n+1}$. K as a simplicial complex has $|K| \equiv \mathbb{S}^n$. Therefore, I can apply this lemma, I started with the map here, here instead of f, I have alpha. So, H be a homotopy given by the above lemma, see alpha is given on A but f is an extension, but is taking values inside $\mathbb{D}^m$, to push it back to $\mathbb{S}^{m-1}$ by the homotopy H, that homotopy is a relative homotopy, controlled homotopy. So, it does not change the function on A, which is alpha. Therefore, the result $g(x) = H(x,1)$ is equal to $\alpha(x), x \in A$. So that is the proof of this corollary 6.3.

So, we will use this one now, that any function $A \to \mathbb{S}^m$ here can be extended to the whole of $\mathbb{S}^n$ provided this n here is less than or equal to m minus 1. So, the homotopy theoretic background is now completed. Now, we have to do purely point-set-topology. So, I recall a few things here, just to make sure that we do not have any confusion.

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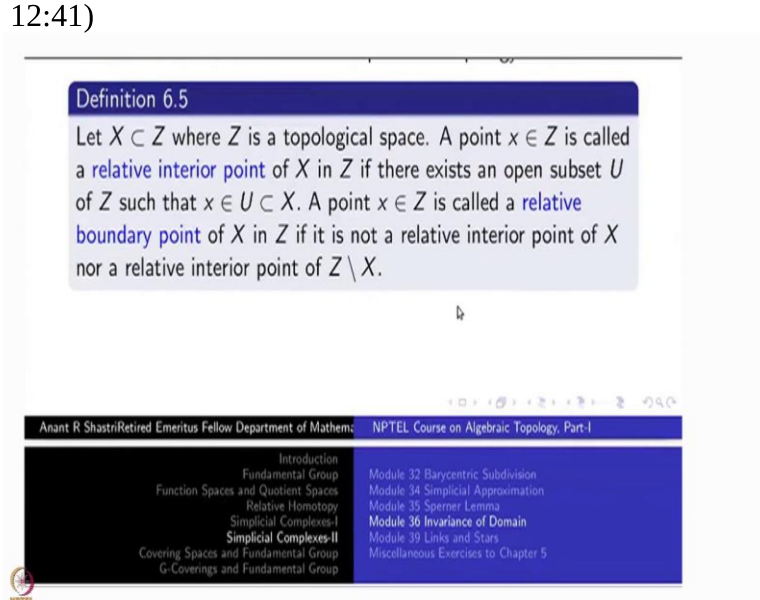

These are elementary things. Take any subset X of a topological space Z. A point x inside X is called a relative interior point of X, if there exists an open subset U of Z, such that

$x \in U \subset X$. Of course, x must be a point of X, if it is an interior point. Not only that, there must be a neighborhood, this neighborhood is not in X, it should be in Z, of x which is contained in X. That is the meaning of relative interior point of X.

A point x belonging to Z is called relative boundary point of X, if it is not a relative interior point of X, nor a relative interior point of the complement, $Z \setminus X$. So, such a thing is called as boundary point, which is the same thing as saying that every open set around x will intersect both X as well as its complement. So, I am just recalling, what is the meaning of boundary point and interior point. The adjuctive `relative' may be redundant but we have put it for emphasis.

Note that we are not looking at manifold theoretic boundary, for example, take $\mathbb{D}^n$ as a manifold. Its boundaries is $\mathbb{S}^{n-1}$. If you include it as a subspace of $\mathbb{R}^n$, then also it is true that this boundary is equal to the relative boundary that we have just defined. That is a different concept of boundary. So, that is not the boundary that we are referring to here, it is relative interior, relative boundary is clearly an embedded notion here.

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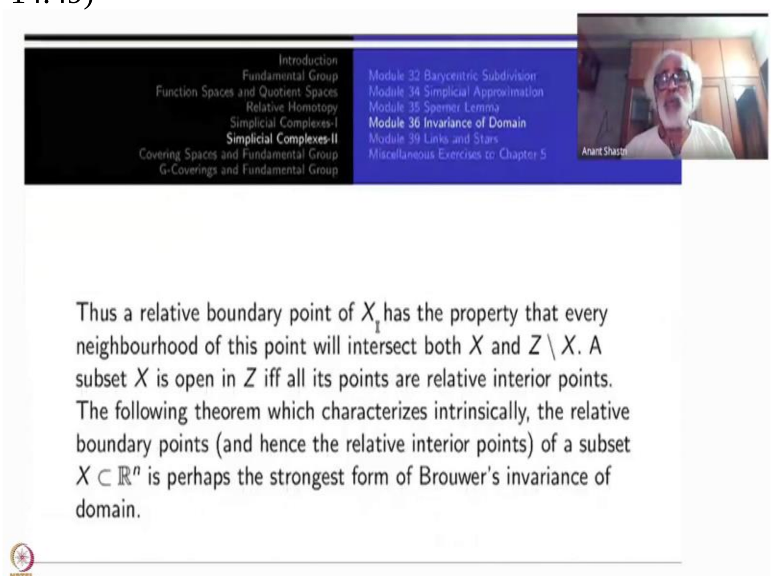

So now, a relative boundary point of X has the property that every neighborhood intersects both X and its complement. That is what I told you. X itself is open in Z if all of its points are relative interior points. That is easy, yeah?

The following theorem characterizes intrinsically the relative boundary points and hence the relative interior points also. Whether you give the characterization for interior points or boundary points, it will be the same, it will be read both of ways, both ways, because it is a characterization of a subset of a Euclidean space. What I am telling you is perhaps the strongest form of Bruower's invariance of domain. Because, this characterization immediately implies Bruower's invariance of domain. Obviously then it is stronger than BID.

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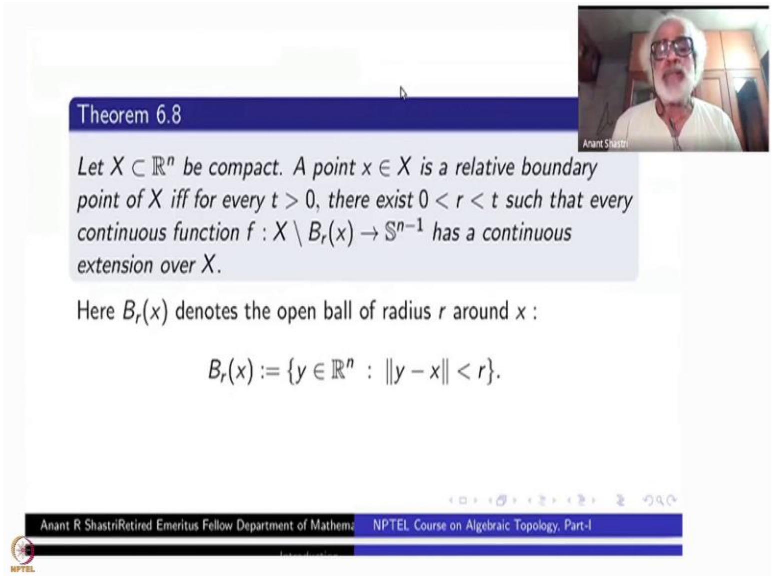

Let us look at the statement of this theorem. Take a compact subset X of $\mathbb{R}^n$. Take a point x in X. It will be a relative boundary point of X, if and only if for every t positive, there exists a r smaller than t, $0 < r < t$, such that every continuous function on the complement of this open ball, $X \setminus B_r(x)$ to $\mathbb{S}^{n-1}$ has a continuous extension over the whole of X. Here, please note that the exponent n is the same.

So, the statement is about the compact subsets, but it can be easily extended to non-compact spaces also. So, let us concentrate on compact set, concentrate on other parts of the theorem rather than why X should be compact and so on. Compactness just helps a bit. So, here $B_r(x)$ is the open ball of radius r around x. So, there are if and only if parts.

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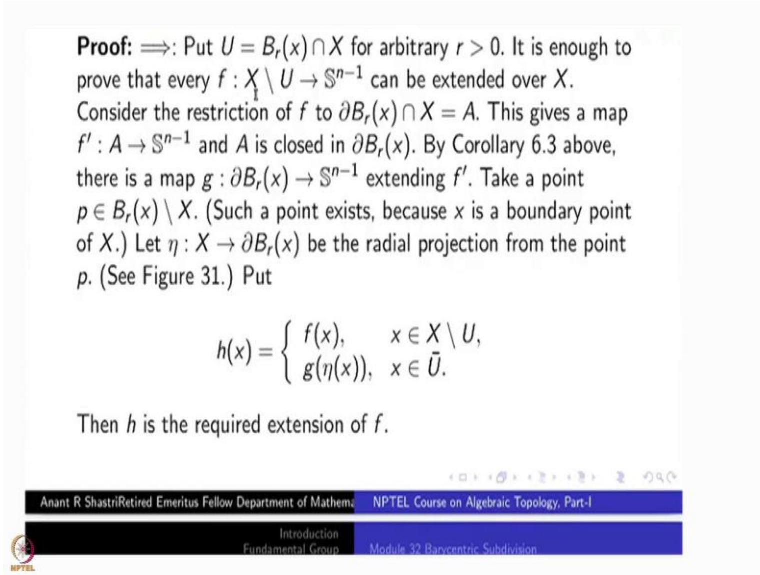

So, I am going to prove the implication here. Take $U = B_r(x) \cap X$. This is just a short notation every time instead of writing this one for some arbitrary r. Take any function $f : X \setminus U \to S^{n-1}$. I will extend it to the whole of X. That is enough because then given any t can choose r to be smaller than that and apply whatever I have done here. So, look at the function f restricted to $\partial(B_r(x)) \cap X =: A$.

Now, this is A is a closed subset of $\partial(B_r(x))$ and this is taking place inside $\mathbb{R}^n$. So, this can be treated as subset of $\mathbb{S}^{n-1}$, a closed subset and this gives a map $f' : A \to \mathbb{S}^{n-1}$. Therefore, by previous corollary, there is a continuous function g :$\partial(B_r(x)) \to \mathbb{S}^{n-1}$which extends f'. on the So, this is the result that we are using here, the corollary 6.3.

So, now, take a point $p \in B_r(x) \setminus X$. How can I say this? Because x is a relative boundary of X, that is what we have assumed and hence every set around x will intersect both X and its complement. So take a point inside here which is not in X. Now, let $\eta : \mathbb{R}^n \setminus \{p\} \to \partial(B_r(x))$ be the radial projection. The radial projections are defined on the entire of $\mathbb{R}^n$ minus p.

What is the meaning of radial projection? Every point v of $\mathbb{R}^n \setminus \{p\}$ lies on a unique ray from the point p and passing through a unique point $u \in \partial(B_r(x))$, somewhat like a polar coordinates with p is the origin. Put $\eta(v) = u$. So, so that eta is the radial projection. After that, all that I do is define

$h : X \to \mathbb{S}^{n-1}$ by this rule: $h(x) = f(x)$ for all points in $X \setminus U$ and $f(x) = g(\eta(x))$ for all points x in the closure of U inside X.

When x belongs to intersection of these sets viz., on $\bar{U} \setminus U \subset \partial(B_r(x))$, then these two will coincide because of what the $\eta(u) = u$ on the boundary. If x is a point on $\partial(B_r(x))$, then $\eta(x) = x$ itself.

So, on the intersection they coincide, both of them are close subsets, therefore, h is a continuous function. And obviously, it is an extension of f. So, what we have done is, every function f defined on the complement of a neighbourhood can be extended to the whole of X, assuming that X is a relative boundary, notice that relative boundary has been used in the choice of the point p.

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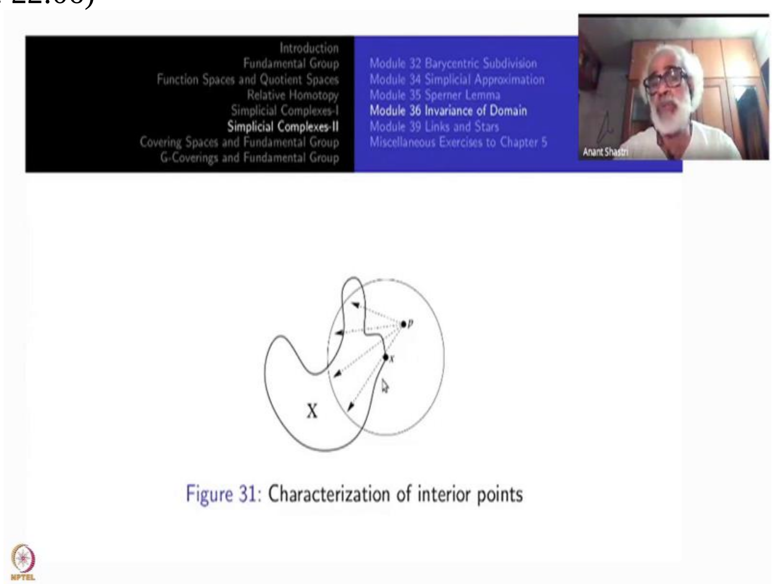

Here is a picture. The point p is an outside point, this is the ball $B_r(x)$. x is a point here in X. Well, I have drawn it on the boundary. So, you may ask oh you have already put it on the boundary. It is a boundary point in what sense? I am using only point set topology here, this open ball will intersect both X as well as complement of X, no matter what r is, x is in the boundary of X, that is all I am using. So, so look at this projection map. And whenever this ball intersects this, this part for example, in both X as well as in the boundary, where does this point go? It will go to this point only. If the point is here, where it will come? It will come to a point here, that is a radial projection. Every point here will come here. So, I am taking this radial projection which is defined on the whole of $\mathbb{R}^n$ minus p restricted to x. So, this will map like this into this one, points here will be mapped here and on in any case the entire thing will be contained inside $\mathbb{S}^{n-1}$. The projection onto this one, this is the picture of the radial projection, so one way we have proved, we have to prove the other way around now.

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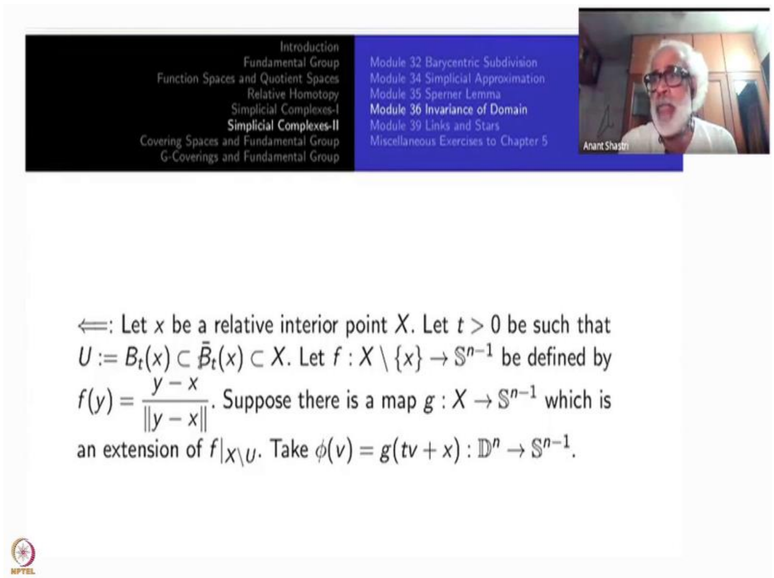

Suppose this criterion is true, then I want to prove that x is a relative boundary point. Instead of that what I am doing is, if it is not relative boundary, it is the point of X not a relative boundary means, it is the relative interior point, then the criterion is false, namely I must find $t > 0$ such that no matter what r, I take, smaller than t, there will be some function which cannot be extended. You have to read the negation of that statement correctly. So, take X to be a relative interior point, let t be any positive number.

Indeed what I am going prove is somewhat stronger. Namely I will give you map $f : \mathbb{R}^n \setminus \{x\} \to \mathbb{S}^{n-1}$. I will choose t>0 such that the close ball $\bar{B}_t(x) \subset X$. Then I will show that for 0<r<t, this map restricted to $X \setminus B_r(x)$ cannot be extended over X. This will prove the claim.

So, let $f(y) = \dfrac{y-x}{\|y-x\|}$. This is again the radial projection, y minus x divided by norm of y minus x. If x is the origin, you would have just taken y divided by norm y. We have studied that one,

$\mathbb{R}^n \to \mathbb{S}^{n-1}$. So, now, because X is treated as an origin, so, you have to say y minus x divided by the norm .

Now suppose $0 < r < t$. The function f clearly restricts to a map $f' : X \setminus B_r(x) \to \mathbb{S}^{n-1}$. It remains to see why we cannot extend it over X continuously.

Suppose, you can extend f' to the whole of X, say there is map $g : X \to \mathbb{S}^{n-1}$ which is an extension of f'. I look at this new map $\phi : \mathbb{D}^n \to \mathbb{S}^{n-1}$ given by $\phi(v) = g(rv + x)$. Take a vector v of modulus less than or equal to 1, multiply by r and add x. You will get a point of the closed ball $\bar{B}_r(x)$. That is contained in X. So we can take $g(rv + x) \in \mathbb{S}^{n-1}$. But now suppose

$\|v\| = 1$. Then $rv + x \in \partial(B_r(x))$. Therefore, $\phi(v) = g(rv + v) = \frac{rv}{\|rv\|} = v.$ This means ϕ is retraction of $\mathbb{D}^n$ onto $\mathbb{S}^{n-1}$, which is absurd, a contradiction to our theorem 6.5 that we have used to prove what? Brouwer's fixed point theorem.

So, we are using Sperner lemma also indirectly here, so we could not have proved this theorem without that.

Thus, we have proved that f restricted to x minus U cannot be extended over X. Extension, if there is such an extension, what we have proved is that there is a contradiction that Dn retracts to its boundary. Now we will take any r less than t, then take V equal to Br of x, then f restricted to v, the same function cannot be extended because even f restricted to the boundary in a smaller thing cannot be extended.

If V is Br of x, then V is a smaller subset than U, X minus U f restricted even that cannot be extended. So, this cannot be extended obviously. If there is an extension of this one, then there will be extension of f of x minus U also, but just now we proved that f of, f restricted of X minus U cannot be extended. So, this completes the proof of the theorem. Let us stop here and we will complete the proof next time.