

**Introduction to Algebraic Topology (Part-I)**  
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**Lecture No. 35**  
**Sperner Lemma**

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Module 35 Sperner Lemma

Anant Shastri

**Theorem 6.4**

**(Sperner lemma)** Let  $\Delta'_n$  be a subdivision of the standard simplicial complex  $\Delta_n$  and let  $\phi : \Delta'_n \rightarrow \Delta_n$  be a simplicial map which, when restricted to the boundary subcomplex  $\mathcal{B}(\Delta'_n)$ , is a simplicial approximation to the identity map. Then the number of  $n$ -simplices of  $\Delta'_n$  mapped onto  $\Delta_n$  is odd.

As a step toward the proof of this, we shall first prove another lemma which is a little more general and elaborate result.

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Welcome to Sperner Lemma. Today, we are going to do a purely combinatorial result, one single result. As a result in topology, but a of a combinatorial nature, it has a very distinctive role, its landmark result which opened the floodgates for many other combinatorial results inside topology, and vice versa. It is very simple to state, it is stated for a particular simplicial complex namely, the standard  $n$ -simplex. So, that way it is very simple, simplest object as such.

What you have to do is to take any subdivision of the standard simplicial complex  $\Delta_n$ , and take a simplicial map  $\phi : \Delta'_n \rightarrow \Delta_n$ , where  $\Delta'_n$  is a subdivision of  $\Delta_n$ . One more condition you have to assume on this simplicial map. It is a simplicial approximation. Restricted to the boundary complex  $\mathcal{B}(\Delta'_n) \rightarrow \Delta_n$ , remember that boundary complex of  $\Delta_n$  triangulates the sphere of  $(n-1)$ -dimension. Take the inclusion map of  $\mathbb{S}^{n-1}$  into  $\mathbb{D}^n$ . That function is simplicially approximated by  $\phi$  on the boundary, inside the simplex it would be any simplicial map. Then the number of  $n$ -simplices of  $\Delta'_n$ , of the subdivision, which are mapped on to the entire of  $\Delta_n$ , this number is odd. In particular there is at least 1. If there are 2 then there will be at least 3. This number is odd, that is the meaning of this. In particular, there is at least 1 and that is very important. Instead of proving that this number is at least 1, we are going to prove that it is odd.

So, what is the meaning of that? We are only counting everything mod 2, and then show that it is 1. So, even the counting is done mod 2, we are not bothered about the actual value whether it is in hundreds or thousands. So, this is what we are going to do. The proof is by a combinatorial method known as 'counting in different ways'. Two different ways, maybe three different ways and so on. So, I am going to state another version of the same lemma, Sperner lemma, which I am going to state and that will be much more elaborate and that will give the proof of this lemma, once you prove the latter lemma. So, that is the idea of the proof.

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**Lemma 6.5**

Let  $\Delta'_n$  be a subdivision of the standard simplicial complex  $\Delta_n$  and  $\phi : \Delta'_n \rightarrow \Delta_n$  be a simplicial map. Let  $L = \Delta_{n-1}$ . For any  $n$ -simplex  $F$  of  $\Delta'_n$ , let  $\alpha(F)$  denote the number of  $(n-1)$ -faces of  $F$  which are mapped onto  $L$ . Put  $s_1 = \sum_F \alpha(F)$ . Let  $s_2$  denote the number of  $n$ -simplices of  $\Delta'_n$  which are mapped onto  $\Delta_n$  by  $\phi$  and  $s_3$  be the number of  $(n-1)$ -simplices of  $B(\Delta'_n)$  mapped onto  $L$  by  $\phi$ . Then

$$s_1 \equiv s_2 \equiv s_3 \pmod{2}. \quad (16)$$

At the bottom of the slide, it says: Anant R Shastri Retired Emeritus Fellow Department of Mathematics, NPTEL Course on Algebraic Topology, Part-I. Below that is another table of contents: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Module 32: Barycentric Subdivision, Module 33: Simplicial Approximation.

So, here is an expanded version of the same lemma. So,  $\Delta'_n$  is subdivision of  $\Delta_n$  and  $\phi : \Delta'_n \rightarrow \Delta_n$  is a simplicial map, no more conditions. Now I am going to denote by  $L$ , subcomplex  $L := \Delta_{n-1} \subset \Delta_n$ . Note that there are  $(n+1)$  faces of  $\Delta_n$  of dimension  $(n-1)$ . In principle,  $L$  could have been chosen to be any one of them.

For any  $n$ -simplex  $F$  of  $\Delta'_n$ , I will denote  $\alpha(F)$ , (I am going to define various numbers here),  $\alpha(F)$  is the number of  $(n-1)$ -faces of  $\Delta'_n$  mapped on to  $L$  by  $\phi$ . You have fixed one  $(n-1)$ -face, of the codomain. For any  $n$ -simplex  $F$  of  $\Delta'_n$ , let  $\alpha(F)$  denote the number of  $(n-1)$  faces of  $F$  which are mapped onto  $L$ . Put  $s_1$  equal to the sum of all these  $\alpha(F)$ 's, where  $F$  ranges over all the  $n$ -faces of  $\Delta'_n$ .

Next, let  $s_2$  denote the number of  $n$ -simplices  $F$  of  $\Delta'_n$ , (this time  $n$ -simplices) which are mapped onto  $\Delta_n$  by  $\phi$ . Finally, there is another number  $s_3$ , the number of  $(n-1)$ -simplices of the boundary

of  $\Delta'_n$  which are mapped onto  $L$  by  $\phi$ . So, three different numbers, we are counting. The claim is that all these three are equal to each other, if you count mod 2. If any one of them is odd, then other two are also odd; if any one of them is even, then the other two are also even. That is the meaning of this,  $s_1$  is congruent to  $s_2$  congruent to  $s_3$  mod 2.

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**Proof:** First note that  $\alpha(F) = 0, 1$  or  $2$ . Also  $\alpha(F) = 1$  iff  $\phi$  restricted to  $F$  is a bijection onto  $\Delta_n$ . Thus the collection of all  $n$ -simplices  $F$  of  $\Delta'_n$  is divided into three groups  $A_0, A_1, A_2$  according to the value of  $\alpha(F)$ . It follows that working modulo 2, we have

$$s_1 = \sum_F \alpha(F) \equiv \sum_{F \in A_1} \alpha(F) = s_2.$$

Now note that  $s_1$  also counts the number of  $(n-1)$ -faces  $G$  of  $\Delta'_n$  which are mapped onto  $L$  except that  $G$  is counted twice iff it is an interior  $(n-1)$ -face. Cutting down all these entries leaves us with only those  $G$  which are in  $B(\Delta'_n)$  and hence with the sum  $s_3$ . This proves  $s_1 \equiv s_3$ .

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So, let us prove this one, that will lead to Sperner lemma. First note that, fix one  $n$ -simplex  $F$  in  $\Delta'_n$ .  $\alpha(F)$  will be either 0, nothing is mapped on to  $L$  or it could be 1, only one of the  $(n-1)$ -simplex goes to  $L$  or it could be two of them, nothing more than that. Because you see, what is  $F$ ?  $F$  is an  $n$ -simplex. So, let us take for example. Put  $n=1$ . Then I have to look at the two 0-simplexes, which are the two boundary points. So, these two points, none of them may go to that point, only one of them may go to that point or both of them may go to that point. That is very easy. Next we take a triangle for example,  $n=2$ . There are three edges. None of them may go over  $L$  only one of them may go over it. If two of them go there, then the third one will have to go to single point, there is no, there is no other choice. The same way, in the general case, if two of the  $(n-1)$ -simplexes, are mapped onto  $L$ , the rest of the  $(n-1)$  simplexes will have to be mapped onto subsets with less than  $n$  elements. Therefore, the first thing to observe is that  $\alpha(F)$  has only three possible values: 0, 1, 2.

Now, what is the meaning of  $\alpha(F) = 1$ . By relabeling if necessary, we may write :  $F = \{u_0, \dots, u_n\}, \phi(u_i) = e_i, 0 \leq i \leq n-1$ . It follows that  $\phi(v_n)$  has no other choice than being equal to  $e_n$ . So,  $\alpha(F) = 1$  iff  $\phi(F) = \Delta_n$ .

Now, the collection of all  $n$ -simplices  $F$  of  $\Delta'_n$  can be divided into three groups, depending upon the value of  $\alpha(F) = 0, 1$  or  $2$ ; say, the three respective collections are  $A_0, A_1, A_2$ . It follows that, if you count modulo  $2$ ,  $A_0$  and  $A_2$  will not contribute anything to the sum  $s_1$ . Only the class  $A_1$  will contribute, one for each member. So, that is what we have. So,  $s_1$  which is sum of all  $\alpha(F)$ 's instead of taking all  $F$ , you can take only those  $F \in A_1$ , so each of them will give you  $1$ , sum total is precisely now equal to  $s_2 \pmod{2}$ . Because  $A_1$  corresponds to those  $F$  which are mapped onto, fully onto  $\Delta_n$ . That is the definition of  $s_2$ . So, we have already proved that  $s_1$  is congruent to  $s_2$  modulo  $2$ .

Next,  $s_1$  also counts the number of  $(n-1)$ -faces  $G$  of  $\Delta'_n$  which are mapped onto  $L$  in some sense. If  $G$  is contained in the boundary, then there is only one  $n$ -simplex  $F$  such that  $G \subset F$ . Therefore,  $G$  is counted at most once. On the other hand, if  $G$  is not contained in the boundary of  $\Delta'_n$  then there will be precisely two  $n$ -simplexes  $F_1, F_2$  of  $\Delta'_n$  such that  $G \subset F_i$ . So, either  $G$  will be counted twice or none at all in the sum  $s_1$ . Therefore, cutting down all these entries, leaves us with only those  $G$  which are in the boundary of  $\Delta'_n$ . On the boundary a  $(n-1)$ -simplex face occurs as a face of an  $n$ -simplex only once. The interior thing will occur exactly in two different places. Two different simplices. This the important geometric fact that is used.

So, when you are counting modulo  $2$ , those things will not contribute anything. So, this last thing tells you that cutting all these entries, the sum total is actually congruent  $s_3$  modulo  $2$ . Thus we have proved that  $s_1$  and  $s_2$  and  $s_3$  are congruent to each other.

If you still have difficulties in understanding the above argument, what you have to do is, do it for  $n$  equal to  $1$ , where it is completely obvious. You work it out, you have to read carefully, I have read it in the beginning at least three times and now I have taught it ten times. So, that is a different thing. So, everybody has to work it out on their own. do it for a single edge, you have to cut it in two number of edges, number of edges because it is subdivision you have to take. And then work out what happened, what is happening to understand these numbers  $s_1, s_2, s_3$ .

If you still have doubts, do it for a triangle, the next, next stage. You can see what happens. Beyond that, you do not have to do, it is not easy either. After that you have to do, by that time you must be very, very confirmed that this argument is works. Now only logic will remain there, after that no, no pictures.

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interior  $(n-1)$ -face. Cutting down all these entries leaves us with only those  $G$  which are in  $B(\Delta'_n)$  and hence with the sum  $s_3$ . This proves  $s_1 \equiv s_3$ .

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We can now prove Sperner lemma by induction: Let  $C_n, n \geq 1$  denote the statement of Sperner lemma. Accordingly we shall temporarily denote the numbers  $s_1, s_2,$  and  $s_3$  by  $s_1(n), s_2(n)$  and  $s_3(n)$ . We need to show that  $s_2(n) \equiv 1 \pmod{2}$ . The case  $n = 1$  is easy, viz.,  $s_3(1) = 1$  since  $\phi$  is identity map on  $B\Delta'_1 = B\Delta_1 = \{e_1, e_2\}$ .

So, now we can prove the Sperner lemma now, by induction. Because we have a passage from  $\Delta_n$  to  $\Delta_{n-1}$  through the above lemma. That is a whole idea. So, for  $n$  greater than or equal to 1, let  $C_n$  denote the statement of Sperner lemma. Accordingly, we shall temporarily denote these numbers  $s_1, s_2, s_3$  by  $s_1(n), s_2(n), s_3(n)$  respectively. And we want to show that  $s_2 \equiv 1 \pmod{2}$ . That is the Sperner lemma. The additional hypothesis on  $\phi$  comes into play now.

For  $n = 1$ , this is very easy, namely  $s_3$  is 1. Since  $\phi$  is a simplicial approximation to the identity map on the boundary, (now you have use this hypothesis)  $\phi$  is actually the identity map on the boundary, in this case.  $\phi(e_1) = e_1, \phi(e_2) = e_2$ . There is no other choice. Now there are many layman kind of argument to see that  $s_2$  has to be odd, you choose your own favourite and we leave it to you. (Refer Slide Time: 14:22)

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Assume that  $n \geq 2$ , and we have proved  $C_{n-1}$ . Since  $\phi$  is a simplicial approximation to the identity when restricted to the boundary, each  $(n-1)$ -face of  $G'$ , where  $G$  is an  $(n-1)$ -face of the boundary will be mapped inside  $G$  itself. In other words, only some of the  $(n-1)$ -faces of  $L'$  are mapped onto  $L$ . Therefore, it follows that  $s_3(n) = s_2(n-1)$ . Now by appealing to (16) and the induction hypothesis we obtain  $s_2(n) \equiv 1 \pmod{2}$ .

So, now, having shown  $C(1)$  true, assume the statement for  $C(n-1)$  is true. We shall prove  $C(n)$ . Since  $\phi$  is the simplicial approximation to the identity when restricted to the boundary, if  $F$  is an  $(n-1)$ -face of  $\Delta_n$  and  $F'$  is the corresponding subcomplex of  $\Delta'_n$ , then each  $(n-1)$ -face of  $G$  of  $F'$  will be mapped inside  $|F|$ . In particular only some of the  $(n-1)$ -faces of  $L'$  are mapped on to  $L$  by  $\phi$ . Therefore, it follows that  $s_3(n) = s_2(n-1)$ .

But now, we can restrict  $\phi : L' \rightarrow L$  and the statement  $C(n-1)$ . That means we have  $s_2(n-1)$  is odd. Therefore,  $s_3(n)$  is odd. That implies  $C(n)$ . I will repeat this part, so what is happening here?  $L$  is the  $(n-1)$ -simplex  $\Delta_{n-1}$ .  $L'$  is its subdivision indices by  $\mathcal{B}(\Delta'_n) \subset \Delta'_n$ . Since  $\phi$  is a simplicial approximation to identity on  $\mathcal{B}(\Delta'_n)$  it restricts to a simplicial map  $\phi : L' \rightarrow L$ . Moreover this restricted  $\phi$  is a simplicial approximation to  $Id_L$  itself. Therefore, we can apply  $C(n-1)$  and conclude that  $s_2(n-1)$  is odd. But the above argument shows that  $s_3(n)$  is actually equal to  $s_2(n-1)$  for the function  $\phi$  restricted to  $L'$ .

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We now apply Sperner lemma to two of the celebrated results of Brouwer—the fixed point theorem and the theorem of invariance of domain. The proof of the fixed point theorem is standard and very quick. Equally quick is the proof of a mild version of invariance of domain, viz., that  $\mathbb{R}^n$  and  $\mathbb{R}^m$  are not homeomorphic to each other for  $n \neq m$ . The proof of the main version takes only a little bit more time.

So, Sperner lemma, can applied now to get many results. Our aim is to prove the Brouwer's invariance of domain. But, first we can and shall prove Brouwer's fixed point theorem for general case. We had proved it for n equal to 2. So, let us, let us be done with that.

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**Theorem 6.5**

For any integer  $n \geq 1$ , the following three statements are equivalent and each of them is true:

- (a) **(Brouwer's fixed point theorem)** Every continuous map  $f : \mathbb{D}^n \rightarrow \mathbb{D}^n$  has a fixed point, i.e., there is  $x \in \mathbb{D}^n$  such that  $f(x) = x$ .
- (b)  $\mathbb{S}^{n-1}$  is not a retract of  $\mathbb{D}^n$ .
- (c)  $\mathbb{S}^{n-1}$  is not contractible.

[refer to cone](#)

For any integer n greater than equal to 1, these three statements are equivalent. I think I have done it for n equal to 2, but the proof is exactly the same in the general case. So, I will repeat it here. First statement is Brouwer's fixed point theorem. Every continuous map on the closed unit disc to unit disc has a fixed point. So, the second statement is, the boundary  $\mathbb{S}^{n-1}$  of  $\mathbb{D}^n$  is not a retract of  $\mathbb{D}^n$ . The third statement is,  $\mathbb{S}^{n-1}$  cannot be contractible.



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**Proof:** (a)  $\implies$  (b): If  $r : \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}$  is a retraction consider the composite  $f$  of the three maps

$$\mathbb{D}^n \xrightarrow{r} \mathbb{S}^{n-1} \xrightarrow{\alpha} \mathbb{S}^{n-1} \xrightarrow{\eta} \mathbb{D}^n$$

where  $\alpha(x) = -x$  and  $\eta$  is the inclusion map. Then  $f$  has no fixed point, contradicting (a).

(b)  $\implies$  (a) The proof here is exactly the same as that we wrote for the case  $n = 2$  in the proof of Corollary 2.2.

[Go to BFTT](#) (b)  $\iff$  (c): We have seen that a space  $X$  is contractible iff  $X$  is a retract of the cone  $CX$ . (See Theorem 3.6.) [Go to cone-retract](#)

Since  $C\mathbb{S}^{n-1}$  is homeomorphic to  $\mathbb{D}^n$  we are done.

You might have forgotten it maybe, so, we will repeat this proof quickly. So, how (a) implies (b)? What we are trying to prove? If not (b) then we will say not (a). Suppose  $\mathbb{S}^{n-1}$  is a retract of  $\mathbb{D}^n$ . If  $r$  is a retraction, then we will get a contradiction to (a). If  $r$  is a retraction, retraction means what? a continuous map such that on the boundary it is identity map. So, take  $\alpha : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$  to be the antipodal map and look at  $f = \alpha \circ r$ . If  $x$  is in the boundary, then  $f(x) = \alpha \circ r(x) = \alpha(x) = -x$ . If  $x$  is in the interior of  $\mathbb{D}^n$ , then clearly,  $f(x) = -r(x)$  is in the boundary and hence cannot be equal to  $x$ .

Therefore  $f$  has no fixed point, contradiction to (a). So, if there is a retraction, that there is a map with no fixed points.

If you take any point on the inside the boundary, inside, in the interior it has gone already to some point in the boundary. Again, under alpha it goes to a boundary point under eta also it goes to boundary point. But the original point was in the interior, so those two cannot be equal anyway, the chance was only in the boundary, but the boundary points goes,  $x$  goes to  $-x$ . So, that is all. So, there is no fixed point at all. That is the contradiction.

Now let us prove (b) implies (a). the proof is exactly the same as that we wrote for  $n$  equal to 2. So, what we have to do? We make a picture draw the line joining  $x$  and  $f(x)$ , extended towards  $f(x)$ , get the point  $g(x)$ ; that will be a retraction. So, that is what we have done.

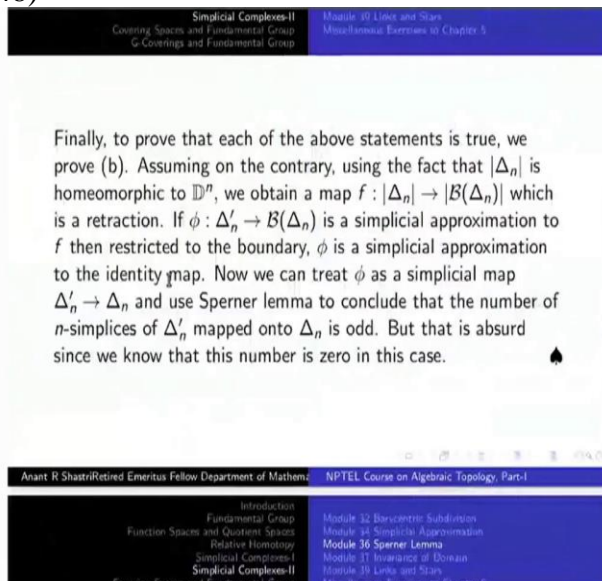


Now let us prove (b) and (c) are equivalent. We have seen that a space  $X$  is a contractible, if and only if  $X$  is a retract of the cone  $CX$ . This is one of the theorems we have proved.

The cone over  $\mathbb{S}^{n-1}$  is  $\mathbb{D}^n$ . So, what does (b) say?  $X$  is the boundary  $\mathbb{S}^{n-1}$ , and  $\mathbb{D}^n$  is the cone  $CX$ . (b) says  $X$  is a retract of its cone, i.e.,  $\mathbb{S}^{n-1}$  is a retract of the cone over it, then  $X$  is contractible. So, so here  $\mathbb{S}^{n-1}$  is not contractible, therefore  $\mathbb{S}^{n-1}$  is not a retract of  $\mathbb{D}^n$ . The same thing as saying that  $\mathbb{S}^{n-1}$  is contractible then  $\mathbb{S}^{n-1}$  is not a retract. Therefore, (b) gives (c) and conversely.

Now how we are going to use this to prove Brouwer's fixed point theorem itself. The above theorem says only that the three statements are equivalent. I have not proved anyone of them. We do not know whether any one of them is true. If you prove any one of them, then all the three gets proved.

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So, finally here, to prove that each of the statement is true, we shall prove (b) that is, there is no retraction from  $\mathbb{D}^n$  onto  $\mathbb{S}^{n-1}$ . If there is one,  $r : \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}$ , take a simplicial approximation to it,  $\phi : \Delta'_n \rightarrow \Delta_n$ . Apply Sperner lemma. Sperner lemma says that, the number of simplices in  $\Delta'_n$  mapped onto the whole of  $\Delta_n$  is actually odd. But since  $r$  is a retraction, the whole of  $\phi(\Delta'_n) \subset \mathcal{B}(\Delta_n)$ , the boundary complex. Therefore, no  $n$ -simplex would have been mapped onto  $\Delta_n$ .

So, Sperner lemma gives you immediately that there is no retraction, of the entire disk on to the boundary. Therefore, Brouwer's fixed point theorem is also proved.

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**Theorem 6.6**  
 For  $n \neq m$ ,  $S^n$  is not homotopy equivalent to  $S^m$ ; in particular,  $S^n, S^m$  are not homeomorphic to each other.

**Proof:** Suppose  $f : S^n \rightarrow S^m$ ,  $n < m$  is a homotopy equivalence. By Corollary 6.1, we know that  $f$  is null homotopic. By pre-composing with the homotopy inverse  $g : S^m \rightarrow S^n$  of  $f$ , this implies that the identity map of  $S^m$  is null homotopic. This is the same as saying that  $S^m$  is contractible and contradicts the above theorem.

Now, we will prove the simpler version of Brouwer's invariance of domain. Many books call this homeomorphic Brouwer's invariance of domain. What it says? It says that  $\mathbb{R}^n, \mathbb{R}^m, n \neq m$  cannot be homeomorphic to each other. That is statement of the next corollary.

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Theorem 6.1

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part I

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By taking one-point compactification we obtain

**Corollary 6.2**  
For  $n \neq m$ ,  $\mathbb{R}^n$  is not homeomorphic to  $\mathbb{R}^m$ .

The above corollary may be called a weak version of Brouwer's invariance of domain. We shall now embark upon proving the main version of the same as stated below.

How do we prove that? Suppose there is a homeomorphism, then you know that you can take one-point compactifications of them, they will be also homeomorphic to each other. What is the one point compactification of  $\mathbb{R}^n$ ? It is  $\mathbb{S}^n$ . For  $\mathbb{R}^m$  it is  $\mathbb{S}^m$ . Now  $m \neq n$  and we have got a homeomorphism between them,  $f : \mathbb{S}^n \rightarrow \mathbb{S}^m$ . In particular,  $f : \mathbb{S}^n \rightarrow \mathbb{S}^m$  will be a homotopy equivalence. This is the theorem that says that this is not possible. So that will complete the proof of Brouwer's invariance of domain, the next theorem.

How does one prove this? Without loss of generality, we can assume that  $n$  is less than  $m$ , by interchanging the  $n$  and  $m$ , if necessary. Now, we have already proved that if  $n$  is less than  $m$ , then any map as above is null homotopic. So, this  $f$  is null homotopic. By pre-composing with a homotopy inverse,  $g : \mathbb{S}^m \rightarrow \mathbb{S}^n$ , you have  $h = g \circ f : \mathbb{S}^n \rightarrow \mathbb{S}^n$  which is homotopic to  $Id_{\mathbb{S}^n}$ .

But  $f$  is homotopic to a constant map.  $h = g \circ f$  is also homotopic to a constant map. Therefore, the identity map of  $\mathbb{S}^n$  is null-homotopic.

But we have seen earlier that for any space  $X$ , if the identity map  $Id_X$  is null homotopic then  $X$  is contractible. Now, part (c) of the previous lemma tells you that, since you have proved (a) already, none of the spheres is contractible.

So, I repeat because of this theorem. What we have proved is, two spheres of different dimensions cannot be of the same homotopic type. In particular they cannot be homeomorphic. From that it follows that the corresponding euclidean spaces cannot be homeomorphic.

So, when this was proved first time, it was a very great result, a landmark result. How Brouwer proved it, more or less through his invention of homology. He had different proofs of this also, by the way. It is proved by using some dimension theory and so on. All these proofs are much, very, very much more complicated.

So, nowadays, homology theory gives you the standard proof of this one. So, what I got is, I got you this one by just using simplicial approximation and Sperner lemma. There is a stronger version. I have obtained you this a weaker version. What is the stronger version? That says that, if you take an open subset of  $\mathbb{R}^n$  non empty open set and suppose this is homeomorphic to another subset of  $\mathbb{R}^n$ . The homeomorphism is only for subset to subset and not necessarily defined on the entire of  $\mathbb{R}^n$ . Suppose one of them is open in  $\mathbb{R}^n$ . Then the other one is also open inside  $\mathbb{R}^n$ . So, that is called actually invariance of domain. Remember, domain was the word used for open and connected subsets, in analysis. So, that will be our next task, proving the full full version of invariance of domain. So, we will take it up next time. Thank you.