Introduction to Algebraic Topology (Part-I)
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Lecture No. 35
Sperner Lemma

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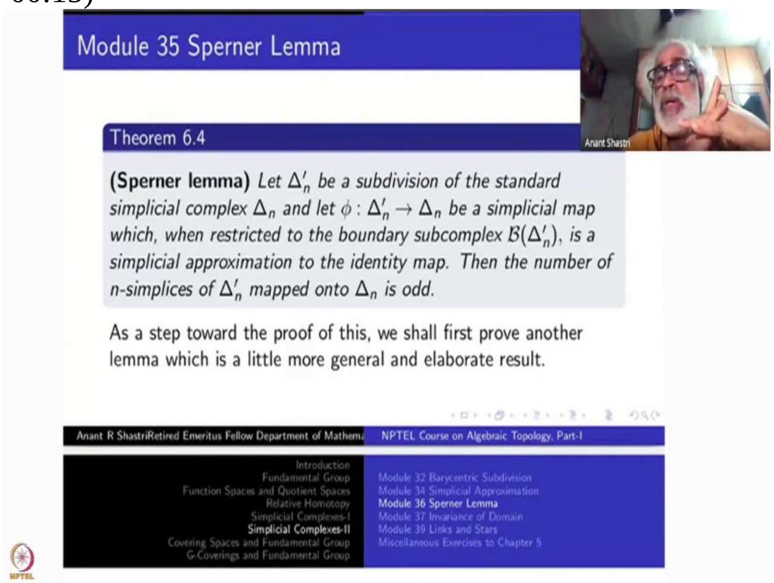

Welcome to Sperner Lemma. Today, we are going to do a purely combinatorial result, one single result. As a result in topology, but a of a combinatorial nature, it has a very distinctive role, its landmark result which opened the floodgates for many other combinatorial results inside topology, and vice versa. It is very simple to state, it is stated for a particular simplicial complex namely, the standard n-simplex. So, that way it is very simple, simplest object as such.

What you have to do is to take any subdivision of the standard simplicial complex Δ_n, and take a simplicial map $\phi : \Delta'_n \to \Delta_n$, where Δ'_n is a subdivision of Δ_n. One more condition you have to assume on this simplicial map. It is a simplicial approximation. Restricted to the boundary complex $\mathcal{B}(\Delta'_n) \to \Delta_n$, remember that boundary complex of Δ_n triangulates the sphere of (n-1) -dimension. Take the inclusion map of $\mathbb{S}^{n-1}$ into $\mathbb{D}^n$. That function is simplicially approximated by ϕ on the boundary, inside the simplex it would be any simplicial map. Then the number of n-simplices of Δ'_n, of the subdivision, which are mapped on to the entire of Δ_n, this number is odd. In particular there is at least 1. If there are 2 then there will be at least 3. This number is odd, that is the meaning of this. In particular, there is at least 1 and that is very important. Instead of proving that this number is at least 1, we are going to prove that it is odd.

So, what is the meaning of that? We are only counting everything mod 2, and then show that it is 1. So, even the counting is done mod 2, we are not bothered about the actual value whether it is in hundreds or thousands. So, this is what we are going to do. The proof is by a combinatorial method known as `counting in different ways'. Two different ways, maybe three different ways and so on. So, I am going to state another version of the same lemma, Sperner lemma, which I am going to state and that will be much more elaborate and that will give the proof of this lemma, once you prove the latter lemma. So, that is the idea of the proof.

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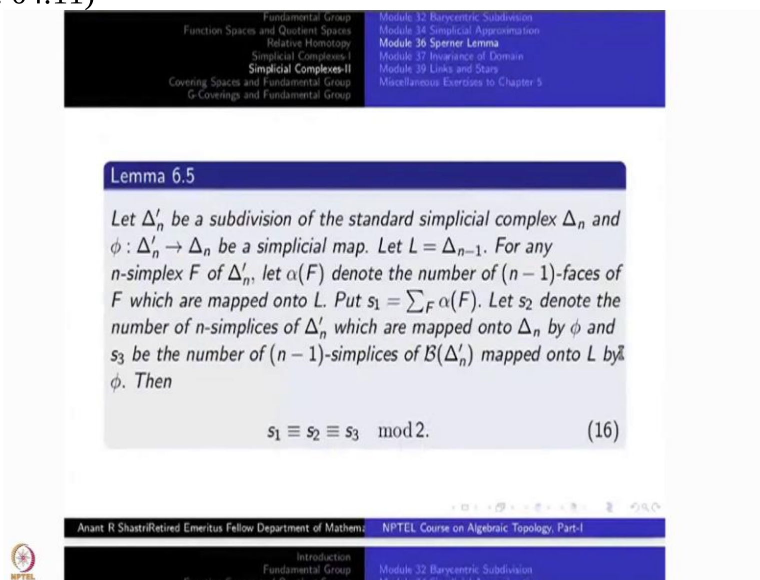

So, here is an expanded version of the same lemma. So, Δ'_n is subdivision of Δ_n and $\phi : \Delta'_n \to \Delta_n$ is a simplicial map, no more conditions. Now I am going denote by L, subcomplex $L := \Delta_{n-1} \subset \Delta_n$. Note that there are (n+1) faces of Δ_n of dimension (n-1). In principle, L could have been chosen to be any one of them.

For any n-simplex F of Δ'_n, I will denote $\alpha(F)$, (I am going to define various numbers here), $\alpha(F)$ is the number of (n-1)-faces of Δ'_n mapped on to L by ϕ. You have fixed one (n-1)-face, of the codomain. For any n-simplex F Δ'_n, let $\alpha(F)$ denote the number of (n-1) faces of F which are mapped onto L. Put s1 equal to the sum of all these $\alpha(F)$'s, where F ranges over all the n-faces of Δ'_n.

Next, let s_2 denote the number of n-simplices F of Δ'_n, (this time n-simplices) which are mapped onto Δ_n by ϕ. Finally, there is another number s_3, the number of (n-1)-simplices of the boundary of Δ'_n which are mapped onto L by ϕ. So, three different numbers, we are counting. The claim is that all these three are equal to each other, if you count mod 2. If any one of them is odd, then other two are also odd; if any one of them is even, then the other two are also even. That is the meaning of this, s_1 is congruent to s_2 congruent to s_3 mod 2.

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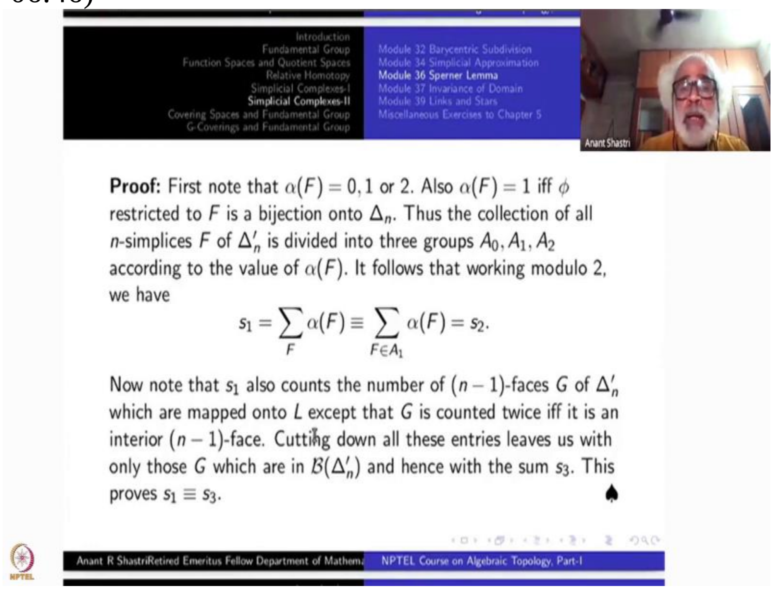

So, let us prove this one, that will lead to Sperner lemma. First note that, fix one n-simplex F in Δ'_n. $\alpha(F)$ will be either 0, nothing is mapped on to L or it could be 1, only one of the (n-1)-simplex goes to L or it could be two of them, nothing more than that. Because you see, what is F? $F$is an n-simplex. So, let us take for example. Put n=1. Then I have to look at the two 0-simplexes, which are the two boundary points. So, these two points, none of them may go to that point, only one of them may go to that point or both of them may go to that point. That is very easy. Next we take a triangle for example, n=2. There are three edges. None of them may go over L only one of them may go over it. If two of them go there, then the third one will have to go to single point, there is no, there is no other choice. The same way, in the general case, if two of the (n-1)- simplexes, are mapped onto L, the rest of the (n-1) simplexes will have to be mapped onto subsets with less than n elements. Therefore, the first thing to observe is that $\alpha(F)$ has only three possible values: $0, 1, 2.$

Now, what is the meaning of $\alpha(F) = 1.$ By relabeling if necessary, we may write : $F = \{u_0, \ldots, u_n\}, \phi(u_i) = e_i, 0 \leq i \leq n-1.$ It follows that $\phi(v_n)$ has no other choice than being equal to e_n. So, $\alpha(F) = 1$ iff $\phi(F) = \Delta_n$.

Now, the collection of all n-simplices F of Δ'_n can be divided into three groups, depending upon the value of $\alpha(F) =$ 0, 1 or 2; say, the three respective collections are $A_0, A_1, A_2.$ It follows that, if you count modulo 2, A_0 and A_2 will not contribute nothing to the sum s_1. Only the class A_1 will contribute, one for each member. So, that is what we have. So, s_1 which is sum of all $\alpha(F)$'s instead of taking all F, you can take only those $F \in A_1$, so each of them will give you 1, sum total is precisely now equal to s_2 mod 2. Because A_1 corresponds to those F which are mapped onto, fully onto Δ_n. That is the definition of s_2. So, we have already proved that s1 is congruent to s_2 modulo 2.

Next, s_1 also counts the number of (n-1)- faces G of Δ'_n which are mapped onto L is some sense. If G is contained in the boundary, then there is only one n-simplex F such that $G \subset F$. Therefore, G is counted at most once. On the other hand, if G is not contained in the boundary of Δ'_n then there will be precisely two n-simplexes F_1, F_2 of Δ'_n such that $G \subset F_i$. So, either G will be counted twice or none at all in the sum $s_1.$ Therefore, cutting down all these entries, leaves us with only those G which are in the boundary of Δ'_n. On the boundary a (n-1)-simplex face occurs as a face of an n-simplex only once. The interior thing will occur exactly in two different places. Two different simplices. This the important geometric fact that is used.

So, when you are counting modulo 2, those things will not contribute anything. So, this last thing tells you that cutting all these entries, the sum total is actually congruent s_3 modulo 2. Thus we have proved that s1 and s_2 and s_3 are congruent to each other.

If you still have difficulties in uderstanding the above argument, what you have to do is, do it for n equal to 1, where it is completely obvious. You work it out, you have to read carefully, I have read it in the beginning at least three times and now I have taught it ten times. So, that is a different thing. So, everybody has to work it out on their own. do it for a single edge, you have to cut it in two number of edges, number of edges because it is subdivision you have to take. And then work out what happened, what is happening to understand these numbers s_1, s_2, s_3.

If you still have doubts, do it for a triangle, the next, next stage. You can see what happens. Beyond that, you do not have to do, it is not easy either. After that you have to do, by that time you must be very, very confirmed that this argument is works. Now only logic will remain there, after that no, no pictures.

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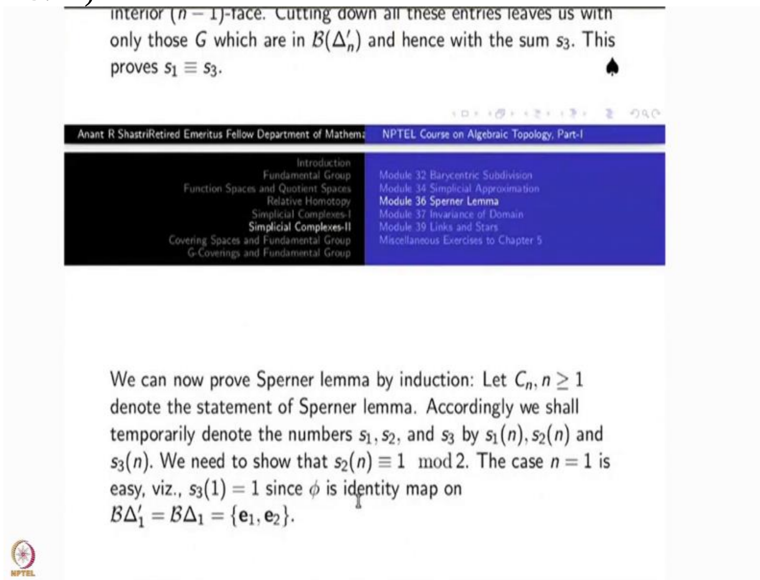
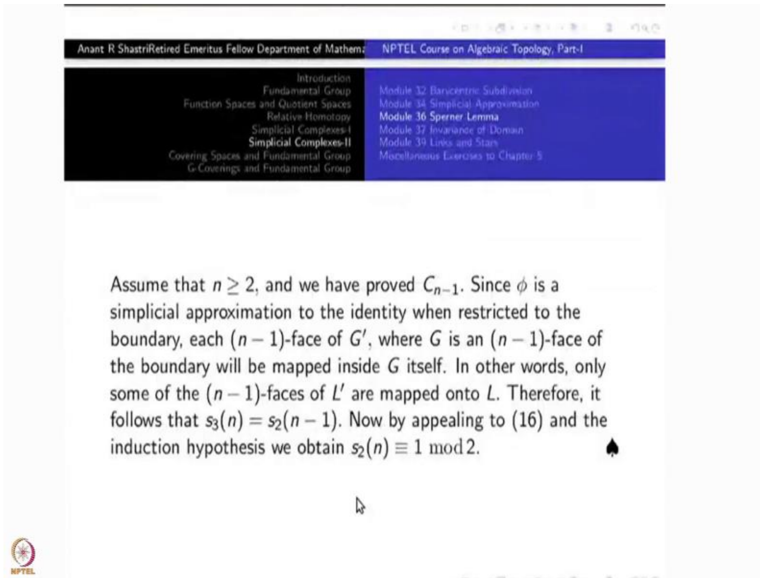


So, now we can prove the Sperner lemma now, by induction. Because we have a passage from Δ_n to Δ_{n-1} through the above lemma. That is a whole idea. So, for n greater that equal to 1, let Cn denote the statement of Sperner lemma. Accordingly, we shall temporarily denote these numbers s_1, s_2, s_3 by $s_1(n), s_2(n), s_3(n)$ respectively. And we want to show that $s_2 \equiv 1$ mod 2. That is the Sperner lemma. The additional hypothesis on ϕ comes into play now.

For $n = 1$, this is very easy, namely s_3 is 1. Since ϕ is a simplicial approximation to the identity map on the boundary, (now you have use this hypothesis) ϕ is actually the identity map on the boundary, in this case. $\phi(e)1) = e_1, \phi(e_2) = e_2$. There is no other choice. Now there are many layman kind of argument to see that s_2 has to odd, you choose your own favourite and we leave it you. (Refer Slide Time: 14:22)



So, now, having shown C(1) true, assume the statement for C(n-1) is true. We shall prove C(n). Since ϕ is the simplicial approximation to the identity when restricted to the boundary, if F is an (n-1)-face of Δ_n and F' is the corrresponding subcomplex of Δ'_n, then each (n-1)-face of G of F' will be mapped inside $|F|$. In particular only some of the (n-1)-faces of L' are mapped on to L by ϕ. Therefore, it follows that $s_3(n) = s_2(n-1)$.

But now, we can restrict $\phi : L' \to L$ and the statement C(n-1). That means we have $s_2(n-1)$ is odd. Therefore, $s_3(n)$ is odd. That implies C(n). I will repeat this part, so what is happening here? L is the (n-1)-simplex Δ_{n-1}. L' is its subdivision indices by $\mathcal{B}(\Delta'_n) \subset \Delta'_n$. Since ϕ is a simplicial approximation to identity on $\mathcal{B}(\Delta'_n)$ it restricts to a simplicial map $\phi : L' \to L$. Moreover this restricted ϕ is a simplicial approximation to Id_L itself. Therefore, we can apply C(n-1) and conclude that $s_2(n-1)$ is odd. But the above argument shows that $s_3(n)$ is actually equal to $s_2(n-1)$ for the function ϕ restricted to L'.

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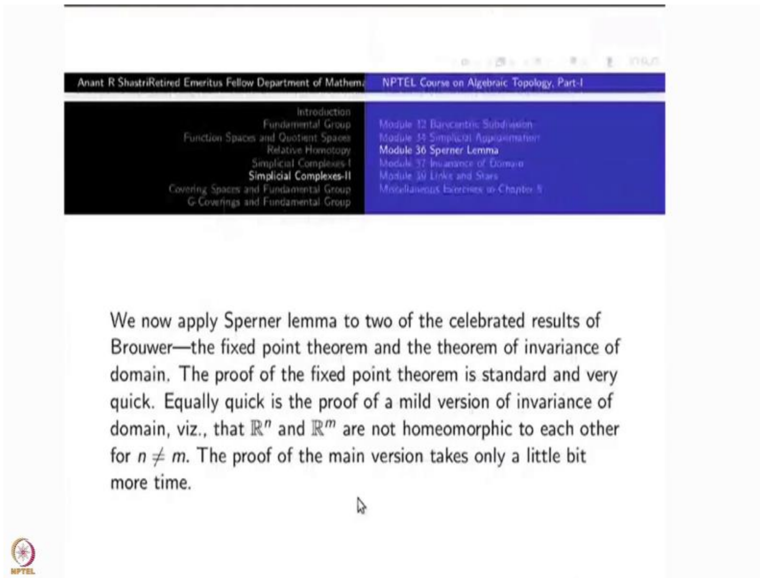

So, Sperner lemma, can applied now to get many results. Our aim is to prove the Brouwer's invariance of domain. But, first we can and shall prove Brouwer's fixed point theorem for general case. We had proved it for n equal to 2. So, let us, let us be done with that.

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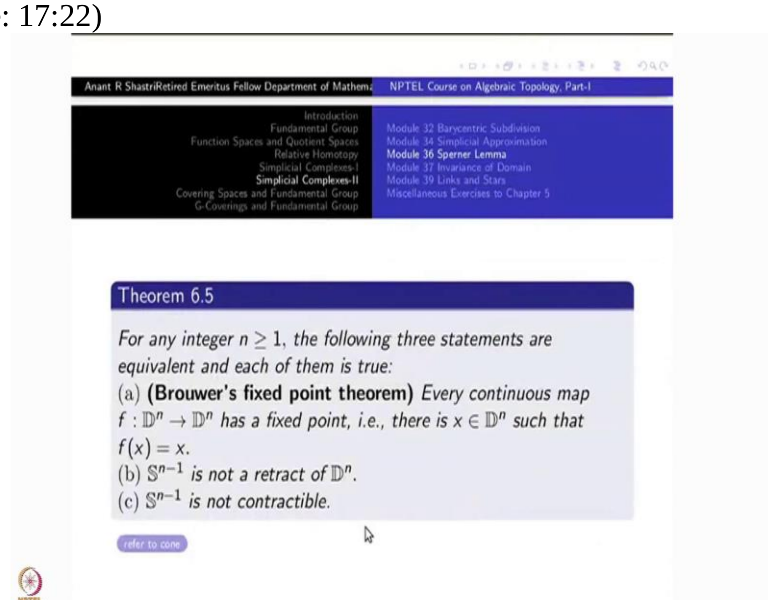

For any integer n greater than equal to 1, these three statements are equivalent. I think I have done it for n equal to 2, but the proof is exactly the same in the general case. So, I will repeat it here. First statement is Brouwer's fixed point theorem. Every continuous map on the closed unit disc to unit disc has a fixed point. So, the second statement is, the boundary $\mathbb{S}^{n-1}$ of $\mathbb{D}^n$ is not a retract of $\mathbb{D}^n$. The third statement is, $\mathbb{S}^{n-1}$ cannot be contractible.

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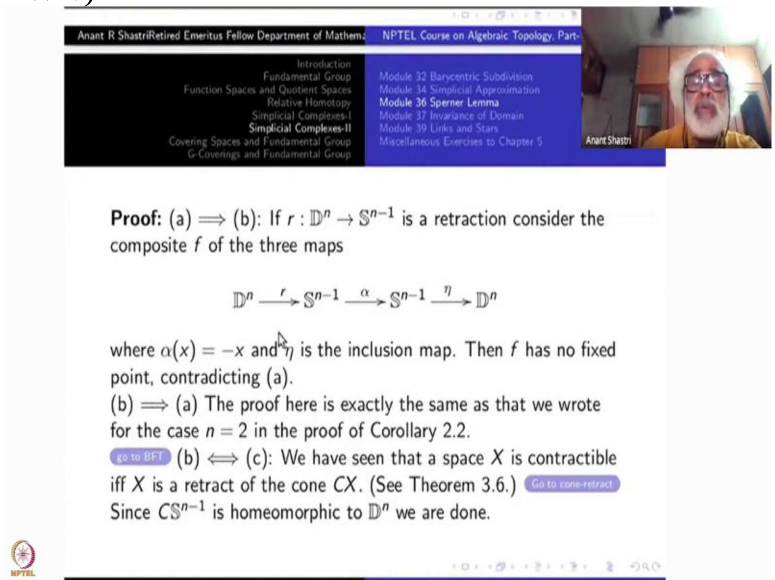

You might have forgotten it maybe, so, we will repeat this proof quickly. So, how (a) implies (b)? What we are trying to prove? If not (b) then we will say not (a). Suppose $\mathbb{S}^{n-1}$ is a retract of $\mathbb{D}^n$. If r is a retraction, then we will get a contradiction to (a). If r is a retraction, retraction means what? a continuous map such that on the boundary it is identity map. So, take $\alpha : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ to be the antipodal map and look at $f = \alpha \circ r$. If x is in the boundary, then $f(x) = \alpha \circ r(x) = \alpha(x) = -x$. If x is in the interior of $\mathbb{D}^n$, then clearly, $f(x) = -r(x)$ is in the boundary and hence cannot be equal to x.

Therefore f has no fixed point, contradiction to (a). So, if there is a retraction, that there is a map with no fixed points.

If you take any point on the inside the boundary, inside, in the interior it has gone already to some point in the boundary. Again, under alpha it goes to a boundary point under eta also it goes to boundary point. But the original point was in the interior, so those two cannot be equal anyway, the chance was only in the boundary, but the boundary points goes, x goes to -x. So, that is all. So, there is no fixed point at all. That is the contradiction.

Now let us prove (b) implies (a). the proof is exactly the same as that we wrote for n equal to 2. So, what we have to do? We make a picture draw the line joining x and $f(x)$, extended towards $f(x)$, get the point $g(x)$; that will be a retraction. So, that is what we have done.

Now let us prove (b) and (c) are equivalent. We have seen that a space X is a contractible, if and only if X is a retract of the cone CX. This is one of the theorems we have proved.

The cone over $\mathbb{S}^{n-1}$ is $\mathbb{D}^n$ So, what does (b) say? X is the boundary $\mathbb{S}^{n-1}$, and $\mathbb{D}^n$ is the cone CX. (b) says X is a retract of its cone, i.,e., $\mathbb{S}^{n-1}$ is a retract of the cone over it, then X is contractible. So, so here $\mathbb{S}^{n-1}$ is not contractible, therefore $\mathbb{S}^{n-1}$ is not a retract of $\mathbb{D}^n$. The same thing as saying that $\mathbb{S}^{n-1}$ is contractible then $\mathbb{S}^{n-1}$ is not a retract. Therefore, (b) gives (c) and and conversely.

Now how we are going to use this to prove Brouwer's fixed point theorem itself. The above theorem says only that the three statements are equivalent. I have not proved anyone of them. We do not know whether any one of them is true. If you prove any one of them, then all the three gets proved.

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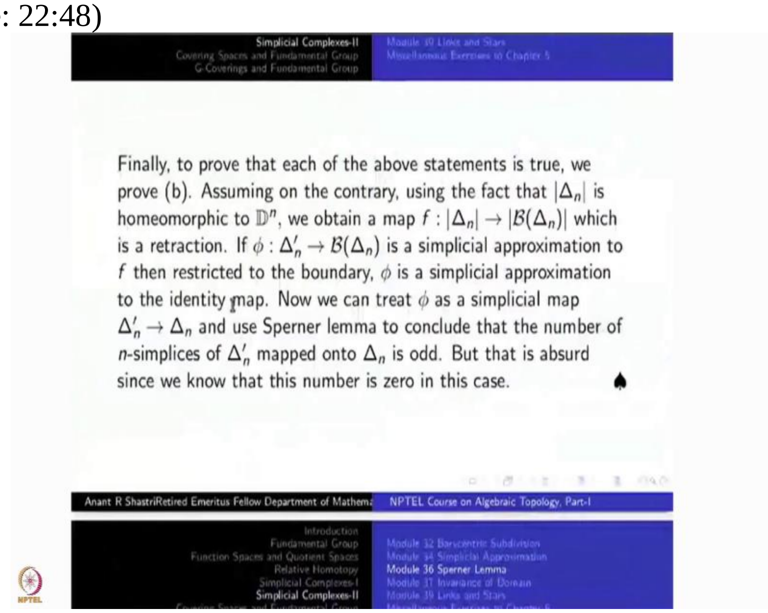

So, finally here, to prove that each of the statement is true, we shall prove (b) that is, there is no retraction from $\mathbb{D}^n$ onto $\mathbb{S}^{n-1}$. If there is one, $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$, take a simplicial approximation to it, $\phi : \Delta'_n \to \Delta_n$. Apply Sperner lemma. Sperner lemma says that, the number of simplices in Δ'_n mapped onto the whole of Δ_n is actually odd. But since r is a retraction, the whole of $\phi(\Delta'_n) \subset \mathcal{B}(\Delta_n)$, the boundary complex. Therefore, no n-simplex would have been mapped onto Δ_n.

So, Sperner lemma gives you immediately that there is no retraction, of the entire disk on to the boundary. Therefore, Brouwer's fixed point theorem is also proved.

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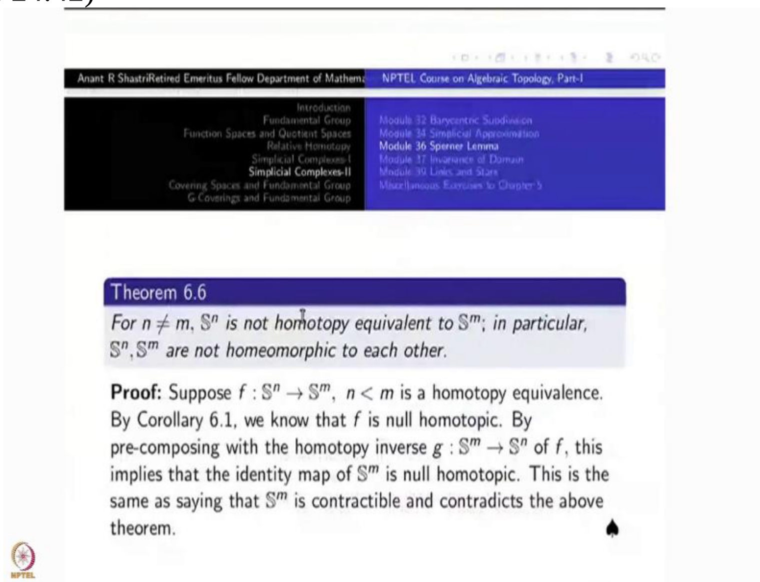

Now, we will prove the simpler version of Bruower's invariance of domain. Many books call this homeomorphic Brouwer's invariance of domain. What it says? It says that $\mathbb{R}^n, \mathbb{R}^m, n \neq m$ cannot be homeomorphic to each other. That is statement of the next corollary.

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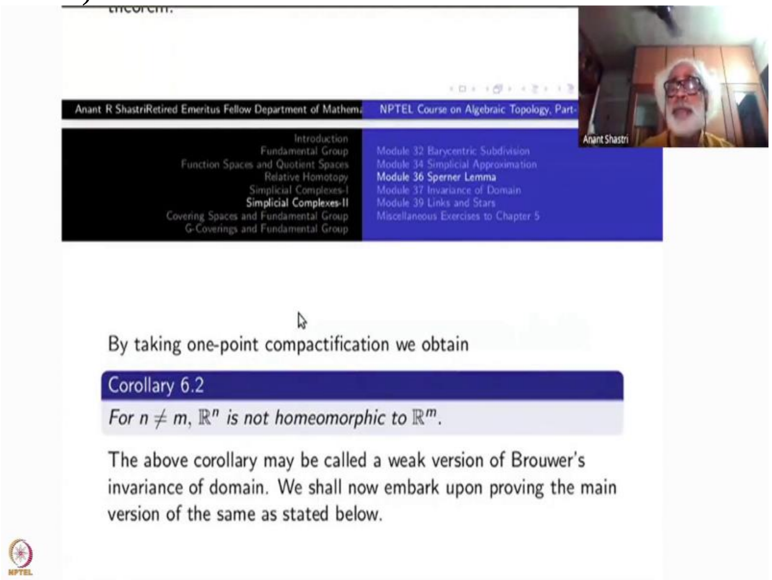

How do we prove that? Suppose there is a homeomorphism, then you know that you can take one-point compactifications of them, they will be also homeomorphic to each other. What is the one point compactification of $\mathbb{R}^n$? It is $\mathbb{S}^n$. For $\mathbb{R}^m$ it is $\mathbb{S}^m$. Now $m \neq n$ and we have got a homeomorphism between them, $f : \mathbb{S}^n \to \mathbb{S}^m$. In particular, $f : \mathbb{S}^n \to \mathbb{S}^m$ will be a homotopy equivalence. This is the theorem that says that this is not possible. So that will complete the proof of Bruower's invariance of domain, the next theorem.

How does one proved this? Without loss of generality, we can assume that n is less than m, by interchanging the n and m, if necessary. Now, we have already proved that if n is less than m , then any map as above is null homotopic. So, this f is null homotopic. By pre- composing with a homotopy inverse, $g : \mathbb{S}^m \to \mathbb{S}^n$, you have $h = g \circ f : \mathbb{S}^n \to \mathbb{S}^n$ which is homotopic to $Id_{\mathbb{S}^n}$.

But f is homotopic to a constant map. T$h = g \circ f$ is also homotopic to a constant map. Therefore, the identity map of $\mathbb{S}^n$ is null-homotopic.

But we have seen earlier that for any space X, if the identity map Id_X is null homotopic then X is contractible. Now, part (c) of the previous lemma tells you that, since you have proved (a) already, none of the spheres is contractible.

So, I repeat because of this theorem. What we have proved is, two spheres of different dimensions cannot be of the same homotopic type. In particular they cannot be homeomorphic. From that it follows that the corresponding euclidean spaces cannot be homeomorphic.

So, when this was proved first time, it was a very great result, a landmark result. How Brouwer proved it, more or less through his invention of homology. He had different proofs of this also, by the way. It is proved by using some dimension theory and so on. All these proofs are much, very, very much more complicated.

So, nowadays, homology theory gives you the standard proof of this one. So, what I got is, I got you this one by just using simplicial approximation and Sperner lemma. There is a stronger version. I have obtained you this a weaker version. What is the stronger version? That says that, if you take an open subset of $\mathbb{R}^n$ non empty open set and suppose this is homeomorphic to another subset of $\mathbb{R}^n$. The homeomophism is only for subset to subset and not necessarily defined on the entire of $\mathbb{R}^n$. Suppose one of them is open in $\mathbb{R}^n$. Then the other one is also open inside $\mathbb{R}^n$. So, that is called actually invariance of domain. Remember, domain was the word used for open and connected subsets, in analysis. So, that will be our next task, proving the full full version of invariance of domain. So, we will take it up next time. Thank you.