Introduction to Algebraic Topology (Part – I)
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Lecture – 32
Barycentric Subdivision

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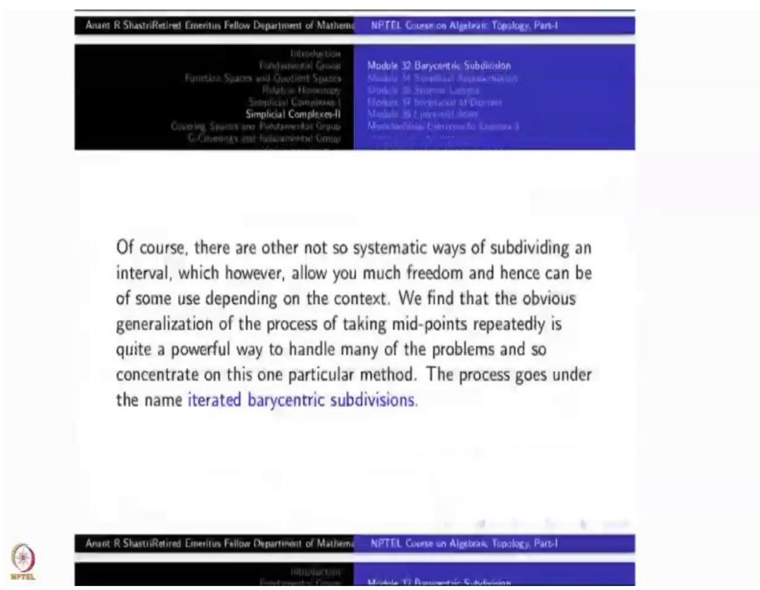

Barycentric subdivision: In real analysis, especially in real analysis of 1-variable, for instance, in Riemann Integration theory, Uniform continuity, etc, especially in integration theory, often you have to subdivide a given interval into finitely many sub intervals and then analyze what is happening in the situation. So this is practice copied in study of polyhedrons also. The idea of polyhedrons is that you can divide them. That is the whole idea.

So there are many, many different ways of doing it. Even in the Riemann Integration theory instead of taking all kinds of subintervals, you can just stick to each time dividing the interval by half, then one fourth, then 1 by 8th and so on. Then you show that if something happens for that subdivision, whole thing is fine. That kind of things are there.

So similar to that we concentrate on one particular kind of subdivision which is called Barycentric subdivision which will be quite powerful and that is the only thing which you are going to actually use here. So what you have, may have to do is barycentric subdivision itself it may have to keep repeating. It is just like, in the case of an interval, namely a closed interval, barycentric subdivision will be just corresponds to taking the interval 0 to 1 means 0 to 1 by 2, 1 by 2 to 1, so introducing the extra points iteratively at midpoints.

If you repeat it then you will be introducing one fourth as well as three fourth. If you repeat you will be introducing all factors of 1 by 8, 3 by 8, 5 by 8, all those points also, and so on. That is called iterated subdivisions, so iterated barycentric subdivisions. So these two things we are going to study here iterated barycentric subdivisions. So that is what we are going to do finally. So today's topic is barycentric subdivision and its consequences.

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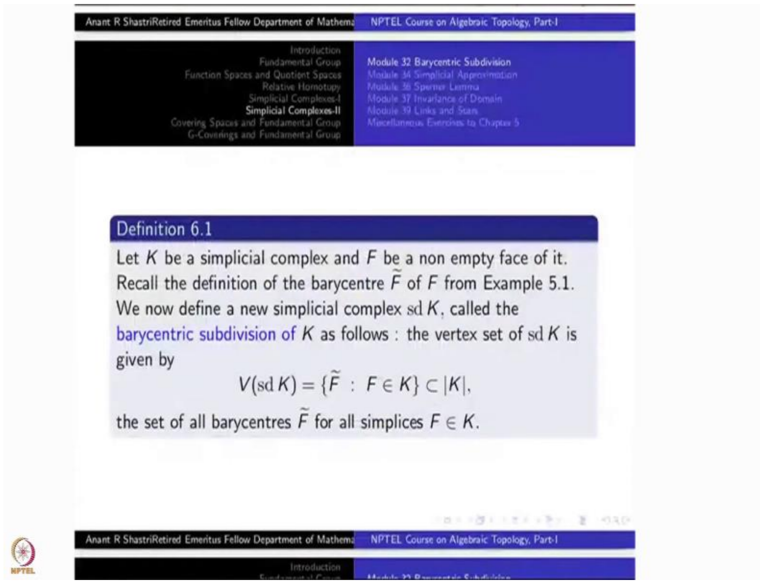

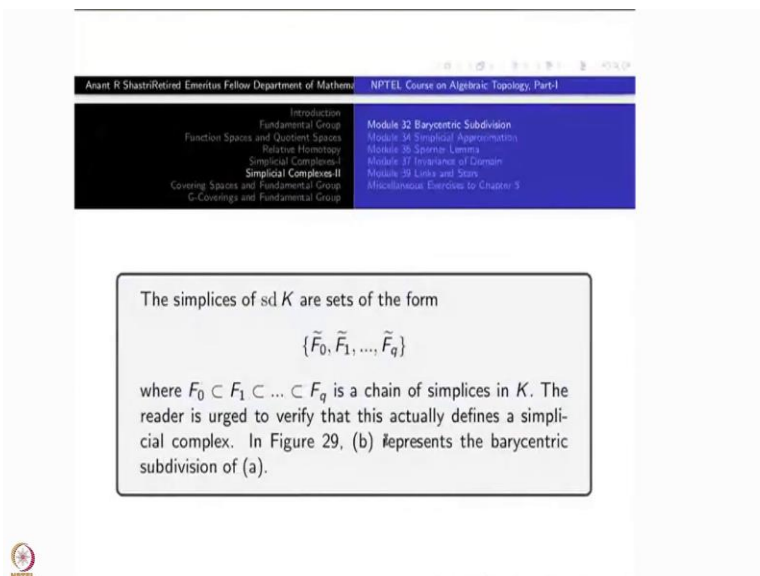

So let us be done with the definition very quickly. Start with a simplicial complex. For each non-empty face, recall what is the definition of barycenter of F denoted by $\beta(F) =: \tilde{F}$ .This is a point in $|F|$, namely the function $\alpha : V \to \mathbb{I}$, such that $\alpha(v) = \dfrac{1}{n+1}$ iff $v \in F$. Here n is the dimension of F. That is the barycenter. So that is all you have to know. This is the very beauty of this concept.

So we now define a new simplicial complex $sd\ K$. That is a short notation for `subdivision'. But it is a very particular subdivision. It is the barycentre subdivision sd. All other subdivisions are not of much importance for us. Called the barycentric subdivision of K, the vertex set of this $sd\ K$ is a subset of $|K|$. What are the points? Namely take $\tilde{F}$ for each $F \in S$. So $V(sd\ K) \subset |K|$ with a 1-1 correspondence with S.

So I have to define what are the simplexes, alright? The simplexes of $sd\ K$are, finite sequences

$\{\tilde{F}_1, \ldots, \tilde{F}_k\}$ where $F_1 \subset \cdots \subset F_k$ is a sequence of faces of K, one contained in the other. Such a thing you have studied, chains of simplexes F naught contained in F1 contained in F2 contained in Fq. No repetitions. It is a strict sequence. For example this could be one single vertex contained in a one simplex contained in a 3 simplex instead of 2 simplex, there can be jumps, contained in a, some 10 simplex, 10-dimensional and so on.

So take such a chain. Just put a twiddle on each of them. You get another sequence here, F naught twiddle F1 twiddle F2 twiddle. So take all these vertices. Declare that as a simplex of sd of K. Automatically if you have a subset of this that will correspond to a sub chain here. Therefore it will also be a simplex so that complete the definition of simplicial complex sd K.

Now what it is this good for? Just like our simplicial complexes you have defined abstractly, this is also defined abstractly except that the points are not abstract points now. They are already points of the geometric realization of K, mod K. So in that sense they are much better geometrically.

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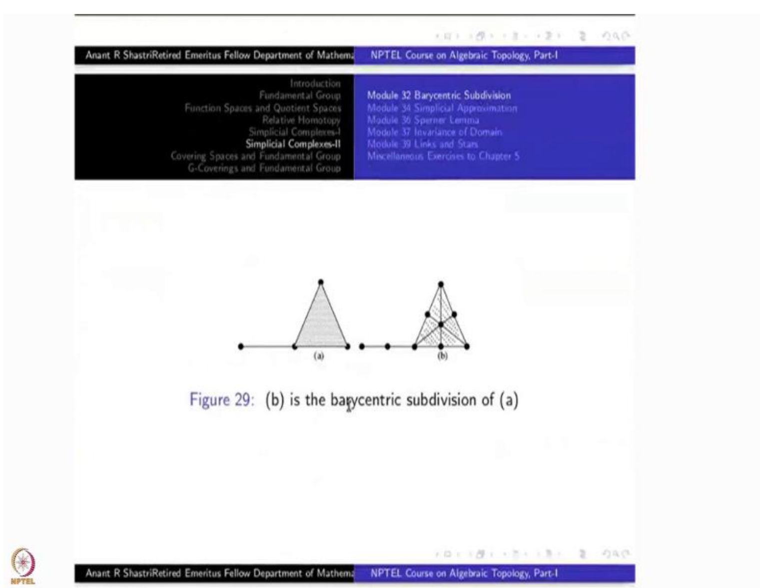

So here is a picture of what we are going to do. This is the original simplicial complex here with one edge and one triangle. Of course this triangle will have 3 edges and so on separately. But I am mentioning only the maximal simplexes here. This is going to be the barycentric subdivision. So I am showing you what are the simplexes, what are the vertices and what are the simplexes.

So for this one simplex here I am taking its barycentre. How did you get this one? This singleton itself is a simplex, 0 simplex. And its barycentre is itself. So that is allowed. Now if you take this simplex, singleton 0 and included in 1 simplex that is a chain of length 2. That will produce the barycentre of this one, and barycentre of this one which is this point. So a 1-simplex from here to here comes up. Similarly another 1-simplex from here to here comes up.

So now you can see that the original interval has been divided into two intervals. So that is the subdivision that is happening. We will make this one a little bit clearer later on but this is a motivation. So what is happening? In here you see that one single triangle is divided into 6 triangles at once. More generally what happens is if you take an n-simplex, in the barycentric subdivision it will get divided into $(n+1)$ factorial n-simplexes , n-dimensional simplexes.

A 1-simplex gets divided into two is 2 factorial 1-simplexes. This is 3 factorial and so on. So let us see what is the correspondence, why is it called subdivision? After all subdivision means the topological space is there. It has to be cut down and that means the pieces together should give you back, the union of all these pieces must be the given simplicial, namely polyhedron, the underlying topological space of the polyhedron. $|K|$ should not change.

In other words $|sd\ K| \equiv |K|$. Let us see how that is true. We do not have any freedom here now. We have to show somehow that mod of sd of K is, or up to some isomorphism, is same thing as mod of K.

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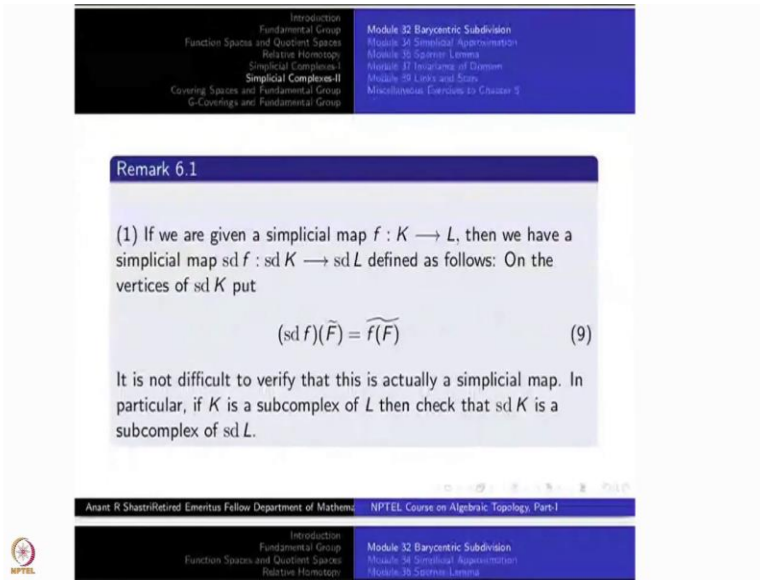

Before that, having defined for each K another simplicial complex sd of K, suppose you have a simplicial map $\phi : K \to L$. Then there will be a corresponding simplicial $sd(\phi) : sd\ K \to sd\ L$. This is simplicial map. It is defined at the combinatorial level itself not at the mod K to mod L. Mod K to mod L that will be $|sd(\phi)|$. That is already defined once $sd(\phi)$ is defined. But this will be $sd(\phi) : sd\ K \to sd\ L$.

How do you define it? First of all you have to define it on the vertex set. Vertex set consists of barycentres of simplexes of K. Take one barycentre like this $\tilde{F}$, send it to the barycenter of the image $\phi(F)$, which is a simplex in L because F is a simplex and ϕ is a simplicial map. This is a simplex in L. So you take the barycentre of this map. So this is the vertex map now. Vertex map you have to verify that it is a simplicial map, should take simplexes to simplexes. What is the simplex on the left hand side? First of all you start a chain of simplexes in K $F_0 \subset F_1 \subset \cdots \subset F_q$, strictly increasing. Look at the image of those, $\phi(F_0) \subset \cdots \subset \phi(F_q)$ will b a chain of simplexes in L

Now what may happen is some of these may become equal; Does not matter. You throw away the repeated ones and obtain proper chain. So you get a simplex of whatever lower dimension because chain may be of the smaller length; does not matter. Take the barycentres. That will be a simplicial map. So this is a simplicial map automatically.

Once you define a vertex map between simplicial complexes you have no other freedom. You have to verify whether this is a simplificial map or not. That is all. In general, a vertex map may fail to be simplicial map that is all. You cannot, there is no further definition to be made about $sd\ \phi$. Once you have defined it as vertex map then you have to verify whether it is simplicial map or not. So that is verified here.

So in particular if K is a subcomplex of L, we can take $\phi = \eta$ as the inclusion map here. Take a subcomplex. Then that is a simplicial map. Not all inclusion maps of vertices may be subcomplexes. But this is subcomplex. Take a subcomplex. And then take the inclusion map. That is a simplicial map. That will give you $sd\ \eta$. That will be automatically an inclusion map So sd of K will become subcomplex of sd of L if K is subcomplex of L, alright.

Now the second remark is, for each n-simplex F, you can think of F as a simplicial complex of dimension n, the full simplicial complex. Then $\mathcal{B}F \subset F$, the boundary of F is a subcomplex of F. So $sd\ \mathcal{B}F \subset sd\ F$. This inclusion map extends to a simplicial isomorphism: $sd\ \mathcal{B}F * \{F\} \cong sd\ F$. $sd\ \mathcal{B}F$ is the base of the cone and it is a subcomplex of $sd\ F$ If you take the cone over $sd\ \mathcal{B}F$ with the apex at $\beta F = \tilde{F}$, the barycenter, what you get is $sd\ F$.

Remember how $\mathcal{B}F$ is this defined? This is a proper subcomplex, every simplex here is a proper face of F, right? Something $F_0, F_1, \cdots F_i$ each $F_i \subset F$ but equal to F, right. So those are the proper sub-faces. If you take such a chain, and then tildes, that will be a subcomplex

$sd\ \mathcal{B}F \subset sd\ F$. Finally, the only simplexes missing in the sub but present in the larger complex are precisely those got by extending the chain by putting $\tilde{F}$ at the end. That is precisely the description of the cone over $sd\ \mathcal{B}F$.

Corresponding to this combinatorial isomorphism we get the homeomorphism $|sd\ \mathcal{B}(F) * \{\tilde{F}\}| \equiv |sd\ F|$. When you take mod of this, that will be isomorphic to mod of sd of F. But what is mod of this? It is the topological cone over $|sd\ \mathcal{B}F|$. Now $\mathcal{B}F$ is s simplicial complex of dimension n-1. Assume, inductively, that we have proved that $|sd\ \mathcal{B}F| \equiv |\mathcal{B}F|$. Then $|F| \equiv C|\mathcal{B}F| \equiv C|sd\ \mathcal{B}F| \equiv |sd\ \mathcal{B}F| * \{\tilde{F}\} \equiv |(sd\ \mathcal{B}F) * \{\tilde{F}\}| \equiv |sd\ F|$.

First we know that $|F|$ is the topological cone over $|\mathcal{B}F|$. By induction, this is the same as the topological cone over $|sd\ \mathcal{B}F|$. But that is the same as the modulus of simplicial cone over $sd\ \mathcal{B}F$. But we have see that the simplicial cone over $sd\ \mathcal{B}F$ is nothing but $sd\ F$.

So we want to verify the same thing for every simplicial complex K, $|sd\ K \equiv |K|$. That is the statement of the next theorem. It assures that topologically the barycentric subdivision does not affect any change. This remark (3) is just a caution of what is going to come but I have told you that key is here itself. The remark 2 tells you the whole story.

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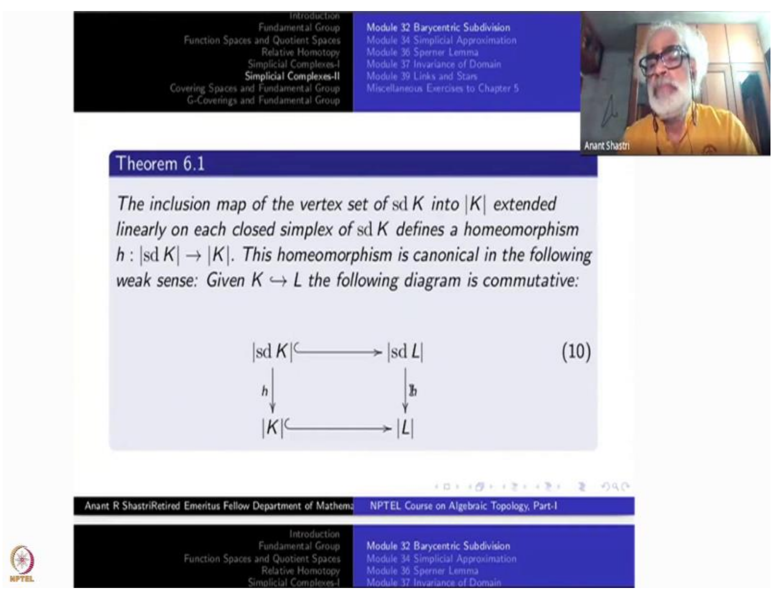

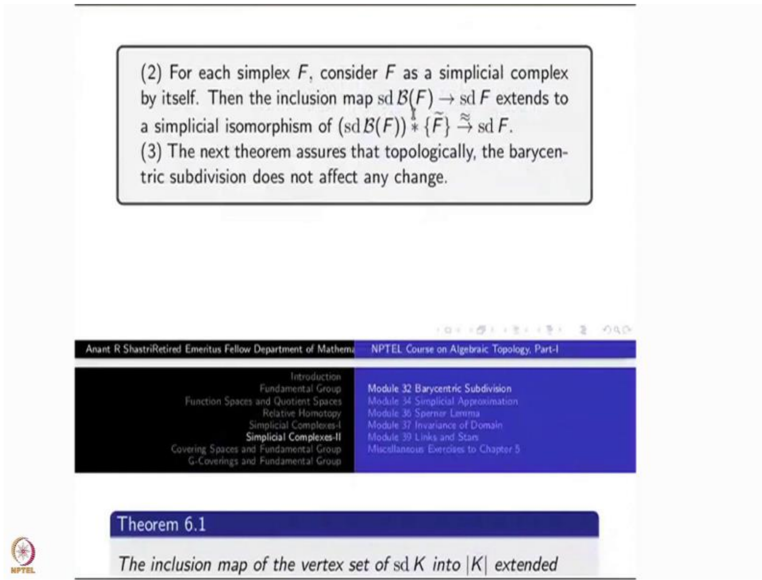

But I am putting it in a neat theorem now. I have already made the key observation, that has already been done into. Now we will summarize this here. There is the inclusion map of the vertex set of $sd\ K$ into $|K|$. Remember every vertex here is some point here, because it is $\tilde{F}$ of some simplex. $\tilde{F}$ means what? It is a barycenter of F. This inclusion map extends, linearly on $|G|$, where G is a simplex of $sd\ K$. You can extend it like that. Then the extended thing defines a homeomorphism of h from $|sd\ K|$ to $|K|$. So this homeomorphism is not something cooked up. It is just the linear extension, by which I mean affine linear extension on each sub-piece here, on each simplex here. Note that each $|G|$ is contained in some $|F|$. The linearity on that coincides with the corresponding linearity on this one.

The linear structure is same. Take α, β in $|G|$ and this part $|G|$ is contained in $|F|$, some simplex here. Then $t\alpha + (1-t)\beta$, for $0 \leq t \leq 1$ has the same meaning whether you take it here in $|G|$ or take it here in $|F|$. They will be the same thing. So that is the meaning of this one. This is already there for each simplex here in the, in this remark.

This homeomorphism is canonical in the following somewhat weak sense because the word canonical is usually used in a much stronger sense, namely functoriality. Here you do not have full functoriality. It is functorial for inclusion maps. Suppose you have subcomplex $K \subset L$. Then the following diagram is commutative.

You take $K \subset L$ which gives $sd\ K \subset sd\ L$. Passing to geometric realization, we get $|K| \subset |L|$

and $|sd\ K| \subset |sd\ L|$. The vertical arrows represent homeomorphisms the same canonical homeomorphism h and the diagram is commutative. That is the whole idea of saying that it is canonical.

It is a homeomorphism here, inclusion map here. So this diagram is commutative whether you include it here this way and come here or this way come here. How to verify this? You have to do it for one simplex each time. That is all. And that is taken care by this, this key lemma, key observation here.

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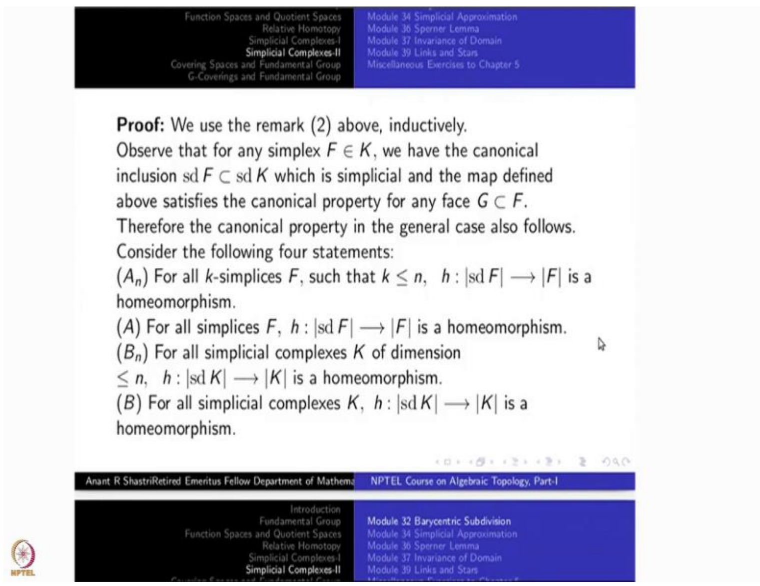

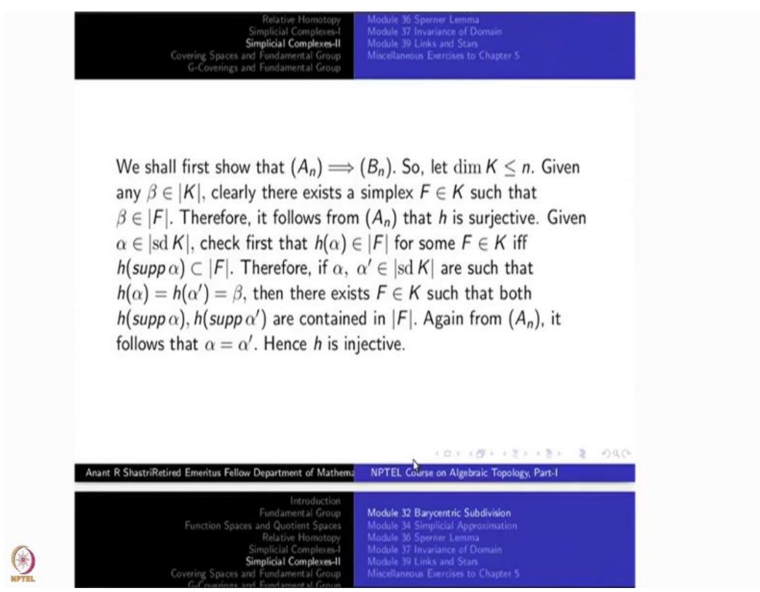

Clearly h is continuous also. To see that it is a closed map, let $G \subset |\mathrm{sd}\, K|$ be a closed set. We must verify that $h(G) \cap |F|$ is closed in $|F|$ for each $F \in K$. But $h(G) \cap |F| = h(G \cap |\mathrm{sd}\, F|)$. Since G is closed, $G \cap |\mathrm{sd}\, F|$ is closed in $|\mathrm{sd}\, F|$ from $((A_n)$ we conclude that $h(G) \cap |F|$ is closed in $|F|$.

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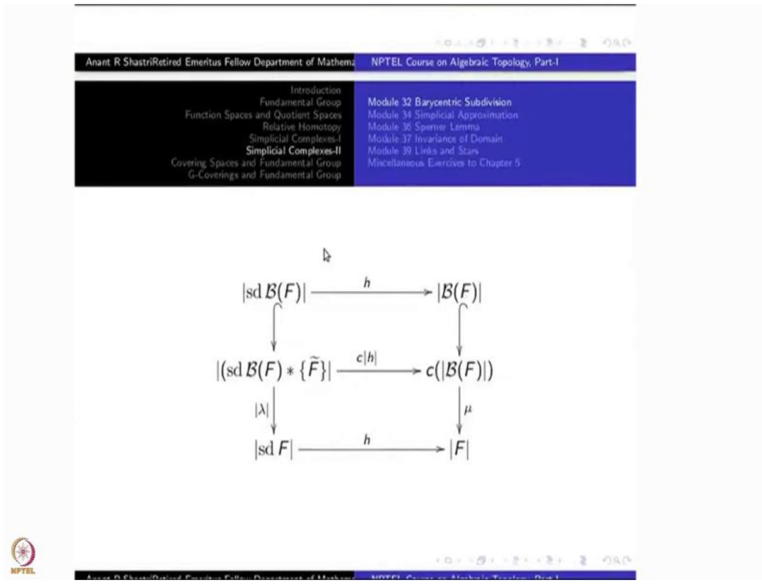
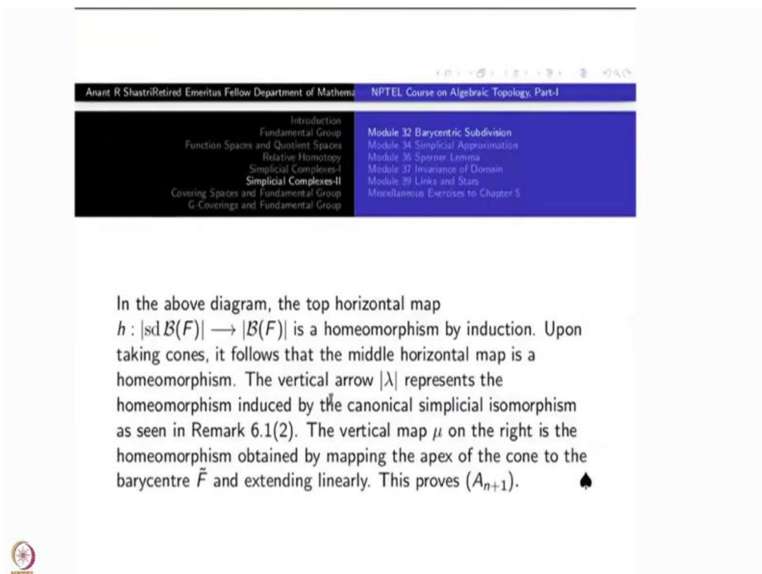
To prove $(A) \Longrightarrow (B)$, we merely replace (A_n) everywhere by (A) in the above argument. Thus, in order to complete the proof of the theorem, we have only to prove (A_n) for all n, which we shall do by induction.

If dimension of F is 0, then both F and $\mathrm{sd}\, F$ coincide and there is nothing to prove. Suppose we have proved (A_n) for all simplices of dimension $\leq n$, for some $n \geq 0$. The theorem is then true for all simplicial complexes of dimension $\leq n$. Let now $\dim F = n+1$.





So let me repeat. Remark 2 is used inductively. This is what. Observe that for any simplex F, here the canonical inclusion $sd\ F \subset sd\ K$ first of all, which is a simplicial map. And the map defined above satisfies the canonical property for each face G contained inside F. You take G contained F from sd of G to sd of K you go. Or first go to sd of F and then to sd of K. They are the same because they are all inclusions. Therefore the canonical property in general case follows. Once you verify it for each faces and sub-faces.

Now I will tell you why sd of K to K itself, h is a homeomorphism. Let us go little slowly here so that we are not making any mistake. So first I say the statement (A_n) is true for all simplices, k-simplices, where $k \leq n$, this is a homeomorphism. So I am trying to build up some inductive hypothesis here. For when n is 0 you can verify it easily. For n equal to 1 also you can easily verify. But when it becomes larger value you do not know. So you would like to make an inductive hypothesis and work out.

Second statement (A) does not involve any n at all. It says, for all simplices F, h is a homeomorphism. So how do you get this one? If you do it for every n here you get this one. That is whole idea, right? That is the meaning of here induction. Now here (B_n) is for all simplicial complexes of dimension less than equal to n, h is a homeomorphism. Then the statement B is for all simplicial complexes K, $h : |sd\ K| \to |K|$ is a homeomorphism, which is the final result we want.

So this is, the statement B is what we want to prove. What we have observed is (A_0) and (B_0). Also we have observed that (B_n) implies (A_{n+1}). Now we shall show that (A_n) actually implies (B_n).

Let me repeat. Take a 0-simplex, what is the barycentric subdivision? It is the simplex itself. There is only one point. So, h is actually the identity map. Therefore (A_0) is true. From this you will conclude (B_0) for all 0-dimensional simplicial complexes. What is 0-dimensional complex? It is just the disjoint union of vertices. When you take barycentric subdivision it will be same set of vertices, nothing more. Therefore h is identity map, right.

So $(A_0), (B_0)$ are verified. Clearly (A_n) for all n implies (A).

Let us see how (A_n) implies (B_n). For dimension 0, we have sen this. Let dimension of K be equal to n>0. By induction viz., (A_{n-1}) implies (B_{n-1}), we have $h : |sd\ K^{(n-1)}| \to |K^{(n-1)}|$ is a homeomorphism. In particular, for every n-simplex $F \in K$, we have the restriction of $h : |sd\ \mathcal{B}F| \to |\mathcal{B}F|$ is a homeomorphism. By (A_n) this extended to a homeomorphism $h : |sd\ F| \to |F|$. In particular since $K = K^{(n)}$ here, this implies that $h : |sd\ K| \to |K|$ is surjective. Now suppose $h(\alpha) = h(\beta)$. We want to show that $\alpha = \beta$. Say $h(\alpha) = h(\beta) \in \langle F \rangle$ for some $F \in S$. Then it follows that $\alpha, \beta \in |sd\ F|$. But then (A_n) implies $\alpha = \beta$. Therefore h is a bijection. We shall now show that h is a closed map which will complete the proof that h is a homeomorphism.

Take a G which is a closed subset of $|sd\ K|$. We must verify $h(G)$ is a closed subset of $|K|$. For that I have to verify that $h(G)$ intersection with $|F|$ is closed in $|F|$ for any simplex $F \in S$. But $h(G) \cap |F| = h(G \cap |sd\ F|)$. Since G is closed, G intersection sd of F is closed. And h restricted to $|sd\ F|$ a is homeomorphism because of (A_n). Therefore $h(G) \cap |F| = h(G \cap |sd\ F|)$ is closed. Thus h is a homeomorphism.

Incidentally, the proof that (A) implies (B) is exactly same as above. Since we have already seen that (B_n) implies (A_{n+1}), the proof of the theorem is complete.

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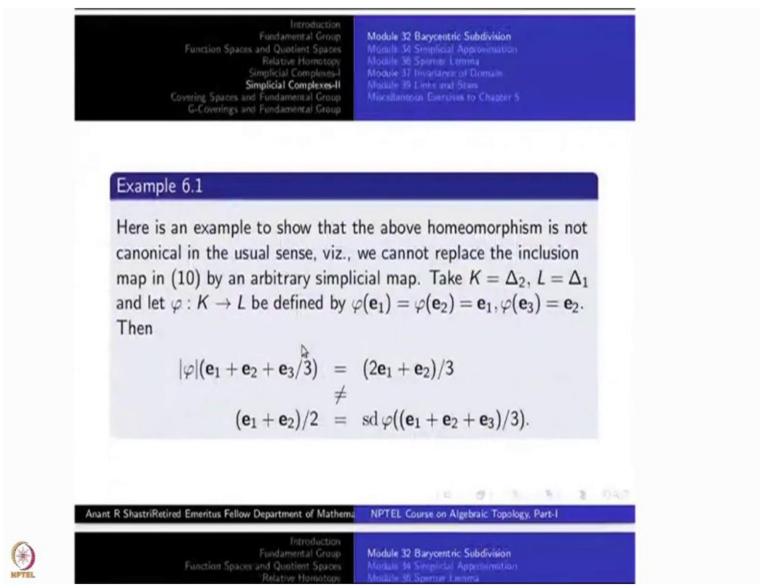

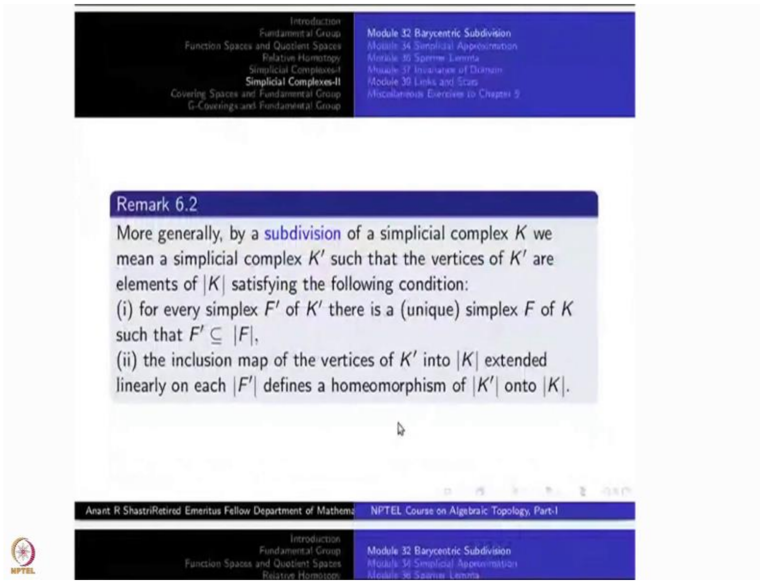

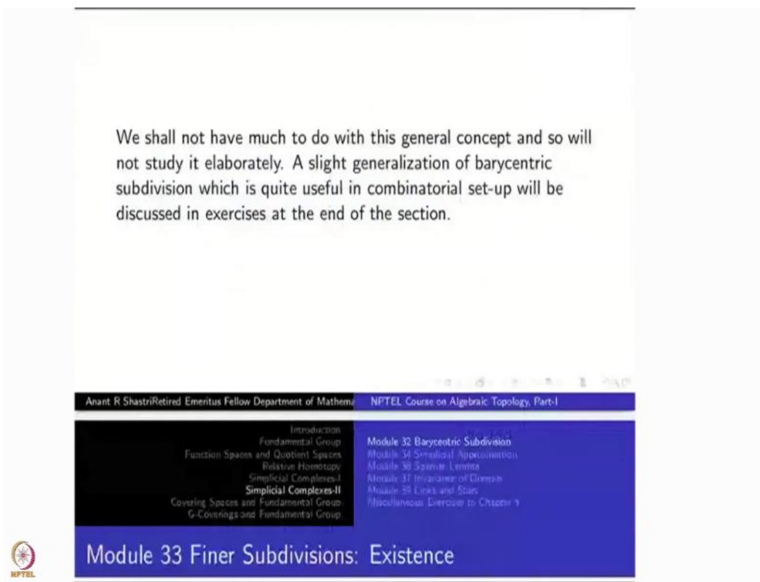

So now I will give you an example that why the canonicalness fails here in general, if you do not have inclusion maps. So very simple example. Take $K = \Delta_2$ a triangle, and $L = \Delta_1$, as a standard 1-simplex. Then I want to make a map $\phi : K \to L$; $e_1, e_2 \mapsto e_1$, and $e_2 \mapsto e_2$. 1 From a set with 3 points I have function to a set with 2 points. So two of the points must go to the same thing. So e1 and e2 go to e1. e3 goes to e2. Now let us compute $|\phi| : |K| \to |L|$. For instance what is $|\phi|(\beta(\Delta_2))$? $\beta(\Delta_2) = \dfrac{e_1 + e_2}{3}$. So we get this is not a barycenter at all. You see $\dfrac{1_1 + e_2}{2} = \beta(\Delta_1)$. On the other hand $\phi(\Delta_2) = \Delta_1$ and hence by definition $sd\ \phi(\beta(\Delta_2)) = \beta(\Delta_1)$. So these two are not equal. So, $|\phi|$ and $|sd\ \phi|$ are not equal. So this is what you have to be careful about.

More generally, I just want to tell you that there are other kinds of subdivisions. Normally, one writes K' for a a subdivision of K. What is the property? The basic property what we observed for barycentric subdivision is taken as a definition. First of all, for every simplex F' of K', there is a unique simplex F of K such that $\langle F' \rangle \subset \langle F \rangle$. And of course $|F'| \subset |F|$ also. Next for each F', the inclusion map of the vertices $F' \subset |F|$ inside mod K can extended extended linearly to a map $\mu : |F'| \to |F|$. That makes sense. Put all of them together that must give you homeomorphism $\mu : |K'| \to |K|$.

So that is the definition of arbitrary subdivision. In this to cut the interval [0,1] you do have to take the point ½. You take 0, 1/3 and 1 that will do. Or take 0, 1/3, 2/3 and 1. Even that is too regular. Indeed, you can take $o < a_1 < a_2 < \cdots < a_n < 1$ any such sequence. That will give you all sorts of subdivision of [0,1]. Then you do not have to write what are the simplices. That is understood. That is the convention we are following in analysis. But underlying that what we have is you have introduced vertices. You have to introduce what are the 1-simplexes you have to tell.

So that is clear in the case of interval because, just take the order. But when you go to triangle just declaring the vertices is not enough, right; whereas in the case of barycentric subdivision it is automatic. We know there is just one single formula. Somebody takes barycentric subdivision of simplicial complex. Somebody else takes it. Both of them will get the identical result.

So the general subdivisions, we will not study them deeply here. But in the exercises, later on, I have included a few things about them because they are also useful when you are studying different kinds of problems, that is all. Alright so let us stop here and next time we will do what is the meaning of finer subdivisions; comparing subdivisions.