Introduction to Algebraic Topology (Part-I)
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Lecture No 34
Simplicial Approximation

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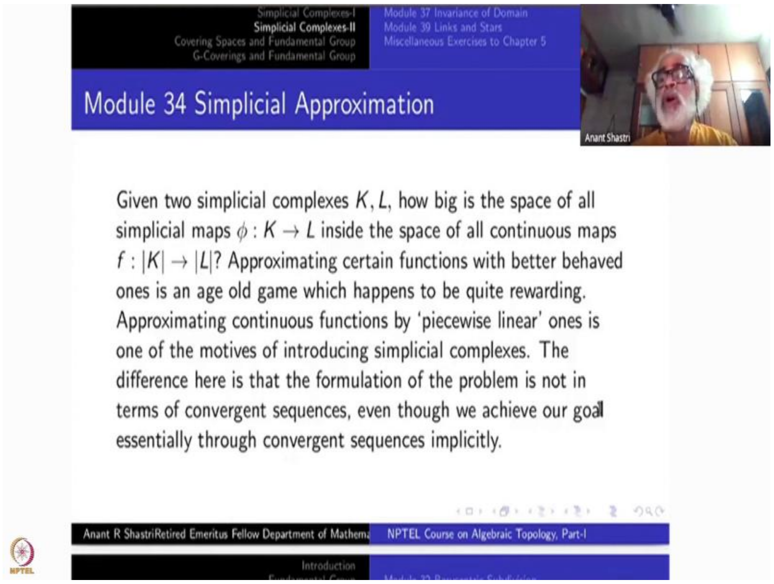

So, now, we are coming to simplicial approximation. Given two simplicial complexes K and L look at all the functions which are simplicial maps ϕ from K into L. Naturally you can think of, take $|\phi|$ and think of this as a continuous function from $|K|$ to $|L|$. But that will be a very special kind of continuous function, which is much smaller than set of all continuous functions. Question is, how big it is? Indeed, what we are going to see is, it is not too small.

Every continuous function can be approximated by a simplicial map that is, simplicial approximation. It result is not all that straightforward like smooth approximations here, the point is that if you keep K and L fixed, you are not going to get simplicial approximations. What you will have to do is, you will have to keep dividing the domain, according to the function that you want to approximate. And this dividing is precisely what we have introduced last time, subdivisions. Especially we are going to use barycentric subdivision and we will solve this problem only for K finite.

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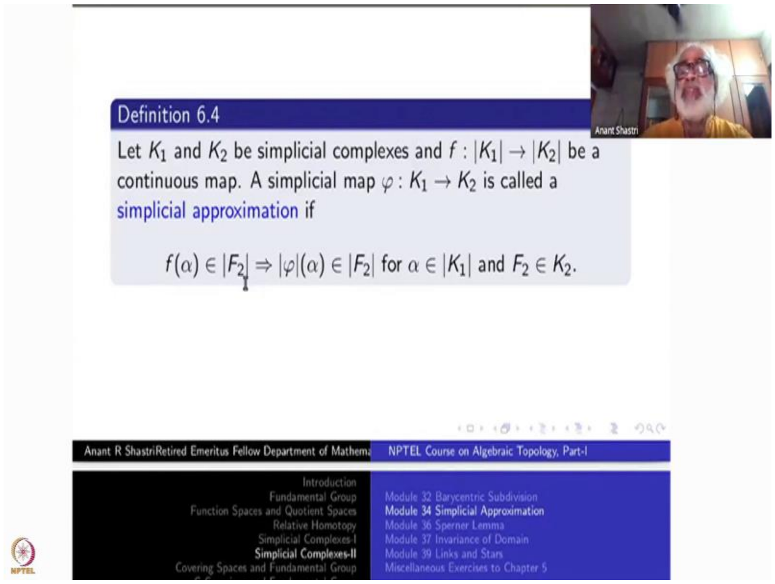

Take K_1 and K_2 be simplicial complexes and let $f : |K_1| \to |K_2|$ be a continuous map. We say a simplicial map $\phi : K_1 \to K_2$, a simplicial approximation to f, if the following condition holds namely, suppose you have a simplex $F_2 \in K_2$ and $f(\alpha) \in |F_2|$ for some $\alpha \in |K_1|$, then the corresponding image of alpha under mod phi should be also inside $|F_2|$, $|\phi|(\alpha) \in |F_2|$.

So, in other words, the image of mod of phi, an image under mod of phi of any of the point from should not go away from the image under f. If it is in a simplex, it should be inside that simplex. This is the, the way we want to do the thing. Here, the far-ness and nearness are measured out of simplicies inside K_2. We do not want to any distance function here.

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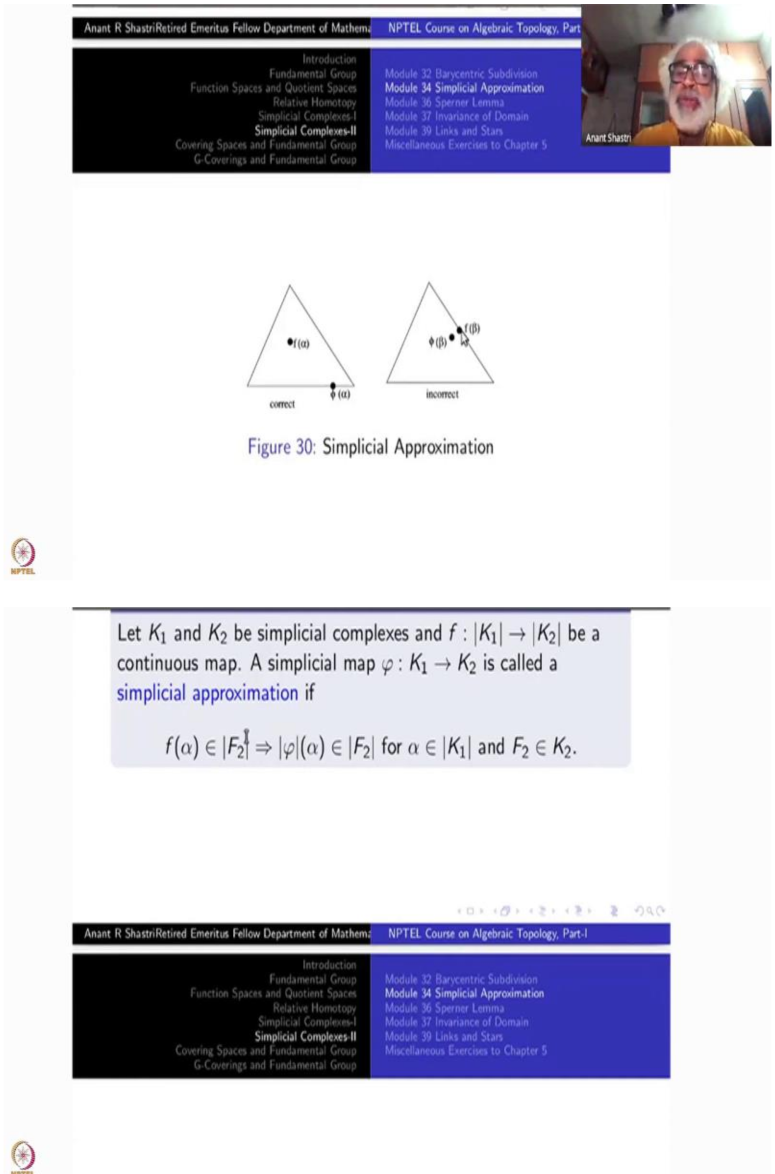

So, here is a picture some, for some alpha, f alpha has come here in that, in that triangle inside this triangle. Now, phi alpha is also in the triangle, it is on the boundary as a triangle, so this is okay. Now here, look at f beta is inside this, one simplex, in the edge. Whereas, phi beta is inside the larger triangle. So, this is where the condition that we started has failed, because what should have happened is phi beta should also inside this f, inside this one simplex.

If you, so you have to choose the condition correctly. You can say f beta is in this triangle. So, phi beta is here. So, that is true, but if you choose f beta to be inside one simplex, then this condition is not satisfied. So, that is the meaning of this condition for every F_2, which contains f alpha, you should have this property. So, this F_2 and alpha, alpha is taken whatever, this F_2 is also floated here, so all those F_2,s have this property, whenever this happens, this should be true, this is the way. I hope this is clear, this point is clear.

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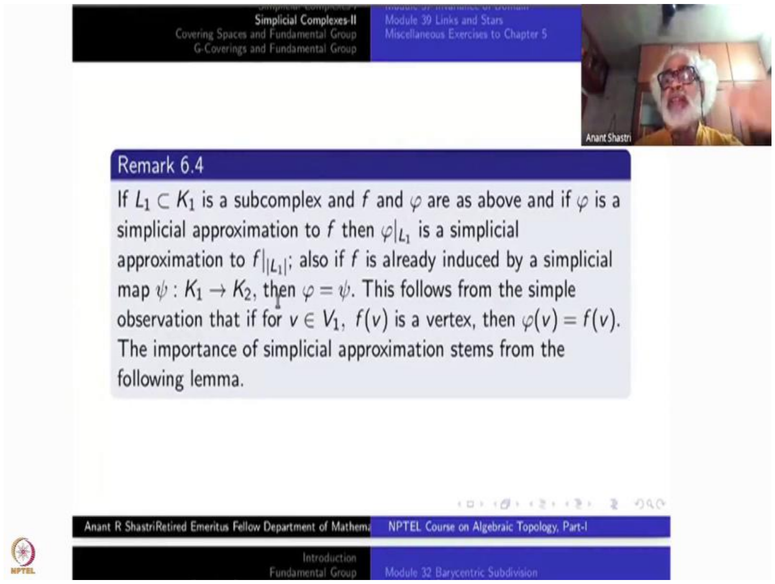

If L_1 is a sub complex of K_1, and ϕ and f are all as above, i.e., one is a simplicial approximation of the other one, (I am not repeating these conditions here), is a simplicial approximation to f, then, ϕ restricted to L_1 will be automatically a simplicial approximation to f on $|L_1|$. Because the conditions are satisfied for the larger simplicial complex, K_1, it will be also satisfied on the subcomplex also. The beauty is that suppose f is already induced by a simplicial map λ on L_1 to K, then any simplicial approximation ψ for f will be equal to λ on L_1. So, psi is giving you f, phi must be equal to psi. This is the meaning here. Whenever the, map is already simplicial, a simplicial approximation to that will be itself. This is quite different from other kind of approximation that we are doing.

So, how does this one follow? This follows by the simple observation that, instead of taking arbitrary point alpha, you take a vertex v in L_1. Then $f(v)$ is a vertex, because f is simplicial on L_1. So, I can take this vertex $f(v) = F$. Then $\phi(v) \in F$, must be also inside that thing that singular vertex, so $\phi(v) = f(v)$.

The importance of simplicial approximation stems from what happens in the homotopy aspect.

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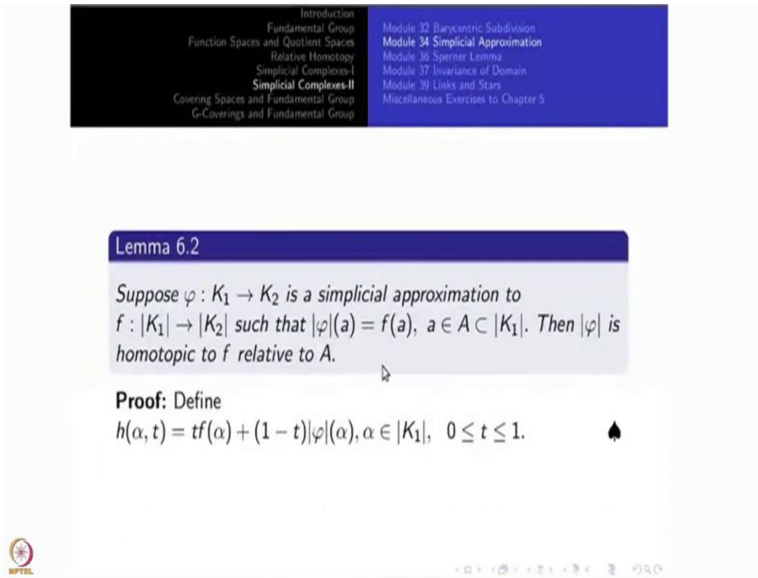

So that is what this lemma says, f is a continuous map and ϕ is the simplicial approximation. Further suppose that on a subset A of $|K_1|$, $|\phi| = f.$ They agree there. This A maybe empty, I do not care. Suppose this happened, then $|\phi|$ is homotopic to f relative to this set A. Obviously, $f(a) = |\phi|(a), A \in A,$ if you want a relative homotopy, first of all, so that is a necessary condition, that is it. Now, how does this follow? Because of the condition that for all $\alpha \in |K_1|,\ f(\alpha), |\phi|(\alpha)$ are both inside a single closed simplex. A closed simplex has a convex structure therefore, so, you can join these two points by a line segment. And that gives you homotopy. So, that is precisely what I am going to do. Take $h(\alpha, t) = (1-t)f(\alpha) + t||\phi|(\alpha).$ This is obviously a homotopy. The right hand side makes sense independent of what simplex $F \in K_2$ you choose, two structures affine structures will be the same, the line segments joining $f(\alpha), |\phi|(\alpha)$ is the same.

So, h is first of all, a well defined function. It is obviously linear restricted to any $|F_1|, F_1 \in K_1.$ So, h is continuous. So, all these things, now I am sure that you can verify by yourself. So, that is why I have written just this homotopy and this will be the required homotopy. For $t = 0$ what happens? It is $f(\alpha).$ For $t = 1,$ it is $|\phi|(\alpha).$ So, it is a homotopy as required.

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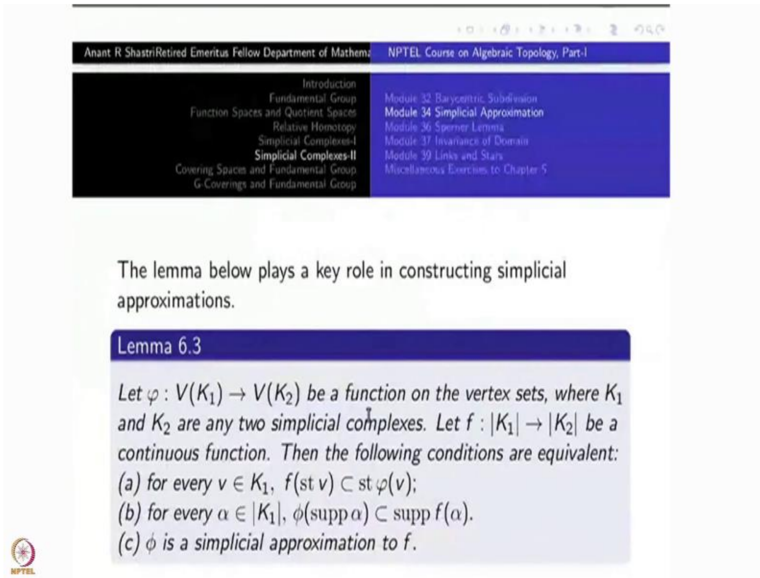

So, now we go toward getting a simplicial approximation, starting with a continuous function. So, a little more elaborate analysis of the definition, that we have given, is ignored in this lemma. And that will lead you to, get the function, the simplicial function that we are, that we are looking for. So, first of all, a simplicial approximation whatever, should be a vertex map. And then it must be a simplicial map and then it must be approximation, that approximation condition should be satisfied.

So, start with the vertex function $\phi : V_1 \to V_2$, first of all, where K_1 and K_2 are simplicial complexes. Suppose $f : |K_1 \to |K_2|$ is a continuous function and this ϕ which is a set function on the vertex sets, satisfies certain properties. So, these (a), (b), (c) are all equivalent. So, what is this first condition (a)? For every $v \in V_1$, a vertex in K_1, f of star of v should be contained in the star of phi v. I recall, what is the star of a vertex; star of a vertex u is all those alphas such that $\alpha(u) \neq 0$. It is denoted by $st\ u$; sometimes called the open star of u; it is star-shaped at the vertex u.

So, f of this open set, f of star of v should be contained in star of phi(v); phi(v) is a vertex in K_2. So, star of that should contain this one, this should not go out of that. Now what is (b)? For every $\alpha \in |K_1|$, support of alpha, support of alpha is a simplex, phi of that is a simplex, I do not know, because phi was only a vertex map. phi of support of alpha must be, not only a simplex, but it must be contained inside a support of f (alpha). Support of any point in $|K|$ is a simplex in K. Once it is contained inside a simplex, this will be also a simplex. It should be contained inside that, it may be equal also, no problem. The third condition (c) is, suddenly phi is simplicial approximation to f. So, these two, first two, conditions are equivalent to the definition. The definition does not even assume that ϕ is simplicial. Here, I do not have the assumption of simplicialness. I am starting only with a vertex map.

After all, in order define a function on the simplicial complex, you have to first get a set theoretic function on the vertex sets. So, that is the only set theory. After that, this gives you immediately that it is simplicial approximation. Either condition (a) or condition (b). And they are equivalent, that is the content of this lemma. Let us go through this proof carefully, because t understanding these statements is important here, not only just a statement, but how it comes out, that will reveal you, what simplicial approximations are about.

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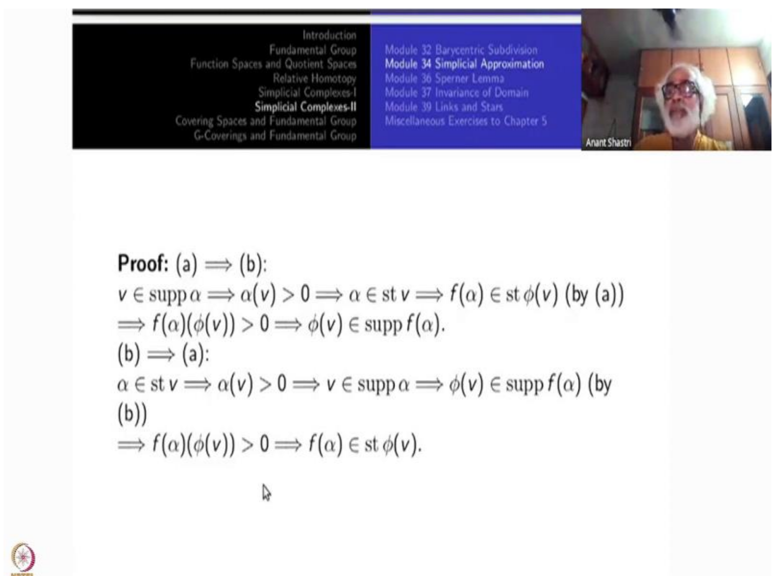

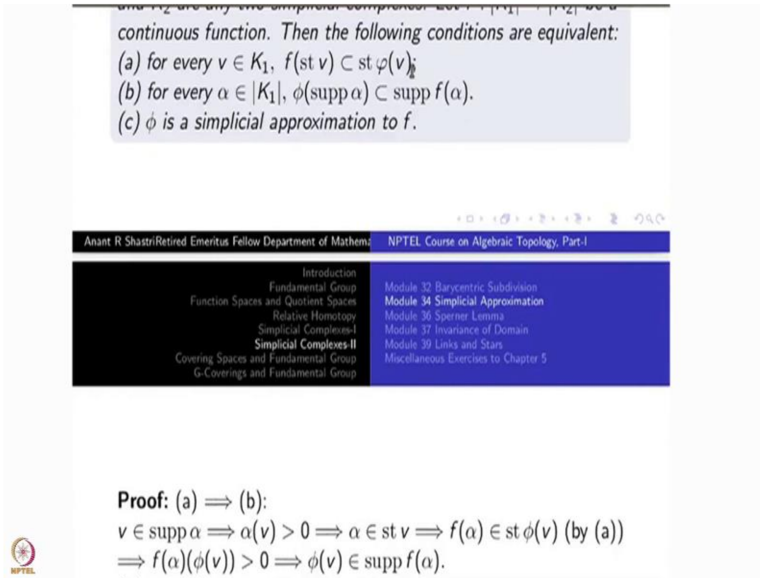

So, let us prove (a) implies (b). The first condition (a) only says that f of star of v is contained the star of phi of v. Then I have to show that, $\phi(supp\ \alpha) \subset supp\ f(\alpha)$. Indeed, the arguments here are reversible and we will get implication (b) implies (a) as well. So, I start with a point $v$inside support of alpha, I have to show that $\phi(v)$ is inside support of $f(\alpha)$.

$v \in supp\ \alpha \Longrightarrow \alpha(v) > 0 \Longrightarrow \alpha \in st\ v$. And conversely of course, up till here, the arrows can be reversed. Now, this implies $f(\alpha) \in st\ \phi(v)$. That is by condition (a). f of star v is this one. So, alpha is here, alpha f, f of alpha will be star of phi v. This condition (a). Now we can go back, something is in star of phi v means, that something evaluated phi v is positive, so, $f(\alpha)(\phi(v)) > 0$. But that in turn, implies that $\phi(v) \in supp\ f(\alpha)$.

So, we have completed proof (a) implies (b). So, these steps can be merely reversed to get (b) implies (a), but only you have to interchange the hypothesis and the conclusion. So, I am rewriting this proof of (b) implies (a). Start with $\alpha \in st\ v$. We have to show that $f(\alpha) \in st\ \phi(v)$. That is what we have to show, using condition (b) of course. So, as before, $\alpha \in st\ v \Longrightarrow \alpha(v) > 0 \Longrightarrow v \in supp\ \alpha$. Now condition (b) gives $\phi(v) \in supp\ f(\alpha)$.

(Earlier it was condition (a), now here we are reverse.) As before, this means that $f(\alpha)(\phi(v)) > 0$, which in turn implies that $f(\alpha) \in st\ \phi(v)$. So, it is exactly reversing the arrows here, (a) implies and implied by (b); (a) and (b) are equivalent. Now, we want to prove, that (a) and (c) are equivalent. If we assume (a), you can assume (b)b also, because they are same equivalent now. If we assume (b), then it is same thing as assuming (a). So, you can take both, advantage of both of them and that is what is going to happen.

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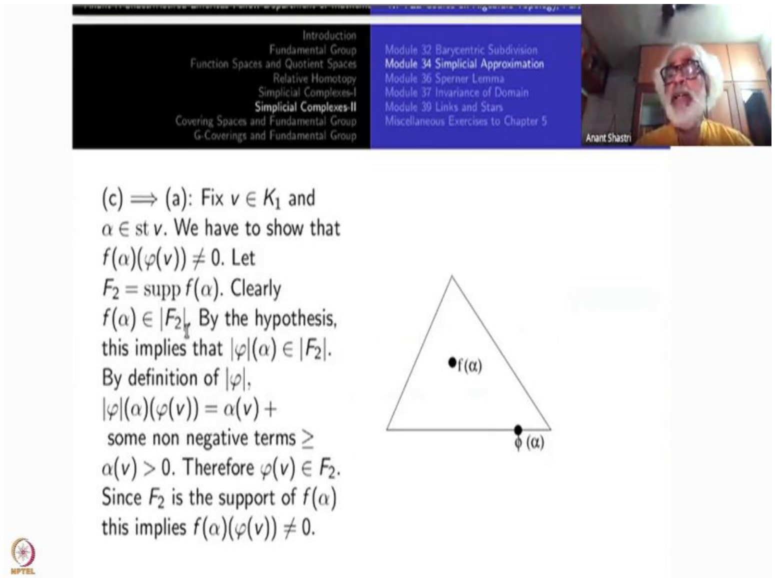

So, first I want to prove (c) implies (a). Condition (c) gives you that, ϕ is a simplicial approximation to f. So, fix $v \in V_1$ and $\alpha \in st\ v$. We want to prove (a). So, what we have to prove? $f(\alpha) \in st\ \phi(v)$. We have to show that $f(\alpha)(\phi(v)) > 0$. So, this is what we have to prove, starting with alpha, let F_2 equal to support of f(alpha.) That means that at any vertex of F_2,

f(alpha) is not 0, that is the meaning of support of f(alpha). So, we have to prove that $\phi(v) \in F_2$.

Clearly $f(\alpha) \in |F_2|$. (It is actually in the interior). Now by hypothesis, simplicial approximation hypothesis now implies that, $|\phi|(\alpha) \in |F_2|$. That means $supp\ |\phi|(\alpha) \subset F_2$. But now look at what is the definition of mod phi? Mod phi of alpha at $\phi(v')$ is the sum of all the alpha(u), where $\phi(u) = v'$. So, in particular, putting $u := \phi(v)$, we have, $|\phi|(\alpha)(u) = \sum_{v' : \phi(v') = u} \alpha(v') = \alpha(v) +$ some non negative terms. Since $\phi(v) = u$, the term $\alpha(v)$ is definitely there in the summation on the RHS. Some more terms $\alpha(v')$, where $\phi(v') = u$, may also be there which are all non negative. That does not matter, it is this alpha v plus something is there. Some non negative terms. Therefore, $|\phi|(\alpha)(u) > 0$. That means $\phi(v) = u \in supp\ |\phi|(\alpha) \subset F_2$. This proves (a).

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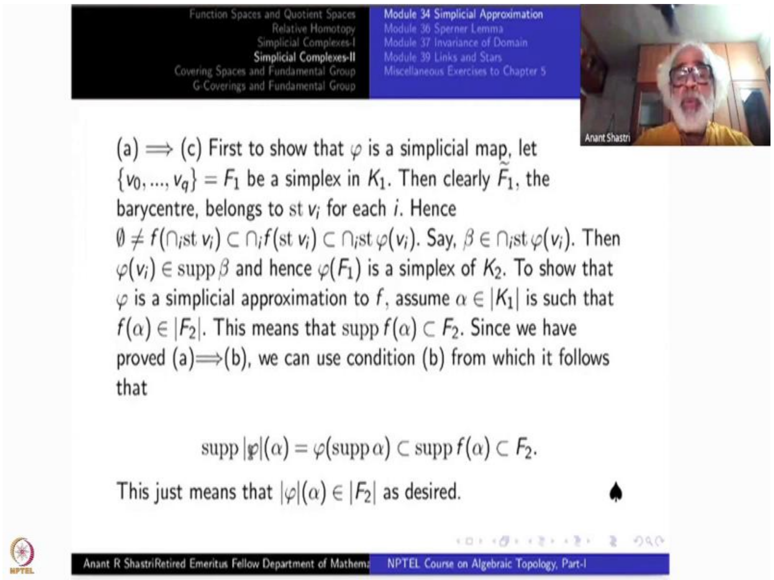

Now finally (a) implies (c). So, as I have told you, if we assume (a), statement (b) also comes to rescue. So, you can use whichever one you like. So, first, to show that phi is a simplicial map. Now, what is given, phi is just a vertex map, satisfying (a). Now, I have to first show that it is a simplicial map. What is the meaning of simplicial map? Take a simplex in K_1, the image must be a simplex in K_2. So, that is what you have to show. Take a simplex $\{v_0, v_1, \ldots, v_q\} =: F_1 \in K_1$.

Then the barycentre of F_1 viz., $\tilde{F_1} = \dfrac{v_0 + \cdots + v_q}{q+1} \in |F_1|$. It is actually in $\langle F_1 \rangle$. Not only that, it actually belongs to $st\ v_i$ for each , because the coefficient, the value of $\tilde{F_1}$ on v_i is equal to $\dfrac{1}{q+1}$. So, this element F1 twiddle operating upon a vi is not 0. So, it is in star of vi for each i. Therefore, this intersection of star of vi's is not empty, $\cap_{0 \le i \le q} st\ v_i \neq \emptyset$. Anyway, $\tilde{F_1}$ is there.

Therefore f of that one will be also non-empty. If we have a non-empty set, its image under a function will be definitely non-empty. Now $f(\cap_{0 \le i \le q} st\ v_i) \subset \cap_{0 \le i \le q} f(st\ v_i) \subset \cap_{0 \le i \le q} st\ \phi(v_i)$. Last inclusion is from condition (a) . f of any intersection is contained in the intersections of the f of those, viz., f of star vi's and each of f of star vi's is contained inside, star of phi vi's, that is our condition (a). Therefore, this intersection is non-empty. This intersection is non-empty means what? There is β in the intersection, which means beta evaluated on phi vi is not 0 for each of them. So, that would mean that,

$\phi(v_i) \in supp\ \beta$, which must be simplex. Because every point of $|K_2|$ is inside some some open simplex. Therefore, $\phi(F_1)$ is also a simplex of K_2. So, you have proved that phi is a simplicial map.

Now, it must be shown that, ϕ is a simplicial approximation to f. So, that part is easier, to show that ϕ is a simplicial approximation to f: take any $\alpha \in |K_1|$ and assume that $f(\alpha) \in |F_2|$. We have to show what? That $|\phi|(\alpha)$ is also in $|F_2|$. This is what we have to do. First of all, $f(\alpha) \in |F_2| \Longrightarrow supp\ f(\alpha) \subset F_2$.

Since we have proved (a) implies (b), we can use statement (b) also, from which it follows that, $\phi(supp\ \alpha) \subset supp\ f(\alpha)$ which, we have seen is contained in F_2. Since for any simplicial map, $supp\ |\phi|(\alpha) = \phi(supp\ \alpha)$, we get $supp\ |\phi|(\alpha) \subset F_2$. But this is the same as saying $|\phi|(\alpha) \in |F_2|$.

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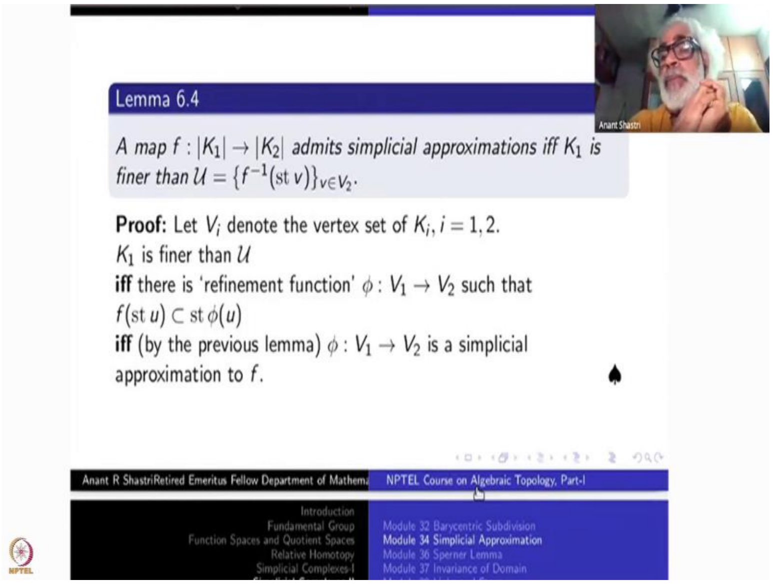

So, now we can harvest this criteria which we have got, we can use that to get another criteria for simplicial approximation. Take any continuous function $f : |K_1| \to |K_2|$. It admits simplicial approximations, (you may not know what they are) if and only if (it is purely condition in terms of f), if and only if K_1 is finer than the open covering, $\mathcal{U} = \{f^{-1}(st\ v)\ :\ v \in V_2\}$, where V_2 denotes the vertex set of K_2. Note that $\{st\ v\ :\ v \in V_2\}$ is an open covering for $|K_2|$. Take f inverse of that, that will be open covering for $|K_1|$. That is members of $\mathcal{U}$ are f inverse of star v,s . The property that K_1 is finer than this, just means that for each $u \in V_1$, $st\ u$ is contained in one of the members here. Remember the finer means that.

So, what is the proof? Here V_i denote the vertex sets of $K_i, i = 1, 2.$ K_1 is finer than $\mathcal{U}$ just means that for each u is inside V_1, $st\ u$ is contained in $f^{-1}(st\ v)$ for some $v \in V_2$. that is the meaning of finer. So, what means that? There is a refinement function $\phi : V_1 \to V_2$ such that $st\ u \subset f^{-1}(st\ \phi(u))$. That is the meaning of the refinement. But now, this is the same as condition (a) in the previous lemma, $f(st\ u) \subset st\ \phi(u)$.

So, in conclusion, any refinement function $\phi : V_1 \to V_2$ is a simplicial approximation. That is from the previous lemma. In this lemma, it is no reference to phi; now we have chosen the function phi here, which is refinement function from one vertex set to another vertex set. There may be many, all of them will be simplicial approximation to f. Now, we know how to get a simplicial approximation to a given map $f : |K_1| \to |K_2|$. We must give $|K_1|$, the underlying topological space, another simplicial structure which is finer than the covering $\mathcal{U}$. Since we have already one structure K_1, we should think how to modify it suitably. And we know the answer, namely, go on taking barycentric subdivisions divide, divide, divide till it becomes very fine, that is the meaning of finer, it might be finer than the covering, that is the last thing.

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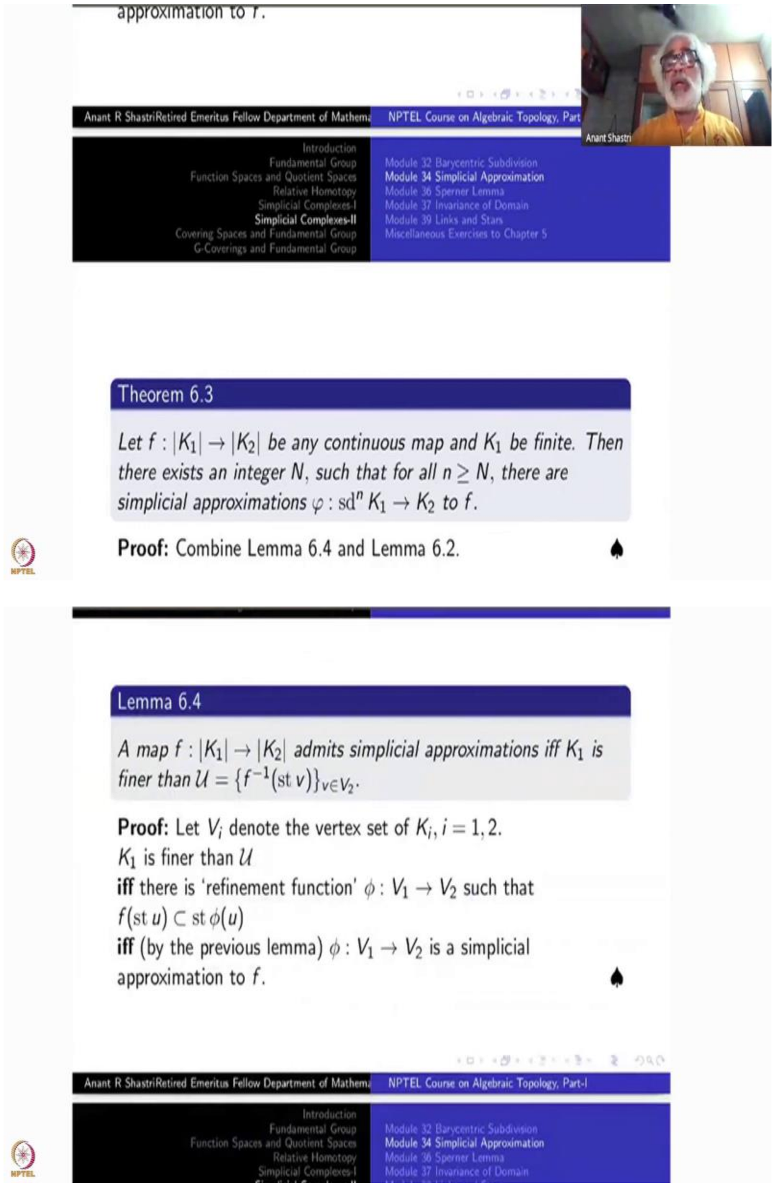

If K1 and K2 are simplicial complex, $f : |K_1| \to |K_2|$ is a continuous function, K_1 is finite, K_2 could be anything. Then their exists an integer N, such that for all n bigger than N, there is simplicial approximation $\phi : sd^n \ K_1 \to K_2$ to the function f. These maps are defined not on K1 but on your finer subdivision and they will be approximation to f.

Once your n is sufficiently large, namely, we know how to estimate this capital N. $\left(\frac{q}{q+1}\right)^N \longrightarrow 0$ and therefore can be make smaller that $\epsilon/2$, where ϵ is the Lebesgue number of the covering $\mathcal{U}$. So, that part we have seen last time. Remember that? So that is the meaning of combining lemma of 6.4 and 6.2. 6.2 about finer subdivision; 6.4 says once it is finer, like this then you are done. So, simplicial approximation theorem is proved.

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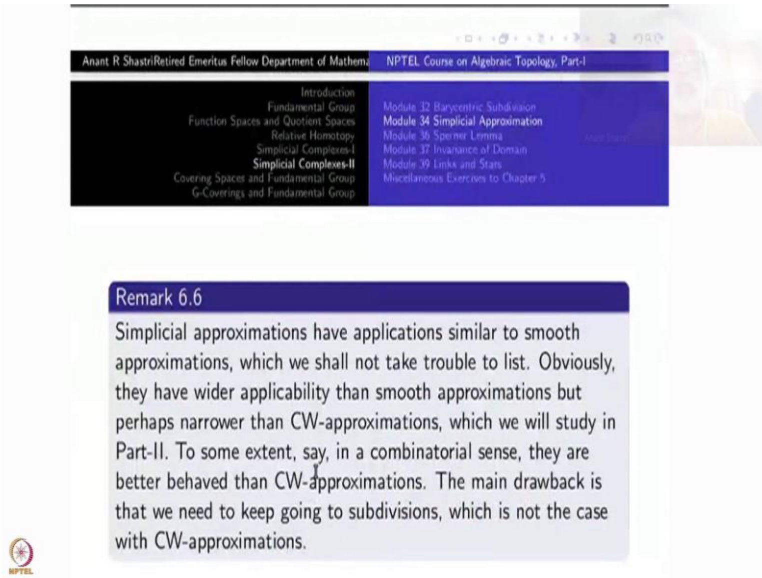

This says similar applications, similar to what? Similar to smooth approximations in differential topology or multivariable calculus that you may have learned. But it obviously has wider applicability because, here we are not taking manifolds at all. Our spaces may have highly cornered points maybe there, non-smooth, triangle, tetrahedron and so on. And things built upon that, very crooked parts could be there.

But there are other types of spaces which are more general, namely, CW complexes and they have also this kind of CW approximations, in some sense they are, they have wider applicability than simplicial approximations. But simplicial approximations have the great advantage, if you K1 and K2 are simplicial complexes, we have to given. And if you specify a few properties of continuous function f, even the computer can find simplicial approximation to you, just, few more, you have to give some effort, what is f we have to give some information about how, what is happening to f, some manageable information, it will do that. This kind of thing is not possible to CW approximations and so on. So, there are a lot of advantages also, as compared to other simply, other approximations.

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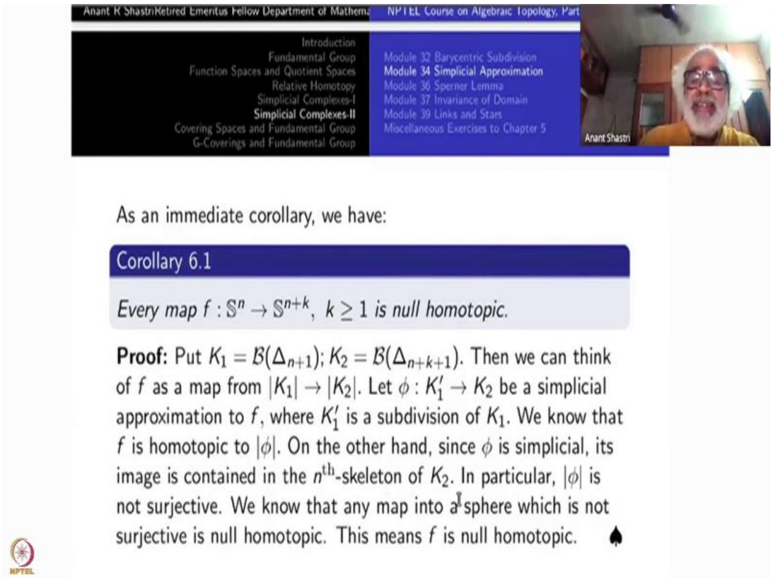

So, some immediate corollary, some of these things can be done by smooth approximations also, it is not that these are the advantages, no. So, this corollary is immediate, namely take any function from one sphere to another sphere of higher dimension, say, $\mathbb{S}^n \to \mathbb{S}^{n+k}$, where k is at least 1. Any such map is null homotopic. Proof is very simple, you have to think of $\mathbb{S}^m$ given by the underlying topological space or geometric realization of the boundary of Δ_{m+1}. This we have seen in already, take K_1 equal to boundary of Δ_{n+1}, then $|K_1| = \mathbb{S}^n$. Similarly take K_2 equal to boundary of Δ_{n+k+1}. Then your f is a map from $|K_1|$ to $|K_2|$. Therefore, after dividing K_1 sufficiently many times, I do not want to write it, say $K_1' = sd^r K_1$, we get a simplicial map $\phi : K_1' \to K_2$, which approximates f. We know that f is a homotopic to $|\phi| : |K_1'| = \mathrm{S}^n \to \mathbb{S}^{n+k} = |K_2|$.

That is the importance of simplicial approximation in homotopy theory. So, to study the homotopy behavior of f, I can pass on to now phi, mod phi. Because these two are homotopic, if I want to show the f is homotopic to constant map, I should show that mod phi is homotopic to constant map. But mod phi has better properties namely because it is simplicial map, its image is contained in the nth skeleton of the image. The dimension of the image of any n simplex here that this maximal simplex is here, cannot be bigger that n. It will be an n-simplex or some smaller dimensional simplex. Because ϕ is a linear map. This part we have seen.

So, it is contained is a nth skeleton of K_2. But K_2 is n plus k dimensional complex. There are n plus k dimensional simplexes also. So, all those things are not covered, which just means that mod phi is not as surjective function. Now, this is one of the simple exercise I had given, you have to better work it out, namely any function into a sphere which is not surjective, that means it misses one point, just one point is enough, then it is homotopic to a constant function. You have seen the proof of this, you have seen this proof at least by now. So, we conclude that f is also null homotopic.

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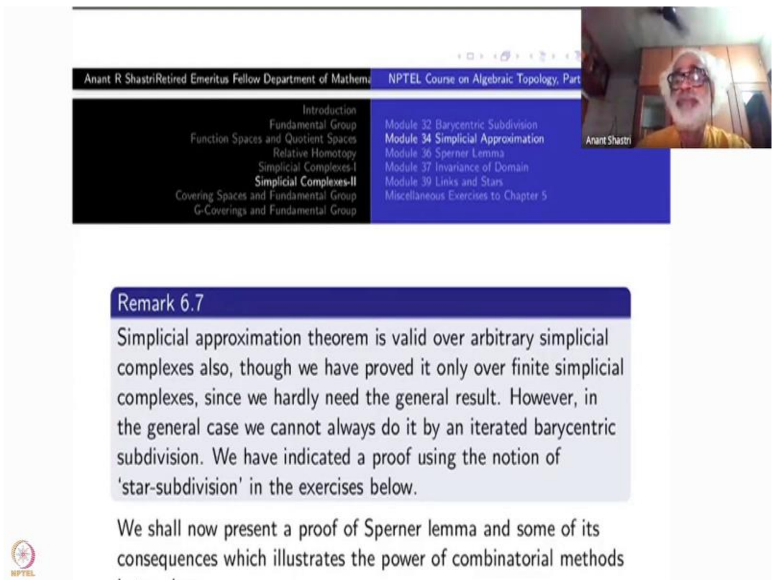

So, simplicial approximation theorem is valid over arbitrary simplicial complexes also, instead of just finite ones. We have assumed that the domain K_1 is finite, that is not a necessity. Only in this proof it was needed. But to prove it in general case, you cannot just use Barycentric subdivision, you have to go for arbitrary subdivisions also. So, let us stop here. Next time we will derive another comment or result out of this simplicial approximation. Thank you.