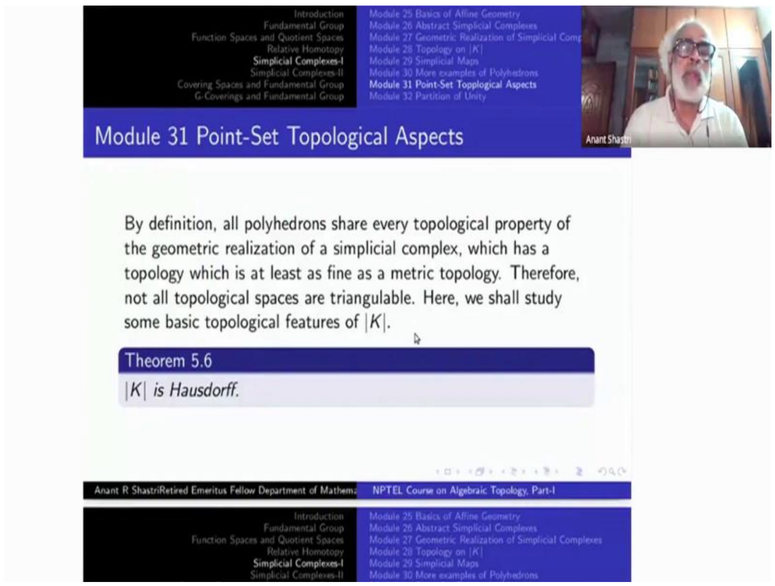
**Introduction to Algebraic Topology (Part – I)**
**Professor Anant R. Shastri**
**Department of Mathematics**
**Indian Institute of Technology, Bombay**
**Lecture – 31**
**Point Set topological Aspects**

(Refer Slide Time: 00:15)

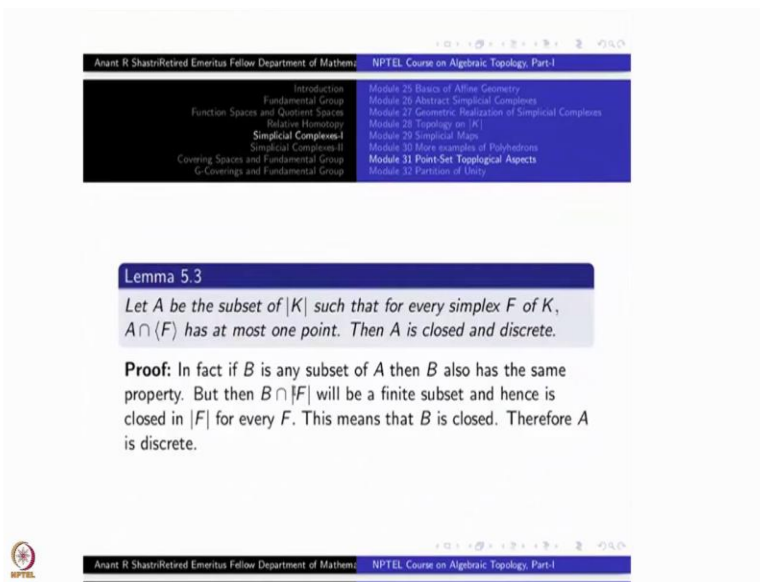



Today we will talk about the point set topological aspects of simplicial complexes. So this is the topic for module 31. By definition, all polyhedrons are homeomorphic to the mod of simplicial complexes, the geometric realization of simplicial complexes, right? So they will share all the topological properties of that.

And we have seen that, in the very definition, that the topology on $|K|$ is finer than the metric topology induced on it by its inclusion in $\mathbb{I}^V$ of those functions which take only finitely many non-zero values. Anything which is finer than a metric topology will have many, many other properties. One such thing is that it must be Hausdorff.

Any space which is finer than a metric space is automatically Hausdorff because the metric space is Hausdorff. So the first theorem is obvious here. $|K|$ is Hausdorff.

(Refer Slide Time: 02:00)


So the next preposition here is the key for several important properties of $|K|$. But this key is totally obvious from the definition, namely every $|K|$ is actually the disjoint union of all its open simplexes. Starts with 0 simplexes. By definition it follows that all 0-simplexes are open (as well as, closed) simplexes. Remember a modulus of a simplex consists of functions which are non-zero at most at the vertices of the simplex, with various properties. Summation should be zero and so on, right?

And the open simplex consists of those which have precisely not equal to 0 on that set; on the set $F$. Everywhere else it is 0. On that set, on all points here, the coordinates is non-zero. That is open simplex. The 0 simplex, the modulus of a singleton nothing but the function which takes 1 there and 0 everywhere, everywhere else, on every other vector.

So automatically singleton, modulus of singleton is both, it is an open simplex as well as closed simplex. So do not confuse it with open subsets and closed subsets. A closed simplex is a closed subset also but open simplex is, need not be open everywhere. They are open inside $|F|$, the closure of $F$, closed simplexes, that is all.

But what I want here is that, $|K|$ is precisely equal to disjoint union of these open simplexes. Take any point $\alpha$ here. It belongs to precisely one of them, namely, $\alpha \in \langle supp\ \alpha \rangle$, its support, look at the support. That is the simplex. Then open simplex therefore contains the point, contains that function and nothing else will contain it. Therefore this is a disjoint union of these things. Is that clear?

Student: Yes Sir, clear.

Professor: We will use this in a very, very certain, very nice way. This lemma we will use that. Take any subset of $|K|$ such that every simplex $F$ of $K$ has exactly one point in the interior of $F$ from A; $A \cap \langle F \rangle$ has at most one point. See, some of them may not intersect alt all. If they intersect, then the number of points of intersection will be 1 and it should be the interior point of $F$, one of the interior points. Assume this.

This is a condition on the subset A, not a condition on $K$ or $F$. For each $F$ this should happen. $A \cap \langle F \rangle$ can be empty or a singleton point. And that point must be the interior here. If that happens then A is a closed subset of $|K|$ and it is discrete. I hope you know what is a discrete subspace of a topological space. So the proof is very straightforward.

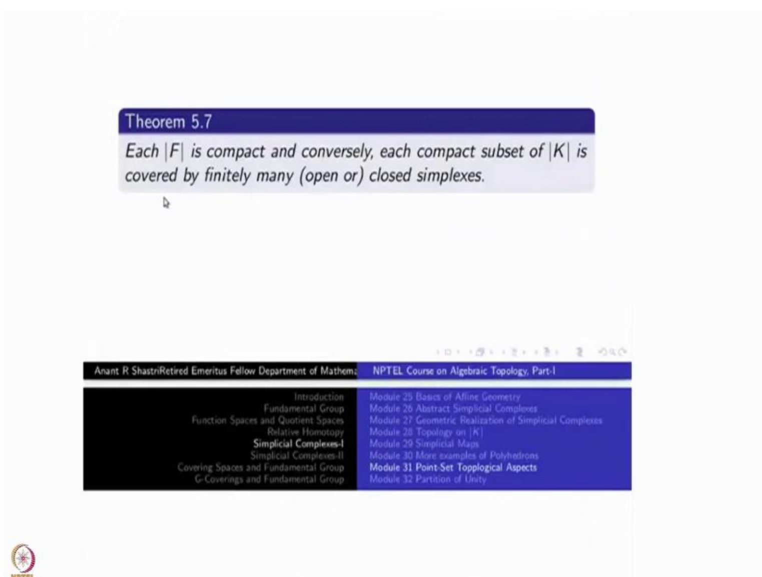
We use the fact that $|K|$ is disjoint union of its open simplexes. Suppose this is true for A, the assumptions on A, I have given. Now you take any subset of B, any subset B of A. That will also have the same property, right? It will have fewer points. So that will have the same property. B $\cap$ $\langle F \rangle$ may even be empty also but it cannot be more than 1 because if A intersection F has 1, so B intersection F will also be at most 1. It is the same property.

In particular, what happens is, if you take the closure of F, closure of F will have all the smaller open simplexes also. Namely take any subset of $F$, proper subset of $F$. All of them will come. That is the union. Closure of F itself is disjoint union of open simplexes, namely all these open simplexes are subsets of $F$ now including $F$. Therefore there will be only finitely many of them. If this has only finitely many open simplex, each of them contributing at most one point B intersection F will be finite subset. Any finite subset of $|K|$ is a closed set because $|K|$ is Hausdorff. So $B \cap |F|$ is closed subset of $|F|$ for every $F \in S$.

But then we know that if this is the property of any subset that B itself must be closed in $|K|$. Intersection with every mod F is closed in mod F means B itself is closed in $|K|$. So what we have proved is that every subset of, every subset B of A, every subset B of A is a closed set. In particular A is closed and it must be discrete. If every subset is closed, every subset is open also. So it is a discrete, so discrete topological space as subspace of $|K|$.

And it is closed. The topology induced from $|K|$ on A is discrete. It is a discrete space. Every point is open in particular, alright. Now this lemma was an easy consequence of the easiest observation here. Now we will derive some consequence of this lemma 5.3 which is not so obvious at all.

(Refer Slide Time: 08:59)




Each $|F|$ is compact. That is obvious because we have seen that it is homeomorphic to a disc, right. But conversely each compact subset of $|K|$ is covered by finitely many open or closed simplexes. A closed simplex is a closed set. And open simplex is not an open set. It is not an open cover for $|K|$. So do not make that mistake.

If we have a compact set  has an open cover then it has a finite subcover. That is not the point here. The point here is that these open sets are not at all open in $|K|$; only maximal simplexes will have open simplexes open inside $|K|$. Yet only finitely many of them will cover it. This is the point. Let us see how.

Once open simplexes, finitely many of them cover it; the close simplexes will follow because you can take the closure also. So there will be finitely many of them. So they will cover it. So I am going to prove this, finitely many open simplexes themselves will cover.

So how do I prove it? Well, $|F|$ is homeomorphic to a closed disc. That we have seen. So it is compact. Now I am going to prove something. Start with any compact set L contained inside $|K|$. Then we can construct a subset A of L such that A intersection with any open simplex  is at most a singleton.

In other words, given $F \in S,$ if $L \cap \langle F \rangle$ is non empty, then  then I pick up one point from this one ant put it inside A.  L is given to be compact.  we are going to construct a subset A of L. Look at $L \cap \langle F \rangle$.  If it is non empty, pick up one point from this and  put it inside A. If it is empty do not worry. So what is this set A? A is subset of L. And for $F \in S$, A intersection with $\langle F \rangle$ is at most a single point. Therefore A has exactly the same property as in this lemma, previous lemma. Therefore A is a discrete set. Therefore being a subset of a compact set, a discrete set must be finite. What is the outcome? Look at, for each point in A, there is one and only one F such that $\langle F \rangle$ is non empty. But if you take all the F they will cover the whole of L. Therefore there are only finitely many F1, F2, F's like this corresponding to elements in A which will intersect L.

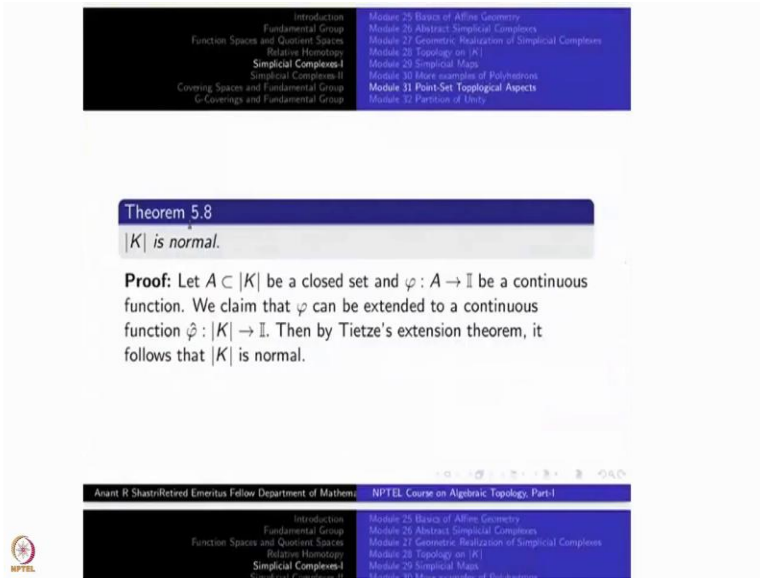
So the first part is over. Take closures here. So they will also cover L. That will give you second part here.

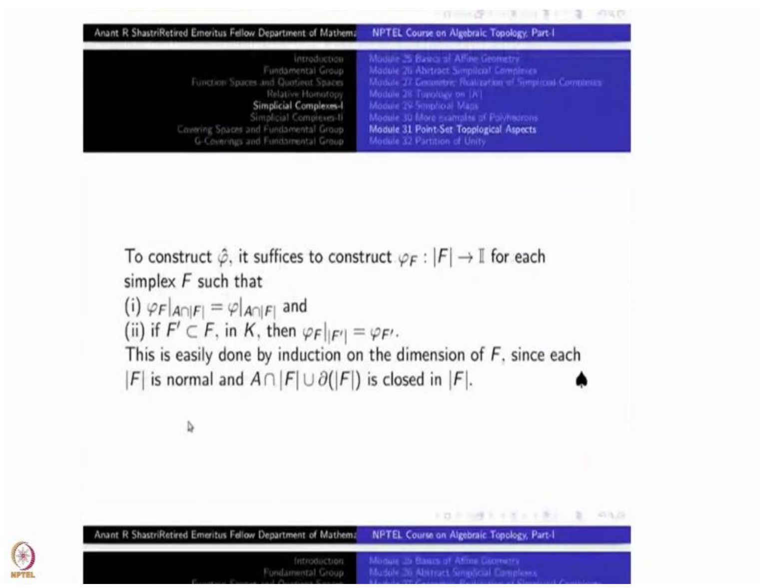
It is not true that number of closed simplexes which meet a given closed simplex is finite in general. No. You can take a vertex. And then there can be infinitely many edges which are incident at that vertex. That is allowed.

Similarly one single edge maybe incident on several triangles, infinitely many triangles. So a simplex you have taken in this very statement, mod F is covered by mod F. So why do you need so closed simplexes, right? But this part is very important. Every compact set is closed by this one. That is the whole thing. But mod F is covered by finitely many open simplexes.

So I should have stated it maybe separately these two things. But mod F is covered by mod F itself, one single simplex. What is the point? That is not the beauty. Beauty is that the proof shows that every compact set is covered by finitely many closed closed simplexes like this.

(Refer Slide Time: 14:41)




The next thing is another important property in topology, that $|K|$ is actually normal. You can also say that it is finer than the metric topology right, so it must be normal. But that is not correct, there is a catch there. Something is normal and you have a finer topology, it may not be normal. Alright metric space is for normal, fine. So what?

But what we have taken is a finer topology. So you have to careful here. You have to prove that $|K|$ is normal. It is not like Hausdorffness. So let us not hand wave here. Let us get a proof of the fact that $|K|$ is normal. What is normality? Given any two disjoint closed sets, you must be able to separate them by open sets-- disjoint open sets.

But there are other criterion. Given any closed subset and a continuous function into $\mathbb{I}$ you can extend it to the whole of $|K|$. That will also prove that $|K|$ is normal. So this is the property given by Tietze's extension theorem. If a space has Tietze's extension property then it must be normal, and conversely. That is what you must have learnt in point set topology. So I am going to use that.

So starting with a closed subset A of $|K|$ and a continuous function $\phi$ from A to $\mathbb{I}$, I want to say that there is a continuous function $\hat{\phi}$ from $|K|$ to $\mathbb{I}$ which extends $\phi$. If I prove this, by Tietze's extension theorem, it follows that $|K|$ is normal. There may be different ways of proving this one but I find this one to be the simplest. So how to construct $\hat{\phi}$?

So again, the recipe we have already declared how to construct continuous functions on $|K|$? Construct them on each simplex, each closed simplex in such a way that on a smaller simplex whatever you have constructed is extended on the larger simplex. If a function is constructed like this then as a function it makes sense first of all.

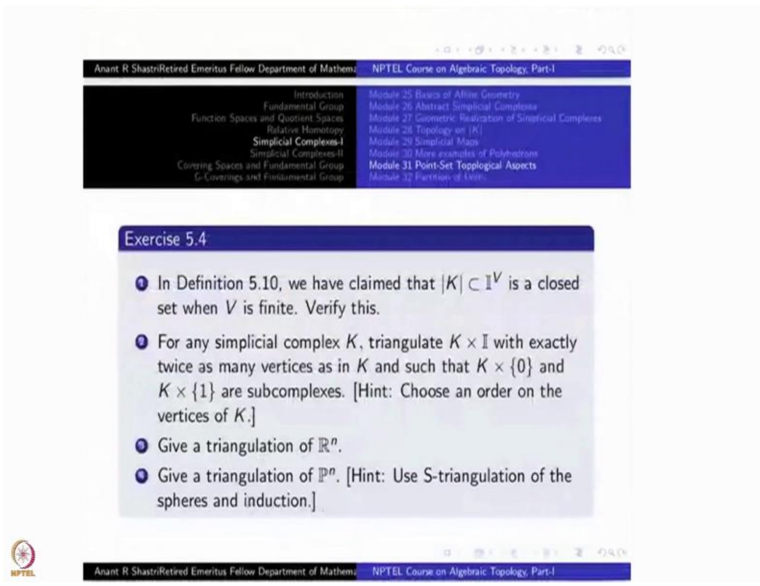
Continuity follows because restricted to each $|K|$ it is continuous. This is the recipe we are going to follow here. This will be the first time perhaps we have doing it like this. So what I am going to do is I will construct a family of functions $\phi_F$, phi indexed by $F$ on $|F|$ to $\mathbb{I}$ for each $F$, $F$ is simplex, such that if you restrict it to $A \cap |F|$, this must be the given function $\phi$. That is the first condition. If it does not intersect I do not care. That could be anything.

Second condition is that if I already constructed it on $|F'|$, for $F' \subset F$, then $\phi_F$ that I am going to construct should be such that when restricted to $|F'|$, it must be must be $\phi_{F'}$. So this what is called compatibility. If you are constructed the function on all the vertices then when you are extending to edge, when you are defining it on an edge, you should take care that on the endpoints it is already the function that you have constructed.

And once all the edges are done on a triangle when you are defining, on the boundary of this triangle it should be already the function that you have constructed, so if you follow that then you are done. So inductively you can do this namely, take $F$, suppose you have already defined it on the boundary which is the union  lower dimensional simplexes, union of lower dimensional simplexes. If you can extend it over F you are done, alright.

This is easily done by induction. That is what I am telling you. Let $F$ be an n-simplex. Since each $|F|$ is normal because it is homeomorphic to a disc, each mod F is normal. By induction the function is defined on $(A \cup K^{(n-1)}) \cap |F|$ which is closed subset. On that closed subset you have a function, continuous function. So apply that mod F is normal. You get an extension $\phi_F$ on mod F by the same Tietze's extension theorem. Put them together, you have the whole function on $|K|$.

(Refer Slide Time: 20:29)




Here are few remarks and exercises. I will go through a couple of them. Exercises are after all, exercises. One of the things that we claim that $|K|$ as a subset of the product space here $\mathbb{I}^V$ is a closed subset if V is finite. You are asked to verify this. We have defined; actually if you just read carefully; when V is finite we have defined the topology in a particular way. Use that to see that it is closed subset.

For any simplicial complex $K$ this is... This is a typo. I have corrected it but today I have taken wrong slide here. Instead of $K \times \mathbb{I}$ it should be $|K| \times \mathbb{I}$. Triangulate $|K| \times \mathbb{I}$ with exactly twice as many vertices as in K such that $K \times 0$ and $K \times 1$ are subcomplexes. Remember $K \times \mathbb{I}$ this denotes the prism over $K$. So I do not want to use that notation here. $K \times \mathbb{I}$ is a simplicial complex whereas $|K| \times \mathbb{I}$ is a topological space, alright.

So I want to give you another triangulation of $|K| \times \mathbb{I}$ in which no extra vertices are introduced. Only from K cross 0 all the vertices of K will be there. A copy of that one K cross 1 again, same thing should be there. For example, if K is a just one edge then K cross I is a square. So then you are allowed to take only 4 vertices, no more vertices and triangulate it.

And we have shown that there are two ways of triangulating it. Either join this diagonal or the other diagonal. There are two diagonals there. You can take any one of the diagonals as another extra edge and you are done. Same thing you should try to do for all simplexes. So that is the exercise here. But there is no canonicalness here now.

And hint is; choose an order on the vertices, some, some linear order. Order means that. That is the hint, like when you have one edge you have E1, E2 or E naught, E1 whatever. So you have to choose an order and then use that order to write down. That will help.

Next, I have given you a triangulation of $\mathbb{R}$. Now you are asked to give a triangulation of $\mathbb{R}^n$. Do that.

Give a triangulation of the projective space $\mathbb{P}^n$. So hint is use S-triangulation, but this is not going to help you much. But it is just a hint that is all. This is not going to help. Use S-triangulation of the spheres and induction. Induction will be correct. But this is not quite, quite the stuff here. S-triangulation will not give you triangulation of $\mathbb{P}^n$.

(Refer Slide Time: 24:30)

**Exercise 5.5**

1. Let $K$ be a simplicial complex. Show that the following conditions are equivalent:
   (a) Each vertex of $K$ belongs to a finite number of edges of $K$.
   (b) Each vertex of $K$ belongs to a finite number of simplices of $K$.
   (c) $|K|$ is locally compact.
   We say $K$ is locally finite if the above conditions are satisfied.
2. Suppose $f : |K| \to \mathbb{R}^n$ is a topological embedding. Show that
   (a) $K$ is locally finite.
   (b) $K$ is countable, i.e., the vertex set of $K$ is countable.
   (c) $\dim K \leq n$.





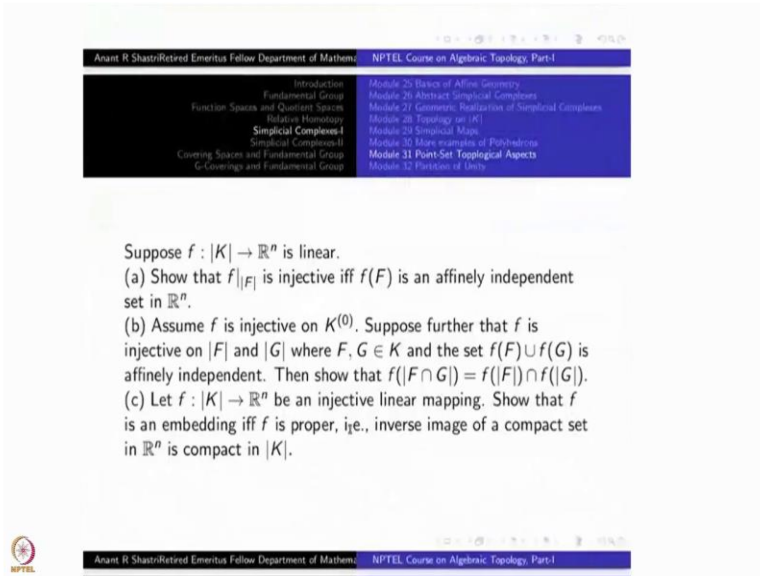

## Embedding $|K|$ in $\mathbb{R}^N$

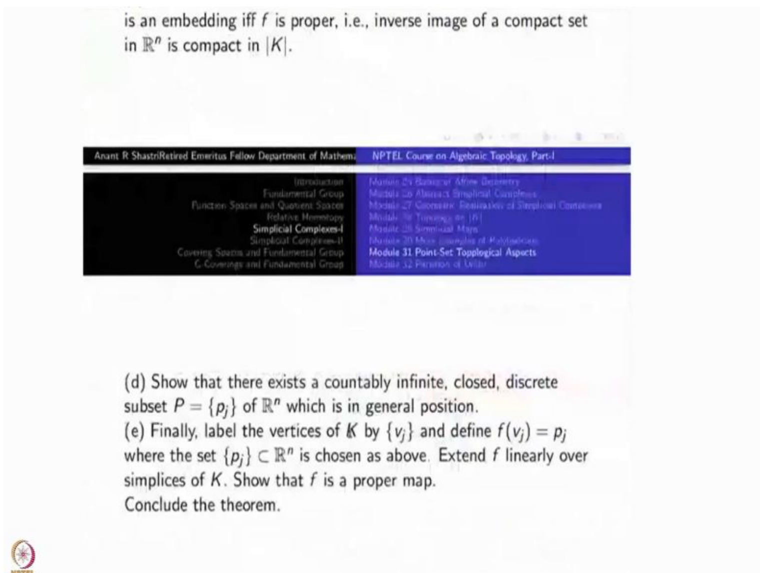
**Exercise 5.6**

A map $f : |K| \to \mathbb{R}^n$ is said to be linear if restricted to each simplex $F \in K$, $f : |F| \to \mathbb{R}^n$ is affine linear.
Prove the following theorem by solving the sequence of exercises below:

**Theorem 5.9**

*Given a locally finite, countable simplicial complex of dimension $k$, there is a linear embedding $f : |K| \to \mathbb{R}^{2k+1}$.*




Here is another concept which I am not very much interested in pursuing. So I have put it in the exercise. Namely, take a simplicial complex $K$. Then the three conditions here are all equivalent. This is an extra condition on $|K|$, okay. This is not a part of every simplicial complex. The first condition says that each vertex of $K$ belongs to only finitely many edges. Only finitely many edges will occur there at each vertex.

Second condition says that each vertex of $K$ belongs to a finite number of simplexes. The third one is a topological condition-- $|K|$ is locally compact. So you see-- this is purely combinatorial condition, first two of them. The third one is topological. You are supposed to prove these three things are equivalent.

Such a $K,$ if it satisfies any one of them, will be called locally finite. Locally finite means all that you have to do is look at every vertex. There must be only finitely many edges emanating from them. Once locally finite is done automatically $|K|$ is locally compact. Or conversely if you have started with a locally compact space X, then if you want to triangulate it you have to choose a K which is locally finite. There is no other choice. Try to prove that.

The next one is something about when can a simplicial complex can be embedded inside $\mathbb{R}^n$. So gives you some condition. Let $f$ from $|K|$ to $\mathbb{R}^n$ be a topological embedding. Topological embedding means what? It is continuous function one-one and onto the image it is a homeomorphism. If you take the image as subspace of $\mathbb{R}^n$, then there will be inverse function which is continuous.

That is the meaning of topological embedding. Show that K has to be locally finite. Second is K must be countable. The number of vertices in K must be countable. That is the meaning of countability. Automatically number of simplexes will be also countable. The vertex set is countable is the meaning of this.

The third thing is dimension of K must be less than or equal to n. So this is harder to prove right now for you. This is harder. So I have, I have some idea why I have given at this stage. Later on this will become easier after we do a little more theory. Dimension of K cannot be bigger than n if it is embedded inside $\mathbb{R}^n$ alright.

So I will continue here. Here it was arbitrary topological embedding. In this exercise you want $f$ to be linear on each simplex. $\mathbb{R}^n$ has a vector space structure. A subset K namely mod F that has a linear structure, affine linear structure. So a function f from mod K to $\mathbb{R}^n$ restricted to each mod F should be affine linear then we say $f$ itself is linear. This is just the abuse of notation, abuse of language.

So here I have stated it as a theorem but because of lack of time, I am not going to treat this one. So following a number of exercise, you can complete the proof of this theorem, namely every locally finite, countable simplicial complex of dimension k, so all these, all these conditions were there in the earlier exercise if there is an embedding.

Now I am reversing. In the converse,  I have put dimension of K to be very small, namely it is less than half you see. Then we can have a linear embedding $f$ from $|K|$ to $\mathbb{R}^{2k+1}$. What is the idea? Idea behind this is  in these steps, alright. So these are steps a, b, c, d. So let me stop here. You can read them by yourself. Thank you.