**Introduction to Algebraic Topology (Part-I)**
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**Lecture No. 33**
**Finer Subdivisions**

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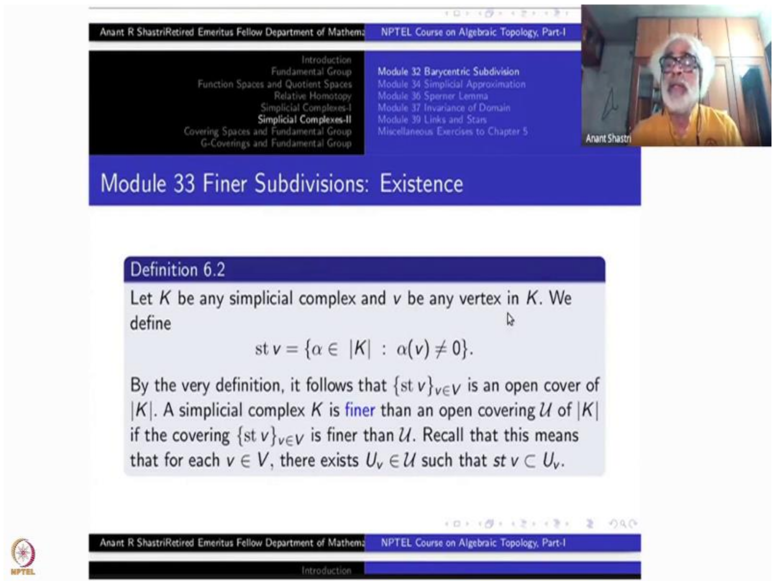


Finer subdivisions. So we shall prove some existence theorem here. This kind of things you must have used several times, for the interval, take an open interval, take a closed interval and take an open covering. Then there is a final subdivision, subdivision such that each interval is contained in one of the open sets. So, that is, you can term as subdivision finer than the open covering.

So, this is what we are going to now generalize, to simplicial complexes and barycentric subdivisions. Let $K$ be any simplicial complex. First let me introduce these notations which will be quite useful, elsewhere also. Let us look at a vertex v, and take all the points inside $|K|$ which can be joined by a single line segment to this vertex v. So, that is going to be a star shaped set. So, we are going to define this star v, except that is going to be the open star, $st\ v$ is defined to be the set of all points $\alpha \in |K|$ such that, $\alpha(v) \neq 0$.

In particular, you know the vertex v is identified with the map which takes 1 at v and 0 at everywhere, all other vertices. So, so $v \in st\ v$. So, if $\alpha(v) \neq 0$, then $\beta = t\alpha + (1-t)v$, this entire line segment will be inside $st\ v$, for we have $\beta(v) \neq 0$. So, $st\ v$ is star shaped at v. In any case, because $st\ v$ is defined by open condition, something is not 0, one coordinate is not 0, coordinate functions are a continuous function after all, it follows that $st\ v$ is an open subset of $|K|$. If you, vary the vertices v, because for each alpha, somewhere alpha will be non-zero, and hence $\alpha$ will be in one of the stars in any case, therefore $\{st\ v\ :\ v \in V\}$ is an open cover for $|K|$.

A cover $\mathcal{U}$ is finer than another cover $\mathcal{V}$ just means that every member in the first cover $\mathcal{U}$ must be contained in some member in the second cover $\mathcal{V}$. This is a general topological notion.

We say a simplicial complex $K$ itself is finer than an open covering $\mathcal{U}$ of $|K|$, if $\{st\ v\ :\ v \in V\}$ which is an open cover for $|K|$ must be finer than the given covering $\mathcal{U}$. So, instead of coverings we are defining, $K$ itself is finer than $\mathcal{U}$, just a short terminology to tell that $\{st\ v\ :\ v \in V\}$ is finer than $\mathcal{U}$. This just that means that, for each $v \in V$, there is some $U_v \in \mathcal{U}$ such that $st\ v \subset U_v$. Every member in this covering must be containing some member in $\mathcal{U}$.

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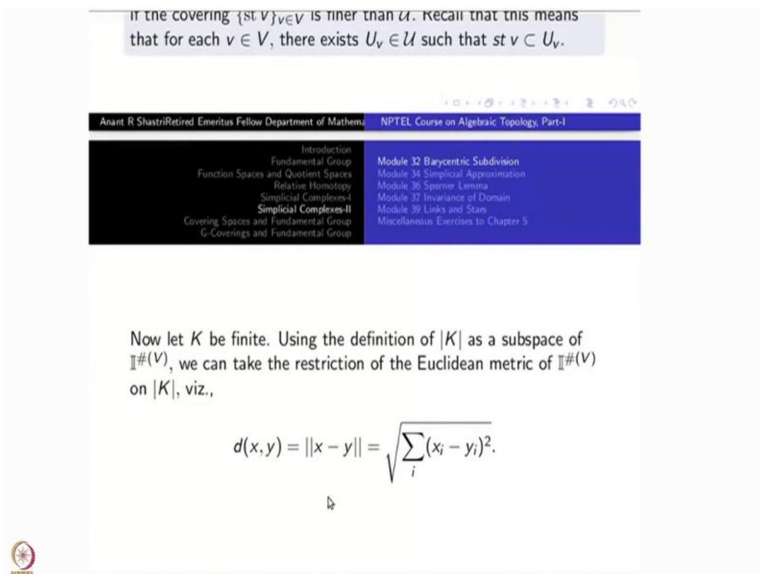


Now, you take a special case, viz., when $K$ is finite. In particular, the vertex set is also finite. Then, look at the definition of $|K|$ as a subspace of $\mathbb{I}^V$. Or, you can just write, as V is finite, $\mathbb{I}^n$ where n is the number of vertices in $K$. So, $|K|$ is subspace of the Euclidean space $\mathbb{I}^n \subset \mathbb{R}^n$. We can take the restriction of Euclidean metric. Namely distance between x and y is just square root of the sum of the squares of the differences of xi and yi. That is the euclidean distance. Because $K$ is finite, recall that the topology on $|K|$ is this metric topology. Only when $K$ is infinite, we are giving a finer topology on $|K|$.

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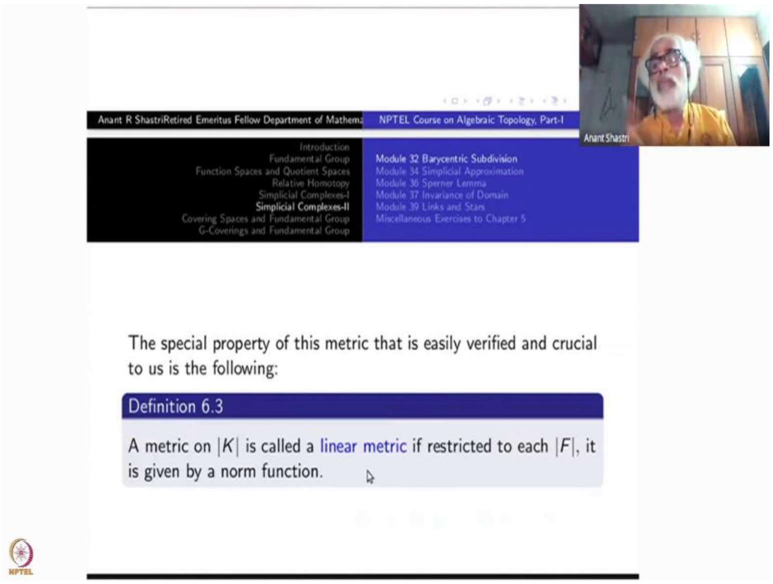


A special property of this metric is the following, namely, it is linear metric in a strong sense. What is that some sense? It is given by a norm in $\mathbb{R}^n$. What we actually want is slightly weaker condition, viz, restricted to each $|F|$, the distance function should come from some norm on $\mathbb{R}^n$. (But this you may ignore.)

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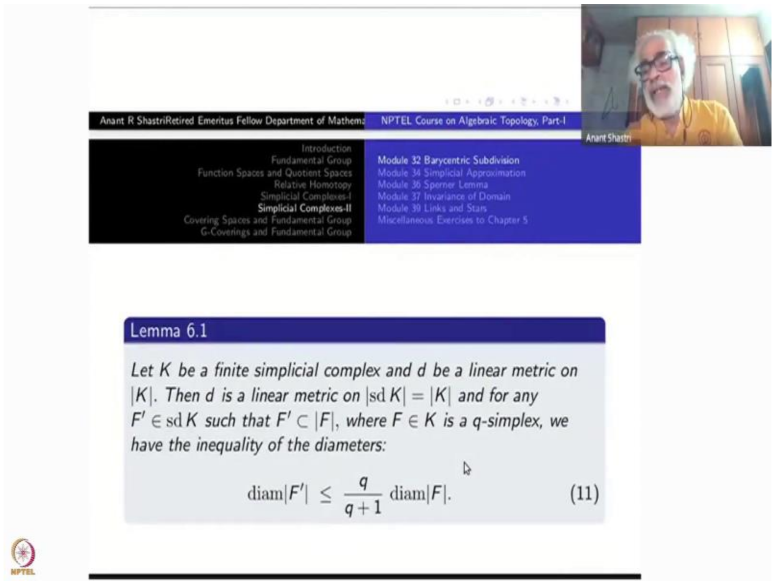


The first lemma is elementary calculus kind of thing here, namely $K$ is a finite simplicial complex. I repeated it, $d$ is the metric that we have taken. Any other metric which has linear property will also do the same thing. That is why I require that $d$ is linear metric.

Then, $d$ is a linear metric on the barycentric subdivision also. And for any $F'$ belonging to $sd\ K$, such that $F' \subset |F|$, $F \in K$, we have the following inequality of diameters: the diameter of $F'$ will be less than or equal to $\frac{q}{q+1}$ times diameter of F.

What is q? q is a dimension of $F$ here. You take a simplex of whatever size inside $sd\ K$, it is contained in $|F|$ for some $F \in K$. That is because of chain condition in the definition of sd. So, obviously, the diameter of $F'$ is less than or equal to diameter of $F$. The important point is the factor $\frac{q}{q+1} < 1$.

For example, if you take q equal to 2 namely a triangle, then this will say that each of the six subtraingles is of diameter is at most two third of the the original one diameter. If q is 1 namely, the simple, then will be the 1 simplex and what is this 1, this is 1 by 2, it is actually equal you can see.

Take one simplex [0,1]. What are the vertices of the sub complex, sd of F, the barycentric subdivision? 0 to half and half to 1. So, each of them has length half, whereas the diameter, original diameter is 1. So, that is the meaning of this one. I have given you examples for this in simple cases, but in general, this is what it is. So, having stated this one, this can be left as an exercise completely. But, since this kind of things are not done in, even in elementary linear algebra or calculus or, or even in topology, so, let me do it here.

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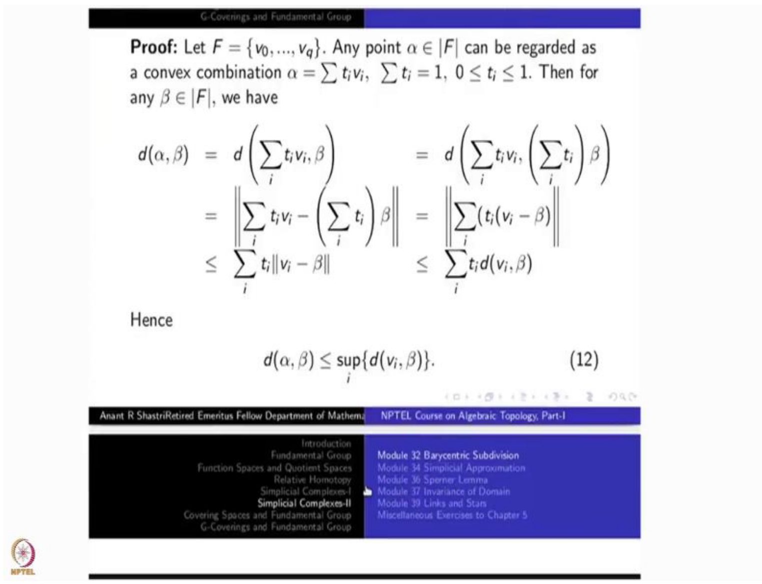


You will see that it is not all that easy, not completely easy. So, let us have a notation: $F = \{v_0, v_1, \dots, v_q\}$; $v_i$ are the vertices. Any point $\alpha \in |F|$ can be thought of as a convex combination of $v_i$'s; this is the definition: $\alpha = \sum_{i=0}^{q} t_i v_i, \ \sum_{i=0}^{q} t_i = 1, \ 0 \le t_i \le 1.$ Now take any other $\beta \in |F|$, We want to estimate the distance between alpha and beta.

I can write beta also as a convex combination but I am just keeping it for a while. The first step is to express $\beta = 1 \cdot \beta = (\sum_{i=0}^{1} t_i)\beta.$ Next, rewrite the distance in terms of a norm. Now, you have two summations here and their difference. So, that is what I am using that this distance is given by the norm. Note that, to begin with both $\alpha, \beta$ are inside $|F|$.

So, what is this? I take norm of this summation minus this summation. We can rewrite it as $\| \sum_{i=0}^{q} t_i (v_i - \beta) \|$. So, the norm of this sum is less than or equal to sum of the norms, by triangle inequality. Next $t_i$ comes out because they are all non negative. The next step is that you can rewrite each norm in terms of distance again. Very simple computation.

Now, how to use this one? Therefore, distance between alpha and beta is less than or equal to $\sum_{i=0}^{q} t_i s$, where $s$ is the supremum of all the these distances. Take the maximum, this is a finite set, so supremum is same thing as maximum, we put the maximum for each of them, use the fact $\sum_t i = 1$. Therefore $d(\alpha,\beta) \leq s = \mathrm{Max}\{d(v_i,\beta), i = 0, 1, \ldots, q\}.$

What is the meaning of this one geometrically? Take any, any simple $F$ and take any two points beta and alpha, the distance between alpha and beta is smaller than the distance between beta and any one of the vertices, take the maximum of that. Largest distance, go all the way to one of vertices. So, one point I have kept fixed, the other point I have replaced by vertices. Now, I want to do the same thing, with $\beta$ and each $v_j$ as well, fixing $v_j$. This are also points of $|F|$ after all. Reverse this, this is symmetric relation. You use this one.

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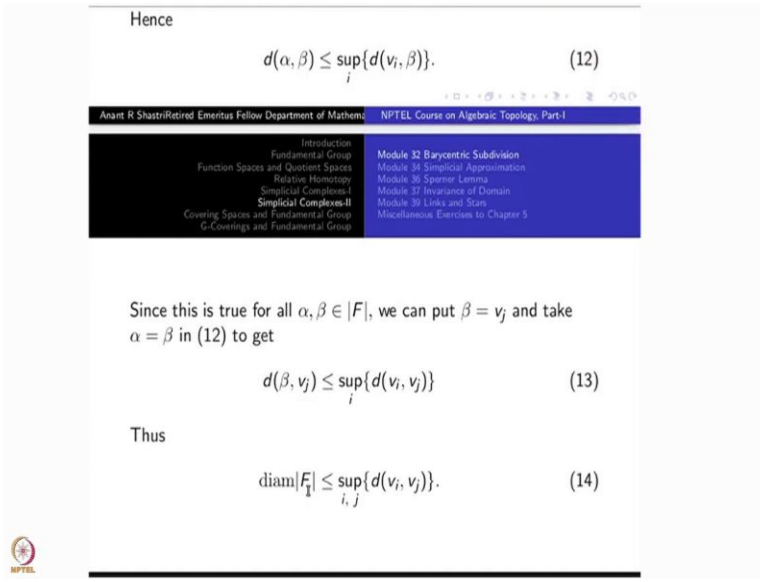


I repeat, what I get is since this is true for all alpha and beta, we can put beta equal to $v_j$ and take alpha equal to beta in this one. Distance between beta and $v_j$ will be less than supremum of distance between $v_i$ and $v_j$ This is true for all betas and all $v_j$'s. So, what does it mean? Distance between any point in the vertex and vertices is smaller than the maximum of the, what are these?

These are edges, length of the edges of the simplex. Therefore, if you maximize the left hand side, what do you get? These were all $v_j$'s is true, each of them is, is true. So, from here you can go to this inequality, alpha beta is less equal to this one. And each of them less than equal this one, so distance between the alpha and beta from (12) and (13) is less than supremum of the distances $d(v_i, v_j)$.

Now, take the maximum on the left hand side. If for everything it is less than equal to this one, the maximum will be also less than equal this one, supremum, this supremum is nothing but diameter of $F$ by definition. So, the diameter of $F$ is actually realized in the length of one of the sides, maybe if I stated this, in the beginning you would have understood it very clearly. So, this is easy to see for triangles, easy means what? If you want to write down, the proof will be like this, but I have done it for any simplex.

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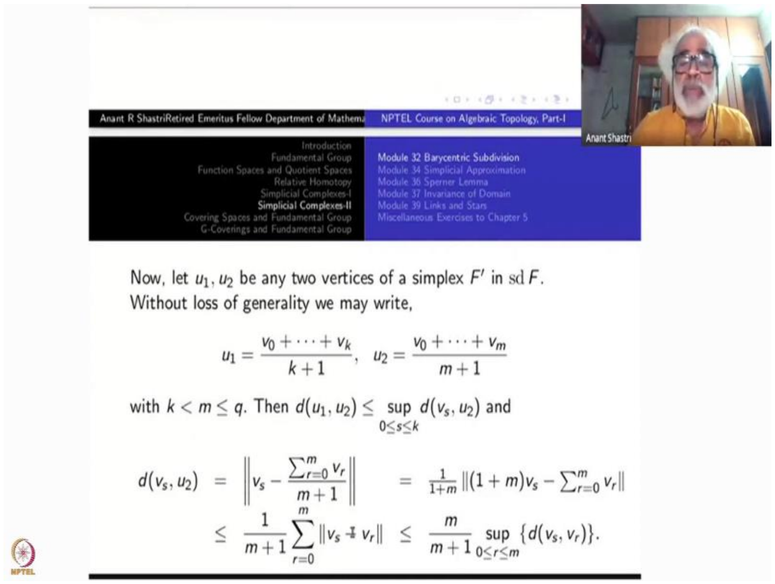


Now, let $u_1, u_2$ be any two vertices of $F'$ where $F'$ is a simplex in $sd\ F$. Without loss of generality, by changing the labeling, we may write $u_1 = \dfrac{v_0 + \cdots + v_k}{k+1}$ and $u_2 = \dfrac{v_0 + \cdots + v_m}{m+1}$ and k<m. This is because, by definition, $F'$ look like $\{F_0, \tilde{F}_1, \ldots, \tilde{F}_p\}$ where $F_0 \subset F_1 \subset \cdots \subset F_p$ are subsets of $F$ and $\tilde{F}_i$ are barycentres of $F_i$.

Then the distance between $u_1$ and $u_2$ can estimated by keeping $u_2$ fixed and using the summation expression for $u_1$. We get $d(u_1, u_2) \leq \text{Sup } \{d(v_s, u_2)\ :\ s = 1, \ldots, k\}$.

Next, we have to estimate each $d(v_s, u_2)$. This I am going to do more carefully now. The distance between $v_s$ and $u_2$, let us write down, is equal to the norm of $v_s$ minus, what is $u_2$? $u_2$ is this expression, I have used this one, $u_2 = \dfrac{\sum_{r=0}^{m} v_r}{m+1}$. So I pull out $m+1$, multiply $v_s$ by $(m+1)$, take minus $\sum_{r=0}^{m} v_r$. So, there are m+1 quantities here and $m > k$ and $s \leq k$.

The beauty here is that one of the $v_r$ will be equal to $v_s$ and the corresponding contribution will be zero. So, you will get at most m of them. Replacing each $\|v_s - v_r\|$ by the supremum $\sigma$ we can replace the RHS by $\dfrac{m}{m+1}\sigma$. That is the beauty.

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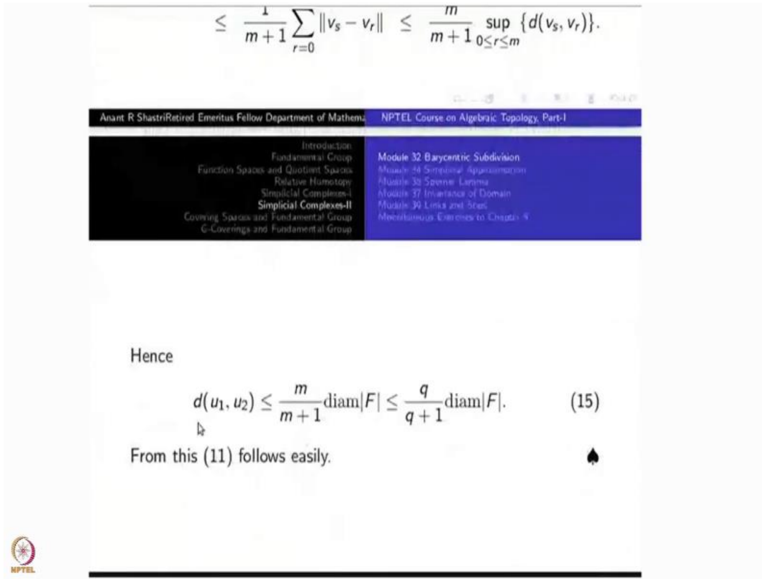


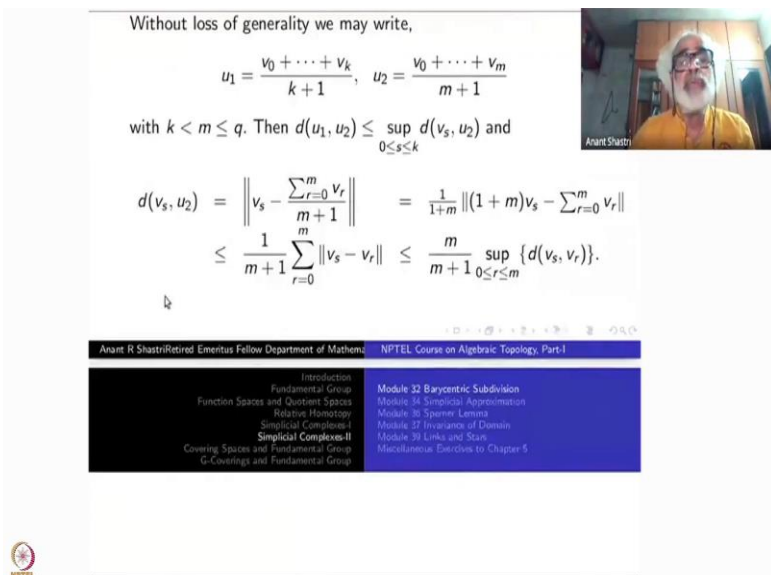


Hence, distance between $u_1$ and $u_2$ is less than or equal to the supremum of these things and each of them is less than equal this one. So, it follows that $d(u_1, u_2) \leq \frac{m}{m+1}\text{Sup }\{d(v_i, v_j) \ : \ 0 \leq i, j \leq q\}$. Further we can replace $\frac{m}{m+1}$ by $\frac{q}{q+1}$ because m is smaller than q. Now we can take the supreum on the LHS and conclude that diameter of $F'$ is less than or equal to $\frac{q}{q+1}$ times the diameter of $F$. We have proved this statement.

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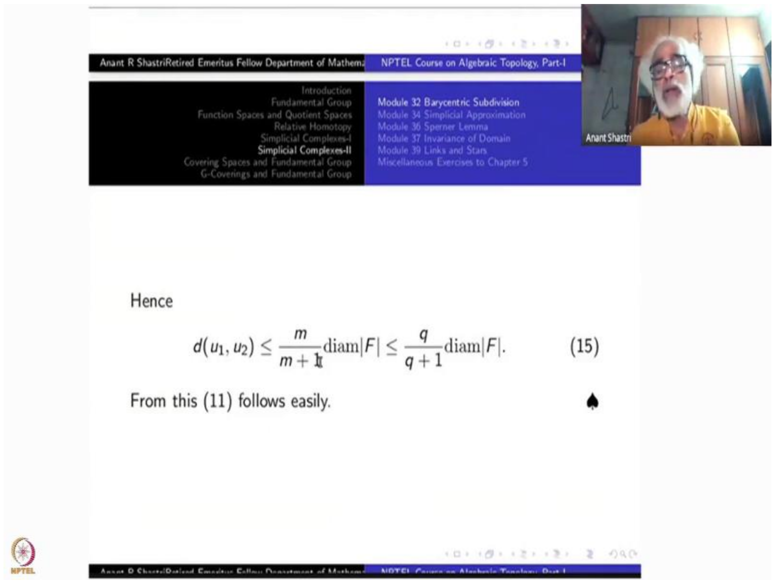


Now, let $K$ be any finite simplicial complex, and $\mathcal{U}$ be any open covering of $|K|$. Then if you divide $K$ sufficient number of times, i.e., apply $sd^n$ which is nothing but $sd \circ \circ \cdots sd$ (n times), we want to say that it will become finer than this open covering.

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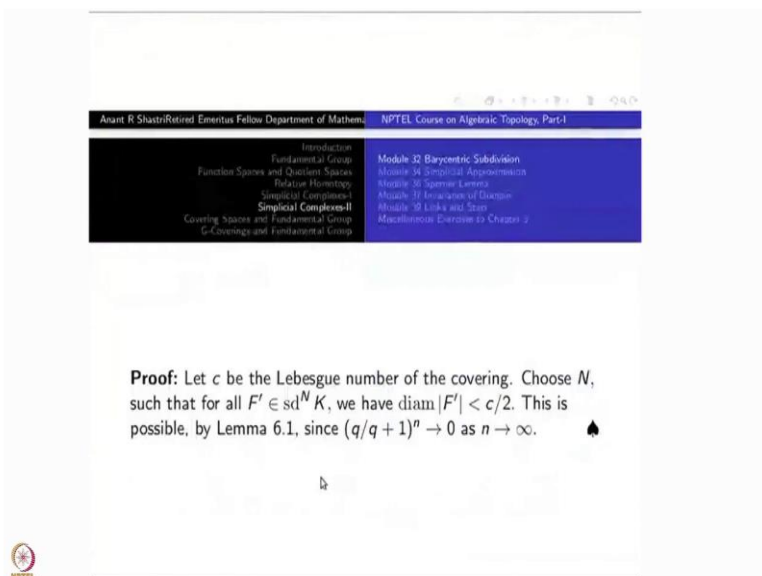


So, here I am using the standard result from topology of metric spaces. On any compact space, we have a open covering, a compact metric space, this is a metric space also, there is a Lebesgue number, the Lebesgue number c>0 is such that if you take any ball or any set of diameter less than equal to c, it will be contained in one of the open set of the cover. So, start with an open covering, choose c to be the Lebesgue number for that covering. Then choose this capital N, such that for all

$F' \in sd^N(K)$, we have the diameter of $F'$ less than c by 2. So, why this is possible? Assume the dimension of $K$ is q. Let us define $\mu(K)$ to be the maximum of the diameters of simplexes of $K$. Then $\mu(sd\ K) \leq \frac{q}{q+1}\mu(K)$. By repeated application of this we get $\mu(sd^N\ K) \leq \left(\frac{q}{q+1}\right)^N \mu(K)$. Since $\left(\frac{q}{q+1}\right)^N$ tend to 0 as $N \longrightarrow \infty$, this can be made less than c by 2 by choosing N to be sufficiently large. Once this smaller, diameter is smaller, then what happens to any star? Star is contained in the union of all the simplexes which have one common vertex. So, diameter of the star will be at the most twice that, that is why I have put c by 2 here. Once the diameter of each of them is less than c, this diameter of the star will be less than c by 2. Therefore, every star which is of diameter c will be contained inside one of the open sets.

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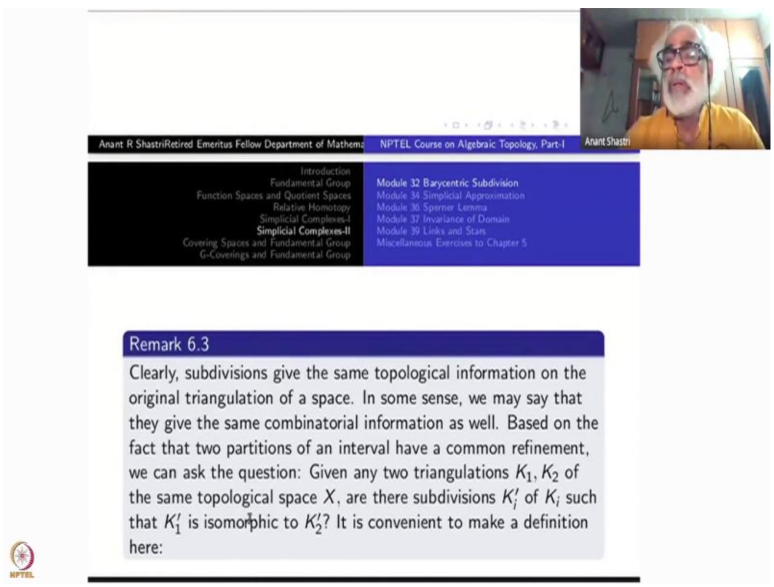


Subdivisions give the same topological information on original triangulated space because the homeomorphism type is the same. They do not change, in some way, you may say that they give same combinatorial information as well but you have to be careful there. Based on the fact that two partitions of an interval have a common refinement. So, it is combinatorial information is also not lost you may say. We can ask the question, given any two triangulations $K_1, K_2$ of a same space compact space X, are there subdivisions $K_1', K_2'$ of $K_1, K_2$ such that, these subdivisions are isomorphic? So, this is true for subdivisions of an interval. If you have two different divisions, you can take a common refinement. So, you can ask similar question for this subdivisions of some,

some subdivision you have taken then you can ask this question. But these things are questions for arbitrary subdivisions, which I have not discussed. I am just telling you some story here that is all.

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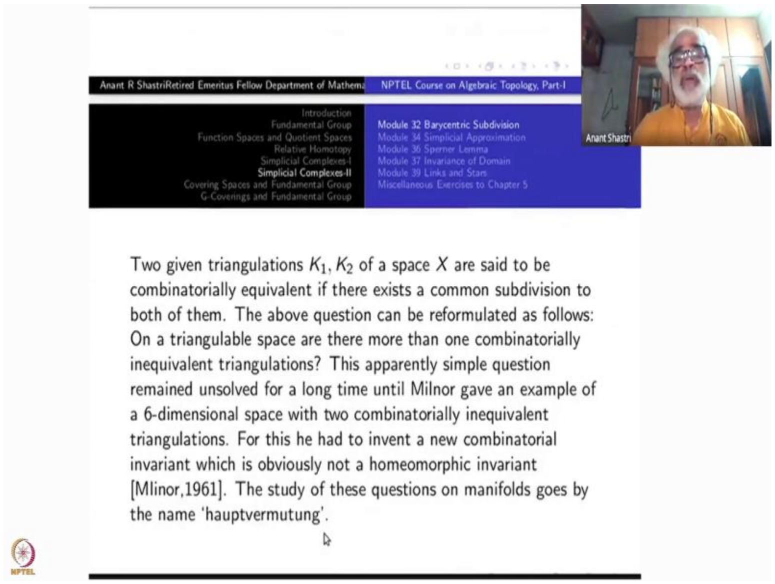


So, two given triangulations of a space  are said to be combinatorially equivalent, if there is a common subdivision to both of them. So, you can reformulate this question claiming that are they combinatorially equivalent. So, this simple question was solved by,  Milnor sometimes in 1961 in the negative.  He cooked up a new invariant for homeomorphism, invariant for this one and show that there are two simplicial complexes, different structures for this one, they have different invariants, so you cannot subdivide them to make them isomorphic.

But for manifolds less than equal to dimension three, the story is different.  This is a deep theorem, which gives a positive answer, so all these things are a big branch of mathematics. I am just trying to tell you so that  you are  aware of that there is lot of mathematics under these things.

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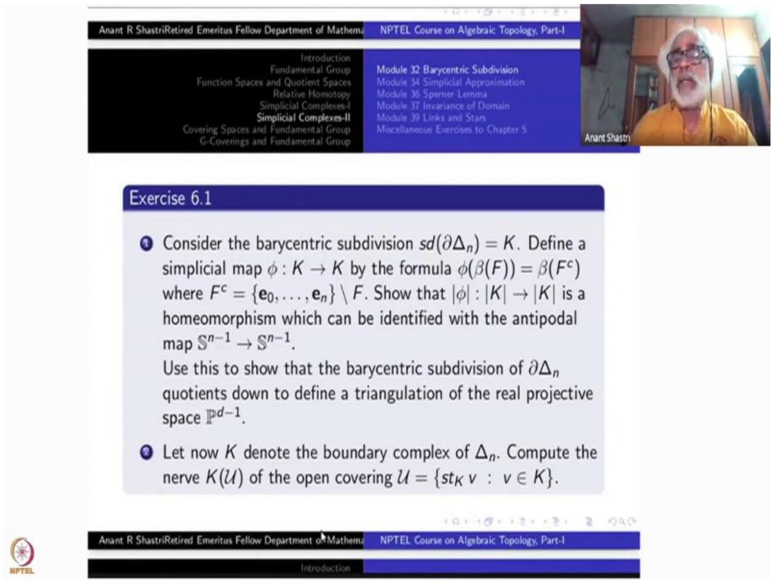


So, I have listed a few exercises, you can go through as soon as, as I keep telling you these things will be again recirculated to you separately. So, you do not have to know all these things, but I will just introduce one more concept here, we will not going to study this one very deeply right now.

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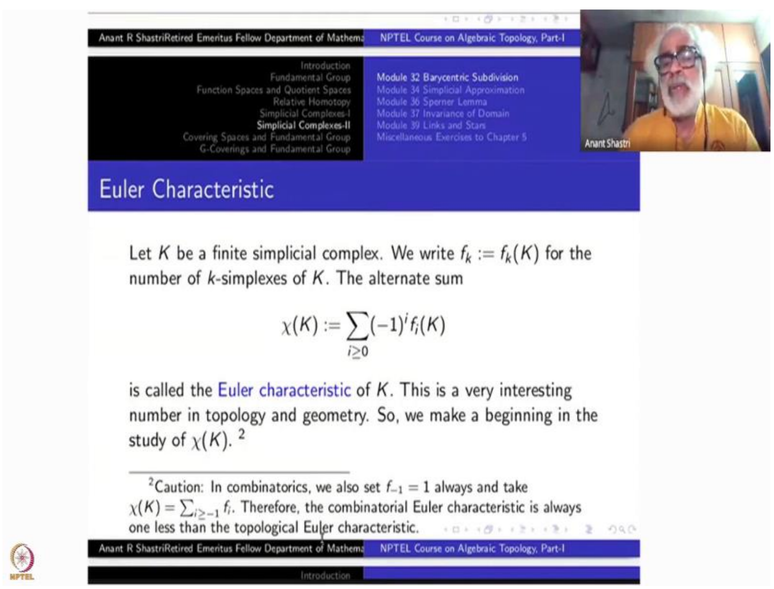


But it is a very, very important concept, what he said is combinatorial concept goes for Euler's that is why called Euler characteristic. What is it, look at a simplicial complex, just count the number of vertices, a number of edges, number of two simplexes, three simplexes and so on. So, that is called fi of K, fi of K is the number of I simplexes of K, take the alternative sum.

That is called the Euler characteristic of K. So, this is some integer, whether positive or negativity, one does not know. This very interesting object has a lot of geometric content in it. And it has lots and lots of applications. Many different (and very, very serious) formulation of this idea have been found, such as, have you must have heard of Poincare-Hopf index theorem, Atiyah Singer index theorem etc. They are all involved with this Ruler characteristic. So, when time permits slowly we will develop this one.

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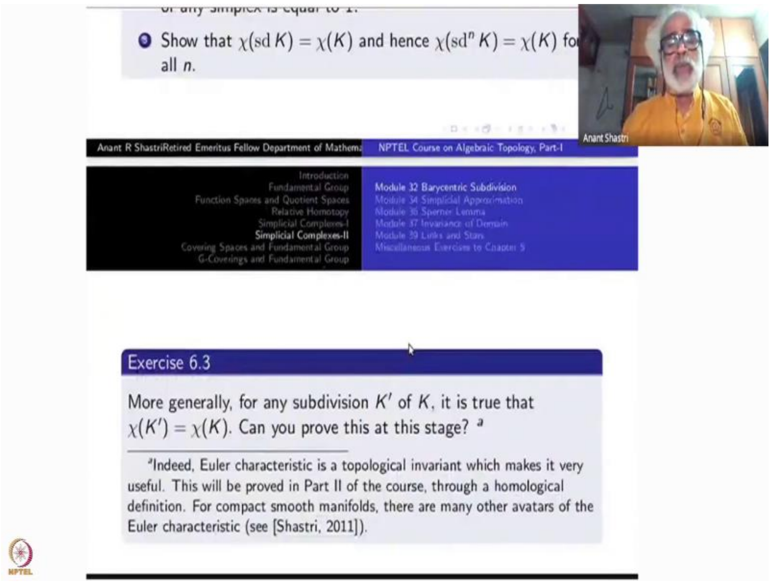


There are some simple exercises here. Based on the, Euler characteristic that is why I have introduced that one here. When time permits, we will do all these things in more detail. We will stop here.