Introduction to Algebraic Topology (Part – I)
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Lecture – 30
Polyhedrons

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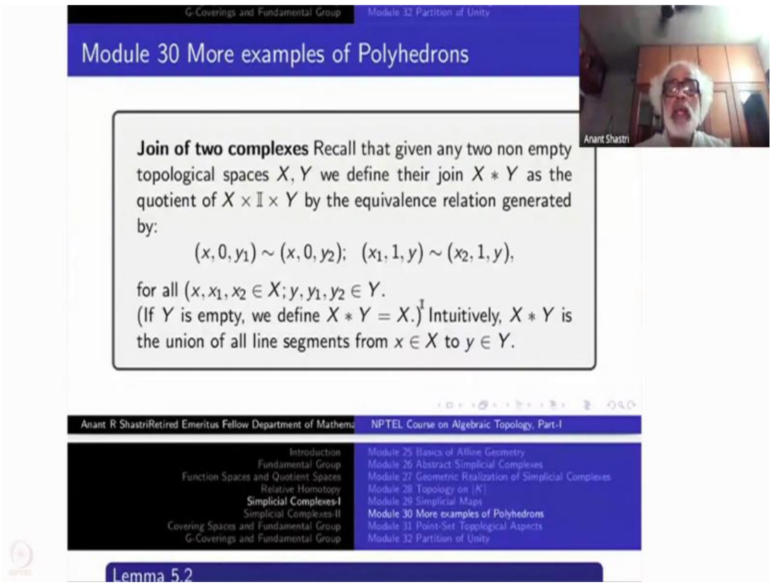

So, last time we defined what is the meaning of a polyhedron or simplicial polyhedron, namely a topological space it can be triangulated. And we saw that the disk is triangulable in such a way that the restricted triangulation to the boundary will give you a triangulation of the boundary of the, boundary sphere. Let us, let us take some more examples now.

So recall that join of two complexes, join of two topological spaces X, Y were defined as the quotient of $X \times \mathbb{I} \times Y$ in which, at the 1-end, i.e., when t=1, all the y1 and y2 are identified for each x and at the 0-end, namely when the t equal to 0, all the x1 and x2 are identified with y keeping fixed. That is the definition of the join. So what we want to show that is, if X and Y are triangulable then its join is triangulable. This is what we are trying to prove now.

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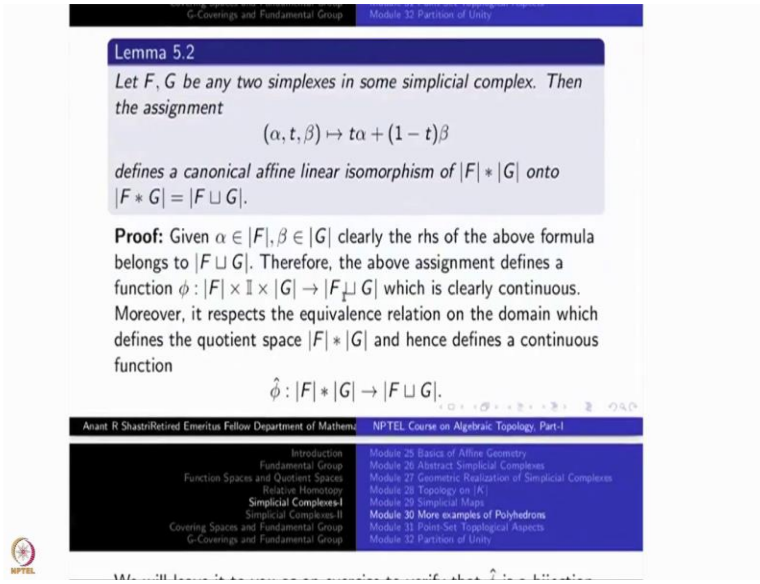

So this is a preparatory lemma here. If F and G are two simplexes in some simplicial complex, then I want to say that $|F| * |G|$ is is triangulated by $F * G$. Remember that if F and G are simplexes, (that means they are finite sets and all their subsets are taken as faces) then $F * G$ is a simplicial complex on the disjoint union F and G.

In this simplicial complex all subsets are there. All subsets of $F \coprod G$ will be there. So they can be written as a subset of F union with subset of G. So there are two different ways of describing $F * G$---just take disjoint union of the vertex sets and then take the full subcomplex over it. Now you take the modulus of that. That is linearly isomorphic to $|F| * |G|$, first take $|F|$ and $|G|$ and then take the topological join. So this is the statement.

Not only that, there is a very natural isomorphism, namely, this map is first of all defined on $|F| \times \mathbb{I} \times |G|$; $(\alpha, t, \beta) \mapsto t\alpha + (1-t)\beta$, the standard construction that we have been following. So given α, β which are in $|F|, |G|$ respectively, the RHS in the above formula belongs to $|F \cup G|$ because you can think of α as a function on $F \coprod G$ taking value 0 on all points of G.

Similarly, β can be thought of as function from F disjoint union G taking value 0 on F. So these two things can be thought of as functions here, and they will be themselves points of $|F \cup G|$. Then you can take the convex combinations inside this big simplex $F \cup G$ which has its own affine structure.

So that is the meaning of the right hand side here. Therefore, the above assignment defines a function $\phi : |F| \times \mathbb{I} \times |G| \to |F \cup G|$ which is by the very formula is continuous, because it is affine an combination. Moreover it respects the equivalence relation on the domain. So what is the equivalence relation here? When t is 0, all alphas are identified irrespective of what beta is for each beta, So this will give beta. When t is 1 all the beta are identified. So this again, respects that relation. So this induces a function $\hat{\phi}$ on the quotient of this space $|F| * |G|$. Whatever is identified here goes to the same corresponding point that is why this is a map here.

Obviously by the definition of quotient topology this will be continuous. So the point that we have to verify is that this is a bijection. The moment it is a bijection, this is a compact space, this is the Hausdorff space, it will automatically a homeomorphism. It is a linear isomorphism, linear in the sense, affine linear, not vector space linear.

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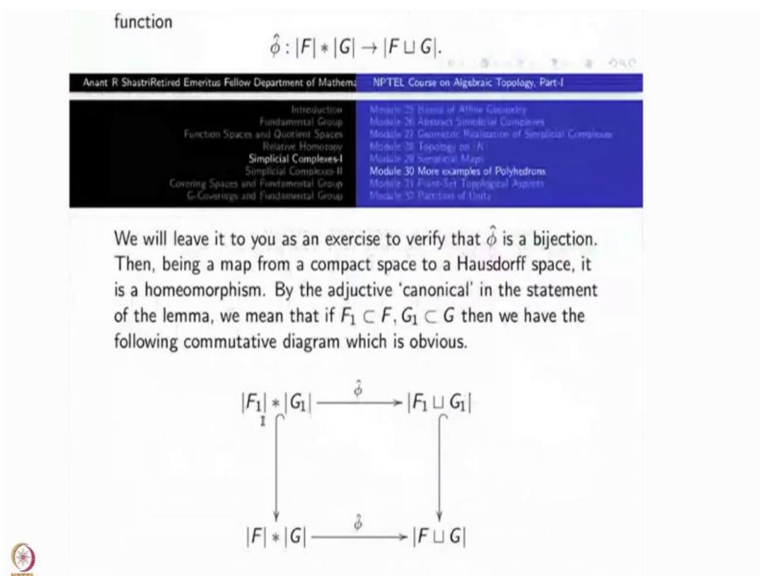

So, what I will do now, I leave it to you to see that this $\hat{\phi}$ is a bijection. This is simpler than what we did the last time, namely Δ_n. So you can just verify it. Take (α', t, β') take this formula. Suppose $\phi(\alpha', t', \beta') = \phi(\alpha, t, \beta)$. Then show that they are equivalent, that the two points are equivalent. That will give me injection here. Injectivity here is not true. Injectivity is here.

And surjectivity is obvious because everything can be written as one point here and one point here, t to 1 minus t. This is easier here than in the case of small delta and beta and so on. So this is an exercise for you.

Finally, there is the word `canonical' here, canonical affine linear isomorphism. What is the meaning of that? I want to explain that one. This is an explanation only. It requires no proof at all. Namely, suppose if you have $F_1 \subset F; G_1 \subset G$. Then we have the following commutative diagram. $|F_1| * |G_1|$, same $\hat{\phi}$ going to $|F_1 \cup G_1|$. This will be a subset of that. So modulus of this will be a subspace of this. Each of them is subspace, so $F * G$ will be containing $F_1 * G_1$. So these are inclusion maps.

Again if we add this here, this diagram is commutative. So the functions here are same as here because they are given by the same formula, t times alpha plus 1 minus t times beta, whether you are here or here, whether you are here or here. That is the meaning of this. But this is an important thing, this property, alright. So that is why I have mentioned it.

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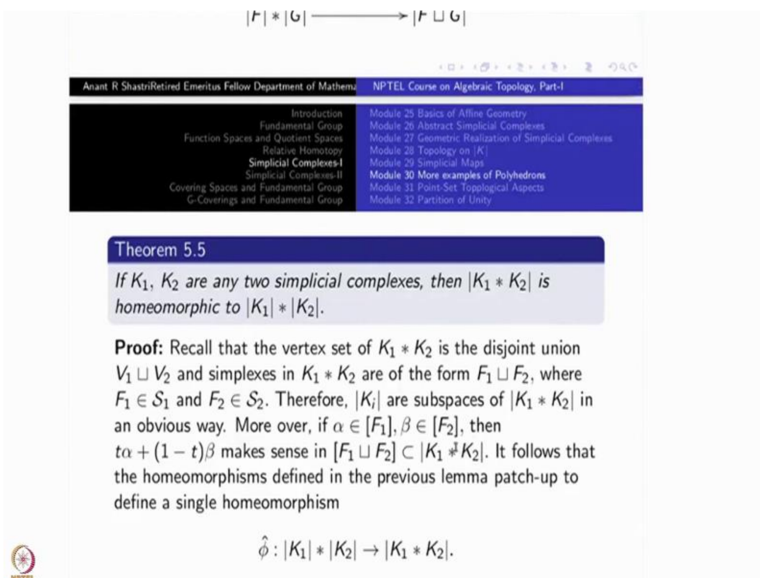

So now we can complete the task of showing that $|K_1 * K_2|$ is homeomorphic to $|K_1| * |K_2|$. As such you do not know whether this is a polyhedron. Once you prove that it follows that this is a polyhedron. That is the whole idea. So once again recall that the vertex set of $K_1 * K_2$ is the disjoint union of the vertex sets V_1 and V_2.

But what are simplexes? A simplex of $K_1 * K_2$ is a disjoint union of a simplex in K_1 and a simplex in K_2. Therefore these $|K_1|, |K_2|$ are subspaces of $|K_1 * K_2|$ in an obvious way, because we have $K_i \subset K_1 * K_2$ as subcomplexes. So moreover, if $\alpha \in |F_1|$ and $\beta \in |F_2|$, then $t\alpha + (1-t)\beta$ makes sense in $|F_1 \cup F_2|$ itself. You do not have to go outside. F1 disjoint in F2 has an affine structure.

That is contained inside $|K_1 * K_2|$. Therefore, the same formula that we have done will work here also locally, for each F1 and each F2 . And if you, from smaller one to larger when you pass, the formula on the smaller one does not change. They are compatible. That is what we have noticed here in this one under canonical property.

Therefore, you can put them all together. These $\hat{\phi}$'s together which are defined for each F1 and F2. That will define a single homeomorphism $\hat{\phi} : |K_1| * |K_2| \to |K_1 * K_2|$, because corresponding images will be also distinct. If F1 and F2, G1 and G2 are different then their interiors will be different. Boundaries will agree with the smaller simplexes. Therefore this itself is a homeomorphism.

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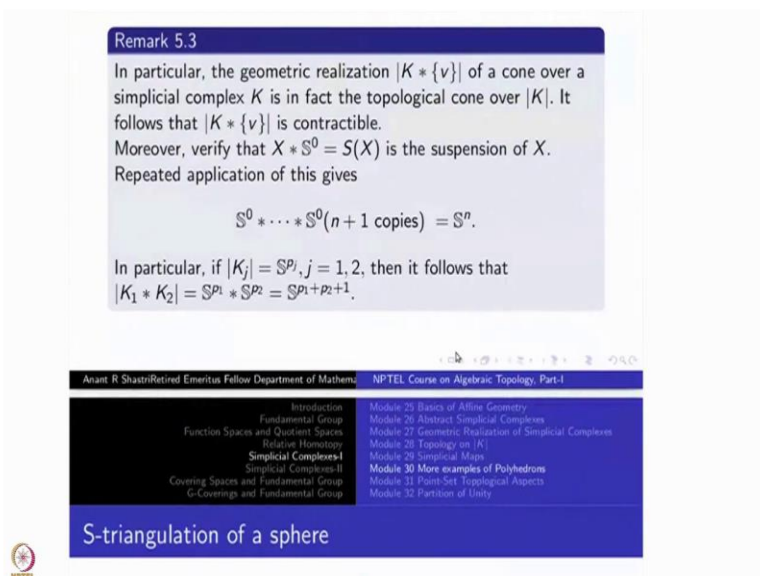

In particular you specialize K_2 to a single point which is nothing but the cone now. If you take two points it will be a suspension. So we get both suspension and cone of a polyhedron will be polyhedrons. Suppose X has already some structure, namely of a polyhedron then the suspension of that will also have polyhedron, namely X star S naught. Mod of x star S naught will be S X. That is the...

So we can define the suspension of X also as a, in the case of polyhedron. In particular we have also observed that if you repeat, repeatedly take the join of $\mathbb{S}^0$ with itself, n plus 1 copies, then it is a sphere. So this will give you another simplicial structure to $\mathbb{S}^n$ than what we have defined in the previous session. This point I want to repeat it. It is, once I have said it, it is over. But I want to repeat it. This is called S-triangulation of a sphere.

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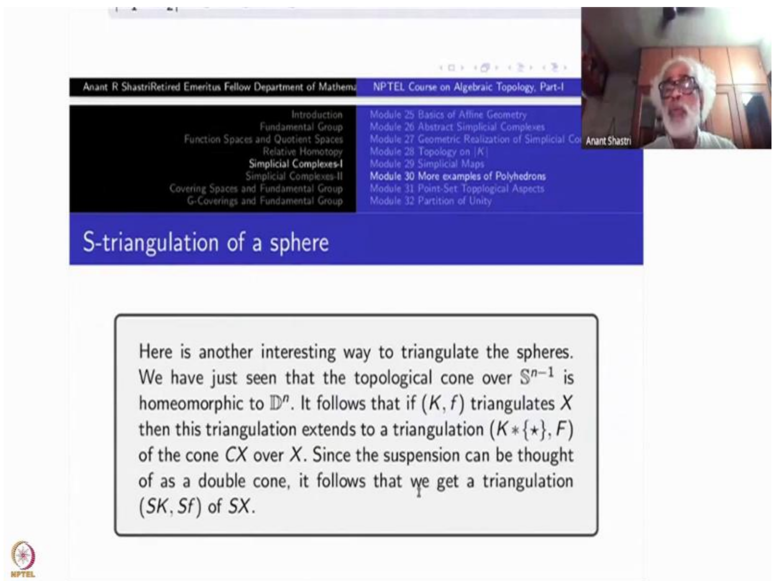

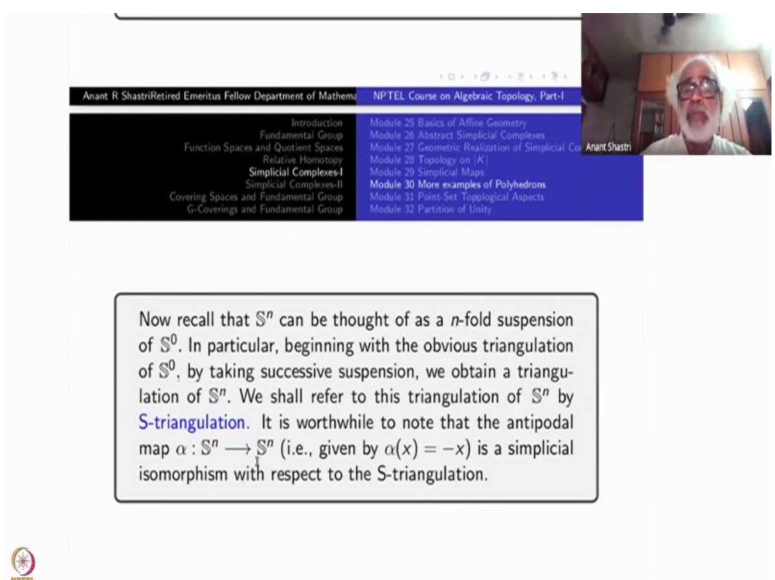

This special case I am going to explain here. So this is an interesting way to triangulate the sphere. We have just seen that the topological cone over $\mathbb{S}^{n-1}$ is homeomorphic to the disc $\mathbb{D}^n$. So it follows that if (K, f) is a triangulation of X, then we get an extended triangulation $(K * \{v\}, F)$ of the cone CX. This capital F is extension of map, not a simplex.

Since the suspension can be thought of as a double cone, it follows that we get a triangulation (SK, Sf) of the suspension SX. This is a more general statement I am making. But here already we have observed that $\mathbb{S}^0 * \cdots * \mathbb{S}^0$ (k-times) will give you the sphere $\mathbb{S}^{k-1}$. So I will come back to that one. Now recall that $\mathbb{S}^n$ can be thought of as n-fold suspension of $\mathbb{S}^0$, in particular beginning with obvious triangulation of $\mathbb{S}^0$.

What is it? Just the 2-point complex, that is all, It is zero-dimensional simplicial complex. By taking successive suspension we get a triangulation of $\mathbb{S}^n$. So we shall refer to this triangulation of $\mathbb{S}^n$ by S-triangulation, namely by taking suspension. It is worthwhile to note that antipodal map, in this case, namely antipodal map means what, $\alpha(x) = -x$. This is a simplicial isomorphism of S-triangulation.

If you take the standard boundary of Δ_n, the antipodal map has no role to play there. It is not ((simplicial. S-triangulation has this property that is very useful.

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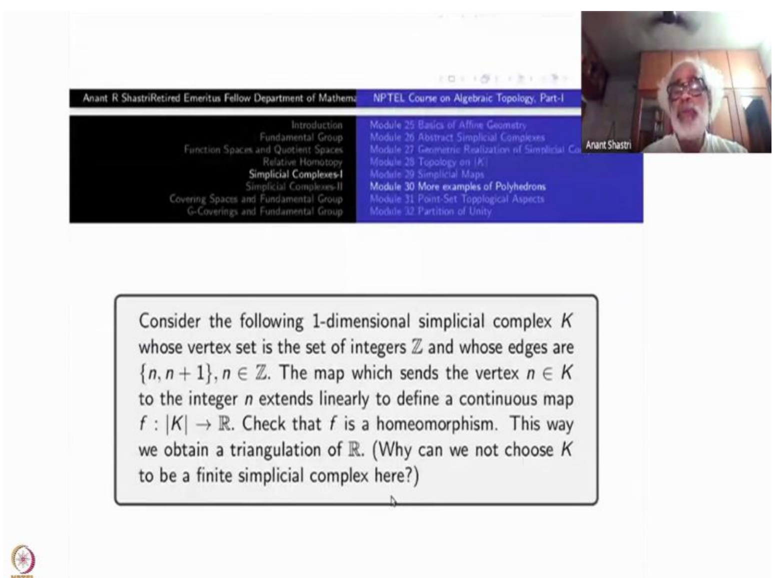

I will give you some more examples here. Now consider the one-dimensional simplicial complex K whose vertex set is set of integers, 1, 2, 3 and so on, 0, minus 1, minus 2 and so on. And the edges, and the one simplexes are precisely consecutive integer pairs, $\{n, n+1\}, \{n+1, n+2\}$ like that, consecutive integers, nothing else. So this is simplicial complex. What is its geometric realization?

It will be the whole of $\mathbb{R}$. Each simplex n to n plus 1 filling up the gap between the interval n to n plus 1. So that is it. So this way we get triangulation of $\mathbb{R}$. The point here is that this is the first triangulation that we have taken to be infinite, a genuine example. Otherwise we had only disks and deltas and so on. The triangulation here, the simplicial complex K could not have been finite because if this were finite then it will be compact whereas $\mathbb{R}$ is non-compact.

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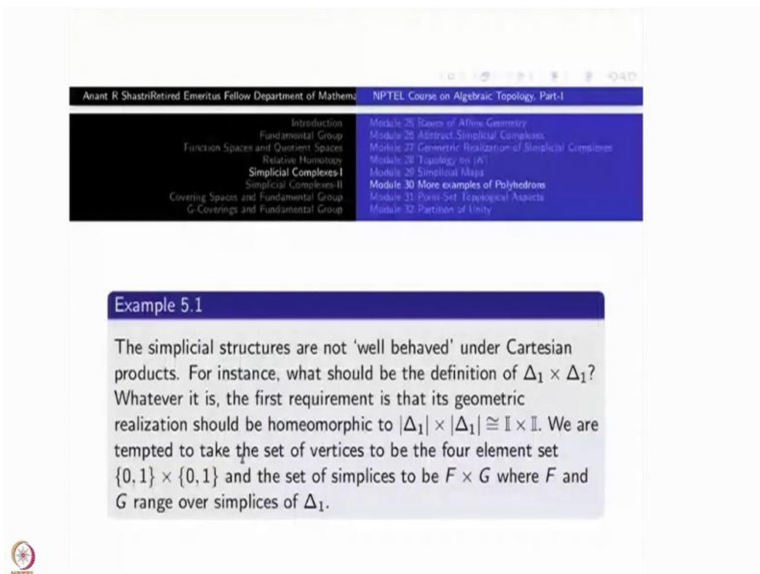

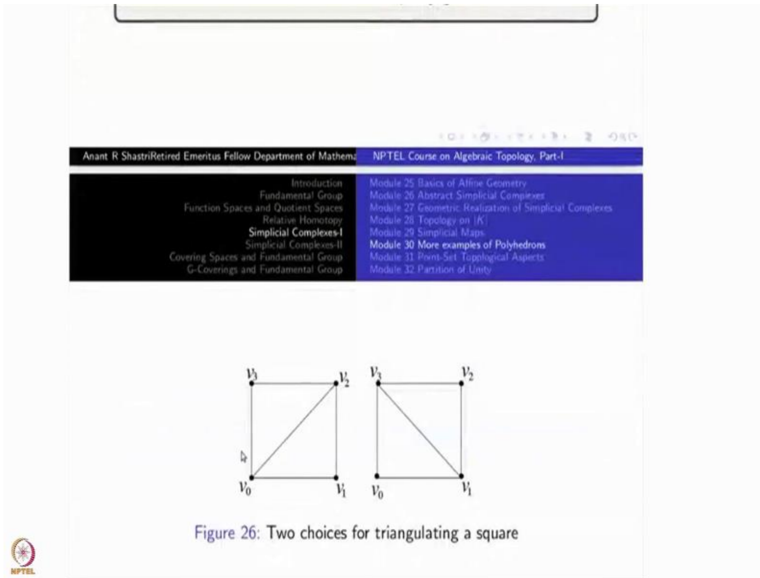

Now I come to a serious problem, namely products. We have already dealt with some way, product with $\mathbb{I}$. But you have to be careful. There it was for the join. It was not actually the Cartesian product of a space with some other space. The problem is what kind of polyhedron will, what kind of simplicial complex will give you the product structure?

Suppose X and Y already have polyhedral structures. Then is there a way to give their product $X \times Y$ a polyhedron structure, such that it has something to do with the original, original complexes on X and Y? So this problem is much deeper and quite difficult. In fact, in general it is not possible. So what I will do is I will specialize to the case when one of the factor is just the interval [0, 1]. Even here we will have problems.

For example, take $\mathbb{I} \times \mathbb{I} = |\Delta_1| \times |\Delta_1|$. What is the best way to triangulate this one? There are 2 vertices, 2 vertices here. So you are tempted to take 4 vertices. Then for each simplex here, 0-simplex cross 1-simplex, 1-simplex cross 0-simplex and so on. If you do that, what happens? the number of vertices taken 4 is lucky. That is right. But number of 1-simplexes are too less or too many. So this is depicted here in this, in this picture.

So V naught V1 is delta 1. V1 V2 is also delta 1. So this is a product. This I cross I, the product space is I cross I. What is the simplicial structure here? I can take $(0,0),(0,1),(1,0),(1,1)$ as vertices. But what should be the 1-simplexes? Naturally the subspace, this line, this line, this line, this line must be there.

But then if you just leave it there, the interior is not a simplex. So you have to either cut it this way or cut it this way, right? Both of them are possible. The fact that there are two different choices creates the problem, which one to take? The moment there is a choice that is going to create a problem.

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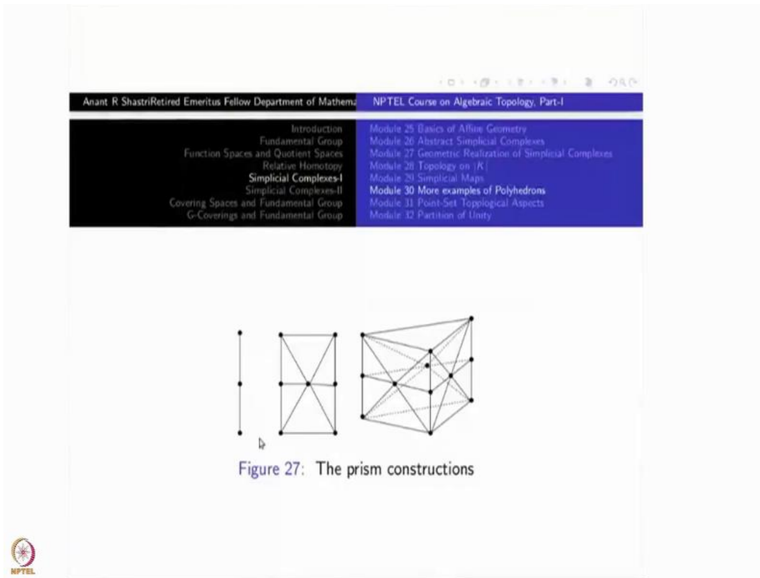

So you want to solve this problem in a canonical way by, maybe by introducing a little more number of vertices and more edges rather than just sticking to this product things like this. So this is what I want to do. So this is point cross $\mathbb{I}$. Point cross $\mathbb{I}$ itself is a simplicial complex, because $\mathbb{I}$ is a simplicial complex. $\mathbb{I}$ is just a 1-simplex.

But I am not going to leave it like that. I am going to divide it into two parts by taking the barycentre of this one as an extra vertex and then declaring these two portions as 1-simplexes. The original 1-simplex actually disappears. It is being cut into two. That is the meaning of it. Alright.

Come to the second space, namely, when it is $|\Delta_1|$. So this is $|\Delta_1|$ cross $\mathbb{I}$. It is a rectangle or a square, whatever you want to think of. Point cross I is this one. So disappears here. Other point cross I also disappears here. $|\Delta_1|$ appears as it is at the zeroth level as well as at the one-th level. So now you have the big square here, empty square. What I am going to do is this. I take the barycenter of this one shifted at the half level. That is going to be an extra vertex.

As soon as you have an extra vertex, join it to all the vertices in the boundary. Join it, if this is simplex, declare this as simplex. If this is simplex declare this as simplex, namely this is the cone construction. Do the same thing for $|\Delta_3|$ also. This is, sorry this is $|\Delta_2|$. There are 2-simplexes. Once you have fixed what the boundary is which is 1-dimensional part, 1 cross I, here the body is fixed.

The bottom is kept as it is. The whole boundary of delta 2 cross I has been triangulated. Take the barycenter of this simplex. Shift it at half-level. And then take the cone over the boundary. So having told this one actually the construction of what I call as the prism construction is over. Why I am calling as prism? Look at this and this delta cross I. This is the prism. This is called the prism construction is over.

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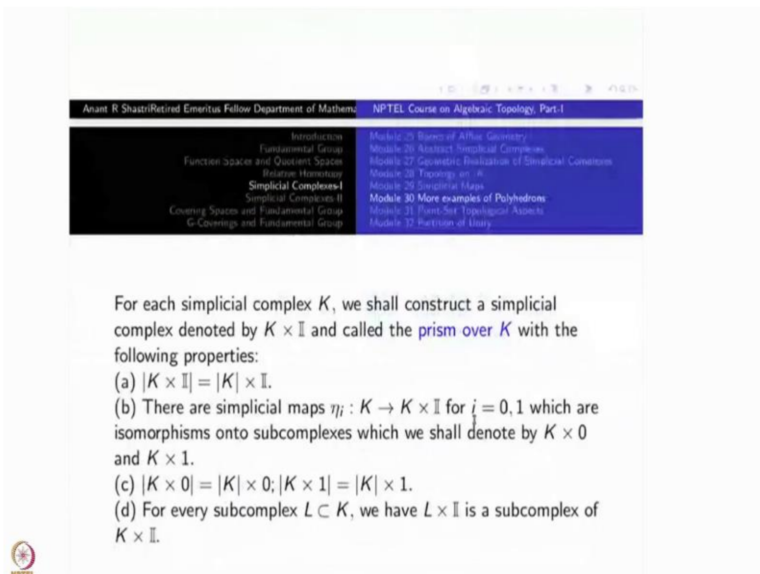

Let me do it systematically once again so that you will understand what is going on. For each simplicial complex K, we shall construct a simplicial complex denoted by $K \times \mathbb{I}$. This is a notation. This is going to be a simplicial complex which I am going to describe now. K is a simplicial complex. $K \times \mathbb{I}$ is not a topological space, it is going to be a simplicial complex, so that if you take its geometric realization $|K \times \mathbb{I}|$ that is a topological space, alright. This $K \times \mathbb{I}$ is just a symbol here. I am going to call it the prism only after I construct the simplicial complex on this one, which I am going to describe. It is called prism over K, like cone over K, how we have constructed, and the suspension over K, this is the prism over K with the following properties.

(a) $|K \times \mathbb{I}|$, the geometric realization is $|K| \times \mathbb{I}$. This is the first property. (b) Then there are simplicial maps, η_0 and η_1 from K, which is simplicial complex to this simplicial complex. So these are simplicial maps, which are isomorphisms onto subcomplexes. That means they are one-one mappings taking simplexes to simplexes in a one-one way. Their images will be denoted by $K \times 0$ and $K \times 1$, respectively.

This are again notations. Just like $K \times \mathbb{I}$ is a notation, $K \times 0$ and $K \times 1$ are also notations. Of course, they are subcomplexes of $K \times \mathbb{I}$. This is the second property I want. I want (a), and I want this property (b) also. I want something more, namely, the underlying topological space of $K \times 0$ is the subspace $|K| \times \{0\} \subset |K| \times \mathbb{I}$, and similarly $|K \times 1| = |K| \times \{1\} \subset |K| \times \mathbb{I}$. This is the third property (c).

Finally I want something more. If we take a subcomplex $L \subset K$, then I must have $L \times \mathbb{I}$, the prism over L, as a subcomplex of $K \times \mathbb{I}$, whatever I want to define. So far, neither I have defined $K \times \mathbb{I}$ nor I have defined $L \times \mathbb{I}$. But they must have these properties. So I want a prism to have all these properties. Now I have put so many conditions which looks like too many restrictions ; we have made your life more difficult.

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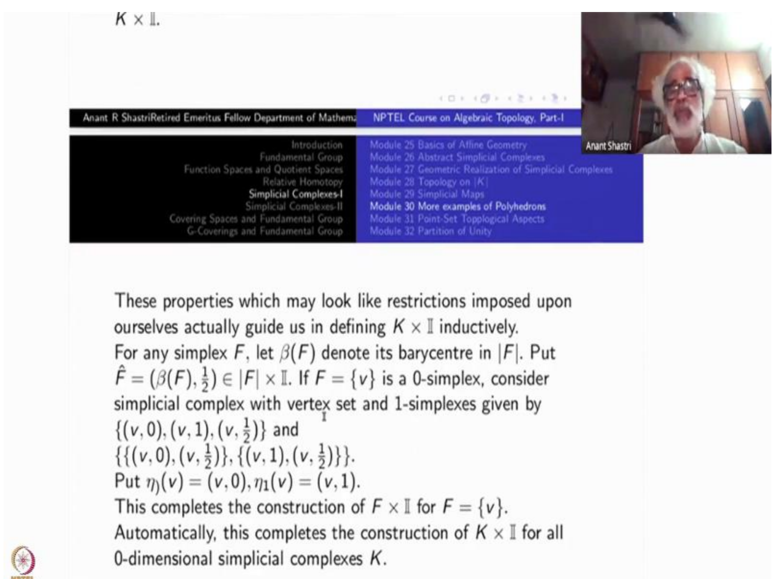

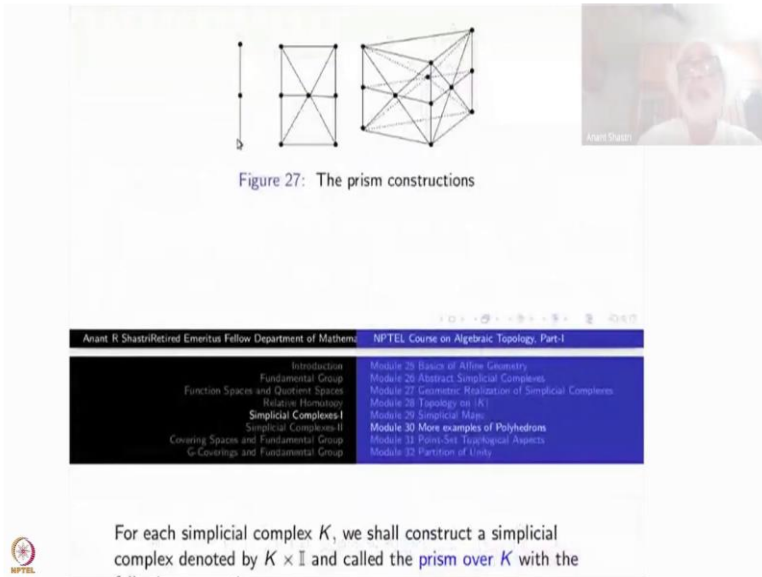

In reality, these restrictions will actually guide you how to construct $K \times \mathbb{I}$. And that is done by induction. So let us have notation: $\beta(F)$ will denote the barycentre of any simplex F. Let us put $\hat{F}$, short notation for the ordered pair $(\beta(F), 1/2)$, belonging to $|F| \times \mathbb{I}$. So I am placing this barycentre at the half-level.

Now suppose $F = \{v\}$ is single vertex, a 0-simplex. Then I am looking at simplicial complex with vertex at precisely equal to this one and 1 simplexes are precisely equal to this one. And this is the simplicial complex. There are no other 2-simplex, 3-simplex and so on. This is going to be my $F \times \mathbb{I}$ where F is a singleton, a 0-simplex. So what are the vertices? $(v, 0), (v, 1)$ and $(v, 1/2) = \hat{v} = (\beta(F), 1/2)$. That is precisely this picture, the first picture here. And $\{(v, 0), (v, /2)\}$ will be one of the 1-simplexes; $\{(v, 1), (v, 1/2)\}$ will be another 1-simplex. So that is what I have listed here. And eta 0 of $\eta_0(v) = (v, 0)$; and $\eta_1(v) = (v, 1)$. Now this completes the construction of $F \times \mathbb{I}$, when F is a singleton. Since there are no subcomplexes, subcomplex is empty, all those conditions (a), (b), (c), (d)'s whatever I have demanded, they are all satisfied.

So the construction for singleton is over. As soon as you have constructed it for singletons you take the union of all these things, it will be constructed for all zero-dimensional simplexes. A zero-dimensional simplex is nothing but disjoint union of singletons. The construction is over for all zero-dimensional simplexes because I am just taking the union. Singleton cross I, disjoint union because singletons are disjoint over K. So the inductive step is over for n equal to 0.

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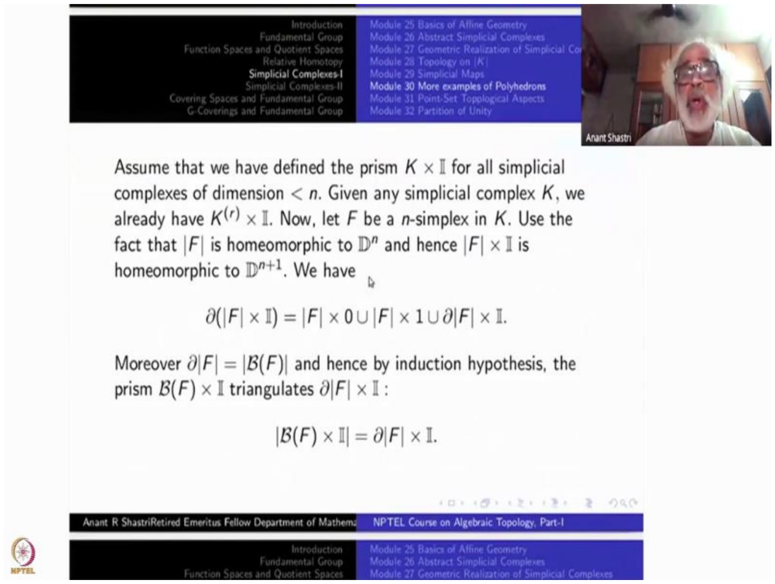

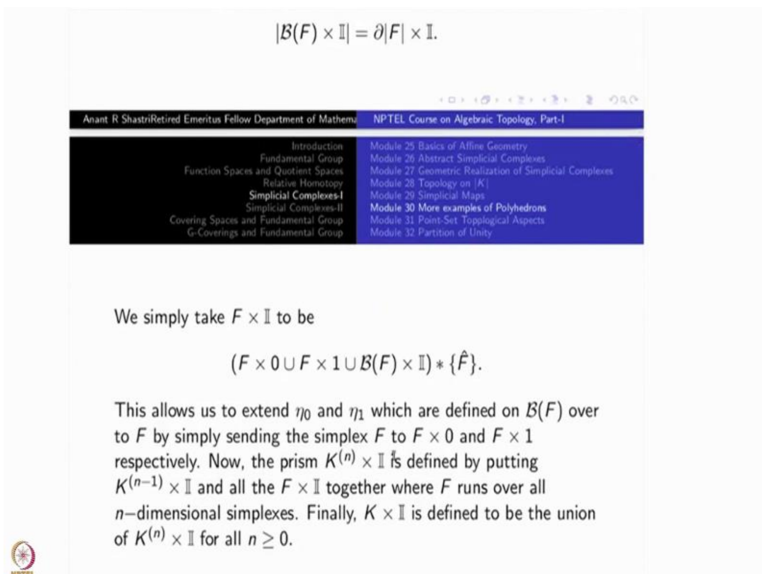

Now assume that we have done it for some n minus 1, all the way up to n minus 1. Then we want to do it for n. What is the meaning of that? Take a simplicial complex K and take a n-simplex F, an n-dimensional simplex in it. On the boundary of this simplex there is already the prism structure: $|\mathcal{B}| \times \mathbb{I} = |\mathcal{B} \times \mathbb{I}|$ by induction.

On the bottom, $F \times 0$, I am not going to disturb it. I am keeping it as $F \times 0$. Similarly, $F \times 1$ also, I am not going to disturb--I am keeping it as $F \times 1$. Taking the union, I complete the picture of boundary of $|F| \times \mathbb{I}$, which is already given a simplicial structure now. Next, you take the cone over this, with the cone point, namely apex point precisely at $\hat{F}$. F hat is what? The barycentre of F comma half. That is what I am going to do.

I repeat. By induction, the prism $\mathcal{B}(F) \times \mathbb{I}$ is defined triangulating $|\mathcal{B}| \times \mathbb{I}$. $|\mathcal{B}(F) \times \mathbb{I}| = |\mathcal{B}(F)| \times \mathbb{I}$. This is by induction. Now you should know that $|F|$ is homeomorphic to $\mathbb{D}^n$, so $|F| \times \mathbb{I}$ homeomorphic to $\mathbb{D}^{n+1}$ which is a cone over its boundary $\mathbb{S}^n$. All this I am using here.

So we simply take $F \times 0$ as it is and $F \times 1$ as it is (they are copies of F) and then the union with $\mathcal{B}(F) \times \mathbb{I}$, so this entire thing from here to here describes the boundary of $|\mathcal{B}(F)| \times \mathbb{I}$. That has been given a simplicial structure. Take the cone with apex at $\hat{F}$ extend this structure to $F \times \mathbb{I}$. So, automatically η_0 and η_1 get extended. That is all.

Once you have done it for each n-simplex F, one by one, them, take the union. simplex we are already constructed. Take the union over all the n simplexes. That will complete the construction for n-dimensional skeleton $K^{(n)} \times \mathbb{I}$. So inductively, extending the prism $K^{n-1)} \times \mathbb{I}$ to $K^{(n)} \times \mathbb{I}$ has been described.

Therefore now the construction is over for all $K^{(n)} \times \mathbb{I}$, where $K^{(n)}$ denotes the nth-skeleton of K. Take the union of all these over n. Because of the property (d) they are compatible. Subspace which is already triangulated gets extended to the triangulation of the larger space, So they will be compatible. So take the union. That is the definition of $K \times \mathbb{I}$ as a simplicial complex which will have its modulus homeomorphic to $|K| \times \mathbb{I}$. Okay this is prism construction, alright.

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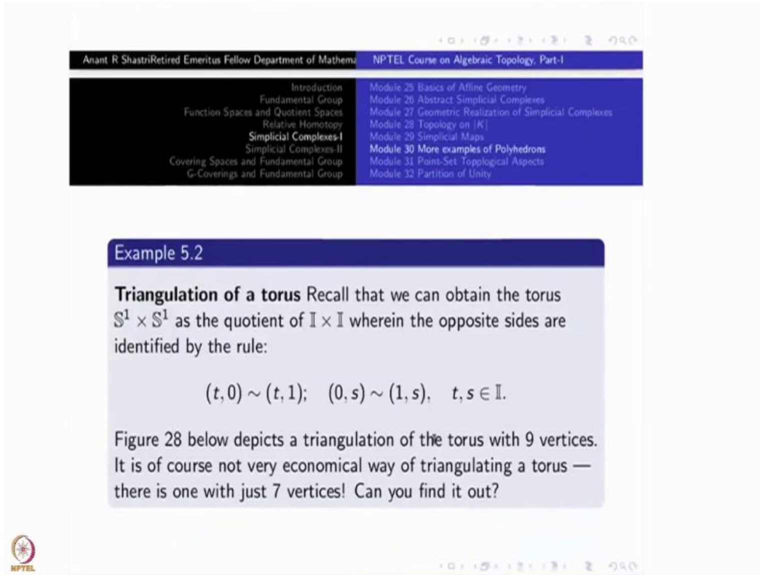

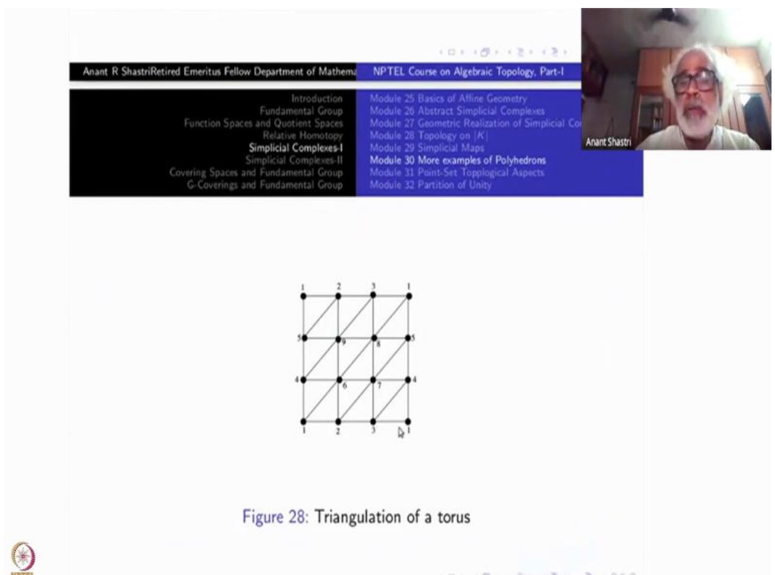

Let me give you another simple thing is a triangulation of a torus. By the definition you must be knowing that the Torus is defined as $\mathbb{S}^1 \times \mathbb{S}^1$. But S1 is defined by the interval and identifying end points. Therefore the Torus can be defined as quotient of $\mathbb{I} \times \mathbb{I}$ by identifying opposite sides; pairs of opposite sides will be identified with the correct orientation. So this you must be knowing already. So in order to give a triangulation on the torus what we do is we choose a triangulation on $\mathbb{I} \times \mathbb{I}$ in such a way that when you quotient out the triangles etc do not collide.

In other words, two vertices which are joined will go to two distinct vertices correspondingly paired. Three vertices which form triangle, they will go to a triangle, no two vertices of a triangle will be identified and so on. Not only that another set of edges will not come on the same side of a vertex. This is what you have to be careful about, all right?

So all this thing can be done. So this is a good example to know when you to deal with quotients, you have to be careful. We cut down $\mathbb{I}$ into three edges, and then identity the end points 0 and 1, you get a triangle, which triangulates $\mathbb{S}^1$. That is the `minimum' way of triangulating a circle-- to take 3 points and 3 edges right? You cannot do with less than that.

So do that. Label the vertices on line seqment at the bottom 1, 2, 3. This is 1 only because this is going to be identified with this 1. Continue like this along the boundary, now take some more vertices, 4, 5, but then again this will be identified with this 1. So this is 1. 3 will be identified with this 3. 2 will be identified with this 2. 1 will be identified by this 1, this entire line being identified with this entire line; 1, 2, 3, 1; 1,2,3,1. Similarly 1, 4, 5, 1 identified 1, 4, 5, 1. These are not new. But inside the square, I am taking 6, 7, 8, 9 here. And then joining them systematically like this. Of course this is not the unique way nor canonical.

There are many ways. Now you see each triangle is uniquely defined when you go to the quotient. This gives you the triangulation of a torus. This is not an economic way of doing it, by the way. Given any surface you can give it a triangulation. That is little more harder to prove. It is a big theorem. Then there are problems like this. What is the minimal number of vertices needed? What is the minimum number of edges needed? And so on.

This is a big industry since several years, almost 30-40 years now. Lot of people are working. One famous mathematician in India has worked all his life on that, namely professor Basudeb Datta and Sarkaria in Punjab, Basudeb Datta at IISc, and some of his students and so on. The torus itself can be triangulated by using just 7-vertices. Try to do that. Let us stop here.