Introduction to Algebraic Topology (Part-I)
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Lecture 3
Bird's Eye-View of the Course

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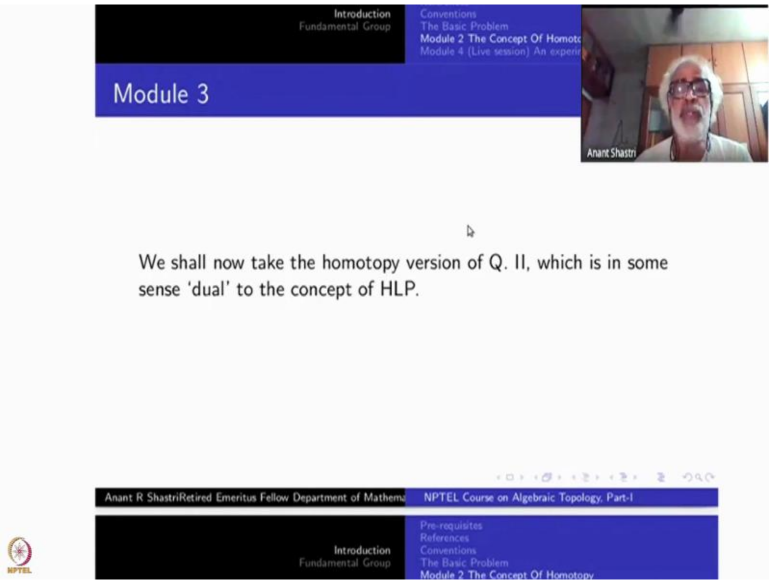

So, in the last module we discussed what is the meaning of homotopy and the homotopy version of the lifting problem. Lifting data implies the lifting problem means the map has lifting property. If the map has lifting property with every space, such a map is called a Hurewicz fibration. This much, we have seen.

Similar and but dual to this is the question number 2, which we want to put in homotopy theory- the homotopy theoretic version of the same problem question number 2, an extension problem. Remember in the extension problem, very closely associated was the quotient problem. Factorization problem which was very easy set-theoretically. So, we are taking only the extension problem here.

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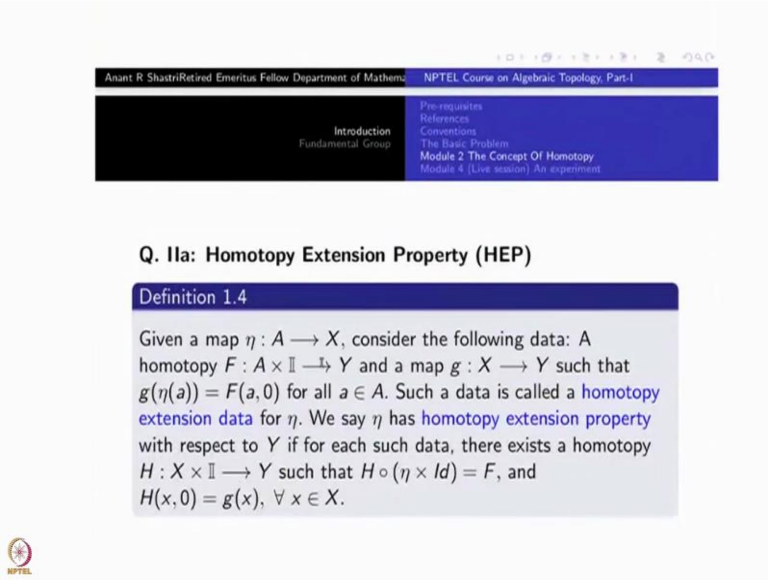

So, start with a map from A to X. So, this is the map which we are concentrating upon. Now, suppose we have the following data. There is a homotopy from A cross $\mathbb{I}$ to Y and a map from X to Y, such that this η composite g or g composite η whatever you want to say is the starting point of this function F, that is, equal to F of a, 0 for all a in A. Such a data is called homotopy extension data.

What does it mean is that think of A as a subspace of X. On the space, on the subspace there is the homotopy of a function which is defined already on X. You take the restriction that is g composite η the starting point of the homotopy F. This is the data. Now, what is the conclusion?

We say η has homotopy extension property with respect to Y, if for each such data, there exists a homotopy H on the whole of X cross $\mathbb{I}$ to Y such that when you restrict it to A cross $\mathbb{I}$ it is the homotopy that has been already given and the homotopy is not of arbitrary map, but of the given function g equal to H of x, 0. So, there is a homotopy of the extended map which extends the original homotopy. So, that is the homotopy extension property. Once again, I will denote, I will represent it by schematically so that you will remember what is the meaning of this one.

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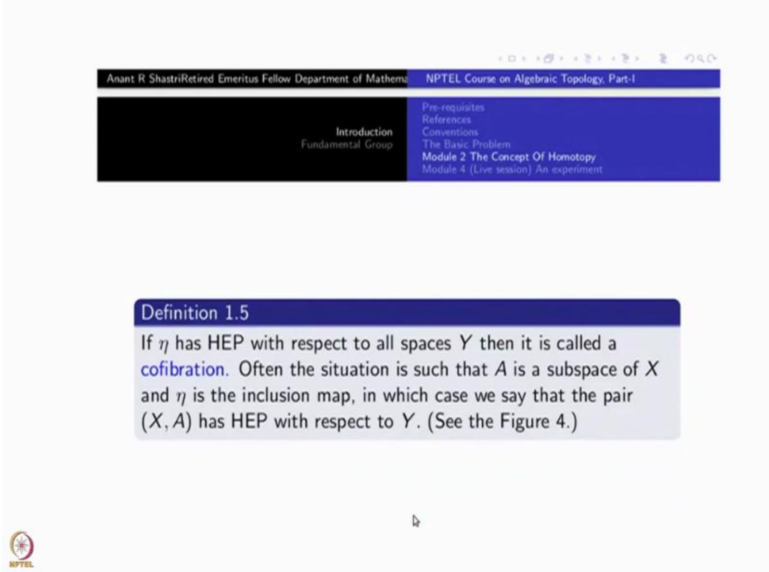

Before that, I have a definition here. If this happens for every homotopy extension data, then η will be called a cofibration. There, we called it as a fibration, P from A to B. Here we have A to X, we are calling it as cofibration indicating that this property is somewhat dual to the property of a fibration.

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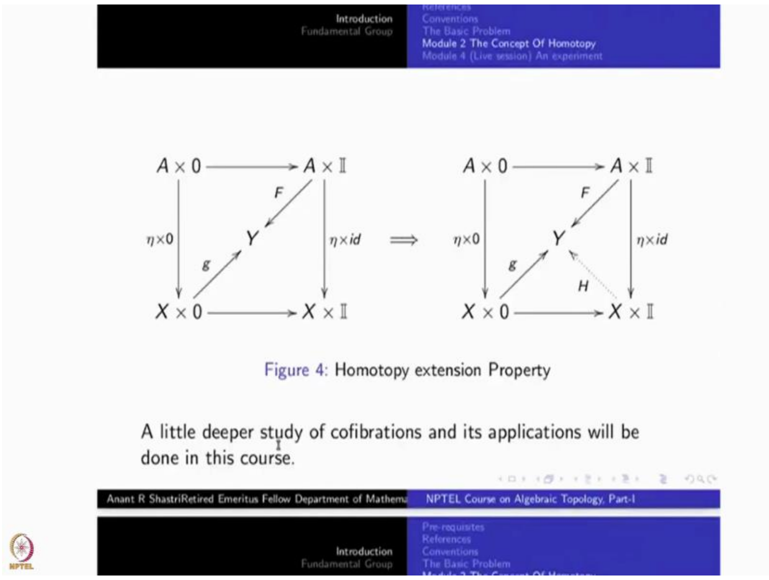

Let us look at the figure here. So, A cross 0 to X cross 0, you have the function.

Function is from A to X. A cross 0 is copy of A and X cross 0 is a copy of X. So, η cross 0 we take here. On X cross 0 you have another function g. So, this part is just a function A to X and a function X to Y. On this part you have a homotopy of this function; the restricted function η, and g composite eta on A cross $\mathbb{I}$. That is a homotopy.

So, this triangle is the given data. This is always given. η is there, so η cross Id from A cross $\mathbb{I}$ to X cross $\mathbb{I}$ and this X cross 0 is contained in X cross $\mathbb{I}$. This part is nothing strange-- what is given is this triangle. This is always there. This triangle is given. Now, the conclusion of this one is that there is a map here from X cross $\mathbb{I}$ to Y, the dotted arrow. So, this map, when you compose it to eta cross I, it is F. So, you can think of this as the inclusion, then this will be the restriction of H, which, on X cross 0, the starting point is the given g.

So, whenever you have this diagram there must be a map here which fits the diagram like this, then it is called homotopy extension property. So, if η has homotopy extension property for every data whatever Y is, whatever g is, whatever F is, then it is called a cofibration.

Once again you will say this is too much to expect. No, the beauty of these things are, they are almost always satisfied or what you will say that whatever interesting spaces come, they will have this property. So, that is why, both these properties, fibrations as well as cofibrations are a part and parcel of algebraic topology. Right in the beginning you should grasp what is happening. Having set up these two problems in the homotopy set-up, we should now study the homotopy a little deeper. So, let us do that one now.

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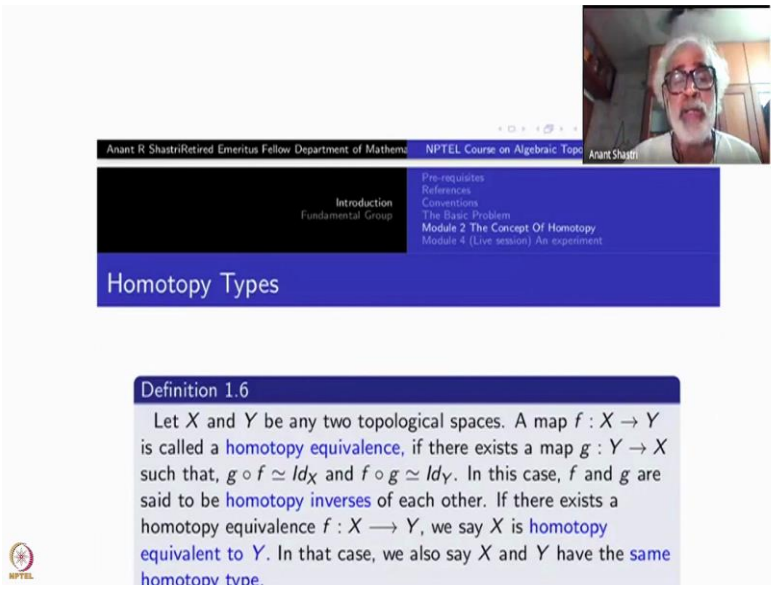

What are called as homotopy types? We had already homotopy classes of functions, maps. Now, we want to do them on spaces. Start with two topological spaces. You know what is the meaning of, saying that they are homoeomorphic. There must be a homeomorphism f from X to Y. Namely there must be g from Y to X which is inverse of f and is continuous. f is continuous; g is continuous. That is a homeomorphism. So, we want to take a weaker equivalence here namely homotopy equivalence. What is that?

So, a map is called a homotopy equivalence if there is g from Y to X such that instead of g composite f being identity, it is homotopic to identity. Similarly, the other way composition:

f composition g must be homotopic to identity of Y. So, instead of equality, we are replacing by homotopy. Such a thing is called homotopy inverse, g will be homotopy inverse of f.

The point is that there may be many g which satisfy this property, yet the homotopy class of g is only one, if at all it exists. There may not be any. Given any function, there may not be any inverses. But if the inverse exists, the homotopy class of that is unique. So, you can call that class as the inverse of the class f, homotopy inverse.

Whenever there exists such an f from X to Y, which is homotopic equivalence, we call X is homotopic to Y. Like the function f is homotopic to g is now the space X is homotopic to Y or X and Y are homotopy equivalent or they have the same homotopy type. So, all these terms are used.

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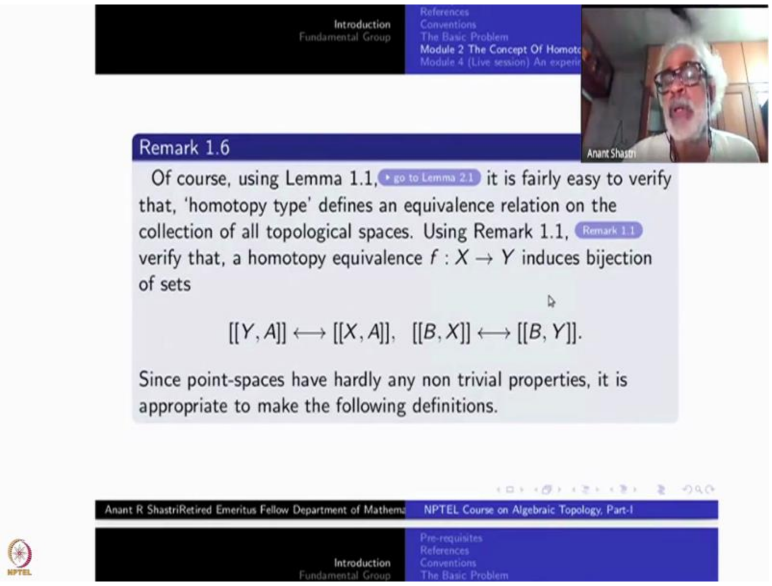

Again, using Lemma 1.1, remember that about compositions of homotopy.

You can go on to verify that if X is homotopic to Y and Y is homotopic to Z, then X will be homotopic to Z. Every space is homotopic to itself because identity map is a homotopy equivalent, inverse of itself.

If f says X is homotopic to Y, its inverse will give you Y is homotopic to X and so on. So, homotopy equivalence is an equivalence relation, therefore we can take equivalence classes, each class is a homotopy type. Once you have that, suppose X and Y are homotopy equivalent, then for every space A, the set Y, A and the set X, A will be in bijection.

So, this is also a consequence of Lemma 1.1. Keep using the fact that compositions are associative. Similarly, all functions from B to X, are in one-one correspondence with all functions from B to Y when you take homotopy classes. The functions themselves may not be in one-one correspondence. When you take homotopy classes, there is a one-one correspondence.

How is it given? Take a homotopy equivalence f from X to Y. Compose with, suppose you have a map from Y to A, compose it with f, you get a map from X to A. To go back you compose it with g. You will go back because g composite f is identity. That is about right. So, these things are all straight forward following our remark after the lemma, Lemma 2.1

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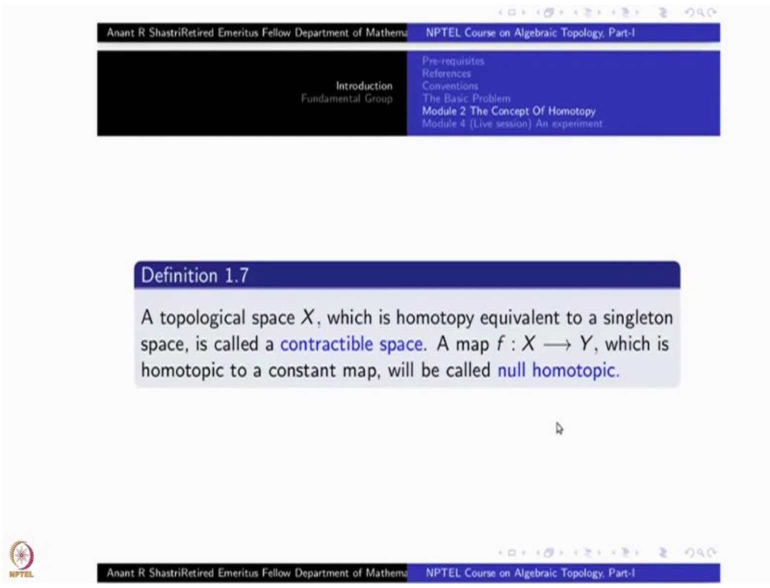

Now we make a special definition here. Anything which is homotopy type of a single point. The single point does not have much properties, topological properties. So, we want to single-it-out. So, such a thing is called a contractible space. Up to homotopy type, there is no difference between any contractible space and a single point. Because they are the same homotopy type. So, all the homotopy theoretic properties will be the same. So, such a thing is a contractible space.

A map f from X to Y which is homotopic to a constant map, (you should also do for maps also,) such a thing is called null homotopy. A constant map will not have much properties. If you compose with a constant map, that will also become constant map, whichever way you compose, post-compose or pre-compose. Once one of them is a constant map, the composition is a constant map.

So, that kind of properties first, the trivial properties have to be first understood carefully. So, there are some easy way of identifying what is a contractible space. Let us go through that one.

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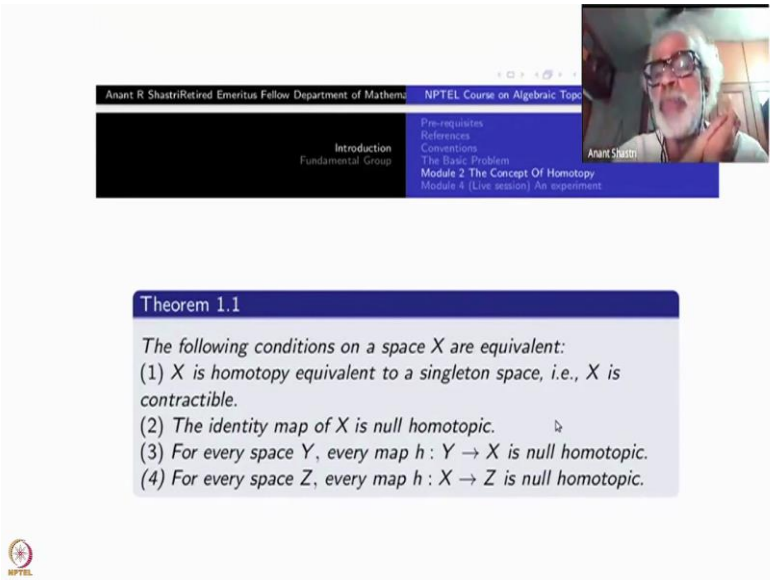

This is the theorem. It is the first theorem of the course. Following conditions on a space are all equivalent. The first condition is X is homotopy equivalent to a singleton space, which, we have named it as X is contractible. That is the first one. The second: the identity map of X to X, it is null homotopic. The third one, for every space Y , every map h from X to Y, is null homotopic. All functions taking values in X, they are null homotopic. Every space and every map h from X to Z is also null homotopic. Every function which its domain is X is a null homotopic. So, both domain and co-domain, both as a domain and as a co-domain, X is behaving very nicely. That is precisely what I told you. If you pre-compose or post-composed a function, it will be constant map and that is reflected here on the spaces.

Whether you start from X or you end in X, if X is contractible, such maps are all null homotopic. So, these are the characteristic of point space and up to homotopy they are the characteristic of contractible space. So, that is the gist of this theorem. So, this being a mathematics lecture, we should examine these things carefully and study the proofs also. Proofs are very easy. Let us go through them.

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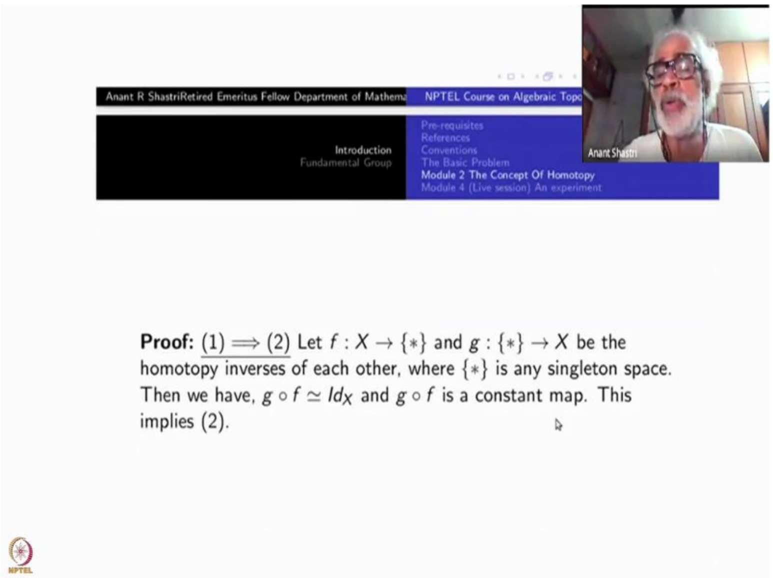

So, let us first prove 1 implies 2. What is the meaning of that? I assume that X is homotopy type of a single point, then I want to show that identity map of X is homotopic to a constant map. That it is homotopy type of a single point means, I am taking Y as a single point space, f is a map from X to Y and g is a map from Y to X. What is Y? Y is a single point. They are homotopy inverses of each other. What does it mean? g composite f is homotopic to identity and g composite f is what? See, first you start with g single point, sorry, first you start with f from X to a single point. Then take g to X. Where does it go? Single point only,--- g composite f is homotopic to identity by definition but g composite f is a constant map. So, identity is homotopic to constant map. So, that is the conclusion 2. Okay? So, 1 implies 2 follows. Now, let us prove 2 implies 1, the reverse. Reversing is also very easy here.

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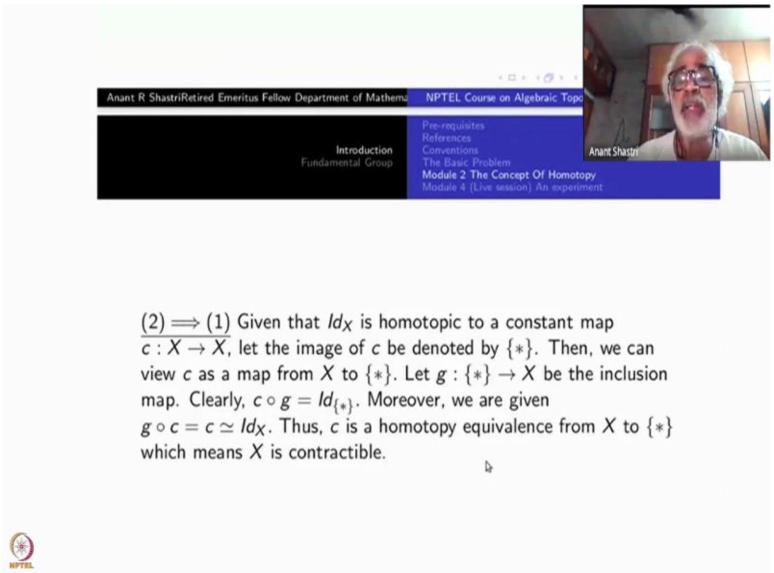

Given that the identity is homotopic to constant map. Let us call this constant map, c from X to {c}. Let us name it, as it is a map from X to X, because it is homotopic to identity. Let us call it as c from X to {c} The image of see is a single point. So, let us call that as Y. Then we can view c as this function as a map from X to single point, because image is single point of that. This is a subspace of Y, subspace of X. But I will call it as Y. Now, let g belong this one, be the inclusion map. Take inclusion map. This is this is subspace of X. Then what happens? If you compose with g, c compose g, g is a constant map, c is a constant map that will be the identity of the constant map. Identity of the single point. Start from single point, it will be come back to single point. Moreover, we are given that g composite c, the other way around, is homotopic to identity. But g composite c is c. Thus, we have what? c is homotopic equivalence from X to single point. So, which means X is contractible.

So, we have got a map X to c single point that itself is a homotopy equivalence is what we have proved. So, X is contractible. So, 1 implies 2, 2 implies 1 is okay. The 3 and 4 as I have told you already, they are built-in automatically. There is no problem. You go to single point and then keep on composing.

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2 and 3 are an equivalence of 2, 3 or 2, 4. They are very straightforward. Let us have some examples now.

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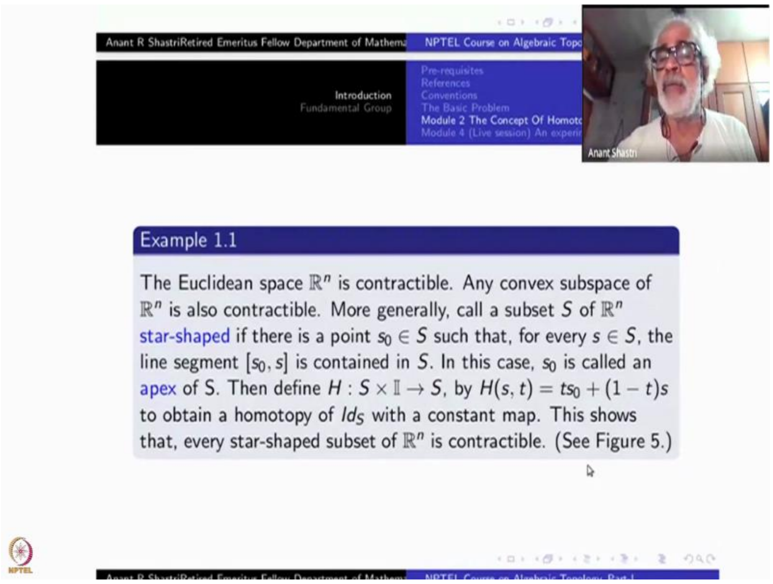

Euclidean spaces $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \ldots$, they are all contractible. In fact, they are all vector spaces. So, what happens? You can join lines. The joining the lines is joining any two points by straight lines. That is the way to get homotopies. More generally, inside a vector space you can take a convex subset.

What is the convex subspace? Given any two points x and y inside A, the line segment between X and Y. So, x and y are vectors. So, t times x plus 1 minus t times y will give you the line segment. That line segment will be entirely inside A. Then you can take that as a homotopy. That is the whole idea.

So, all convex subsets are contractible. In fact, we can generalize it to what are called as star-shaped subsets. Namely take a subset S it is called star-shaped, if there is one point s_0 such that every element in S can be join to this s_0 inside S. It is the same thing as saying that the line segment [s_0,s] is contained in S. In this case s_0 is called the apex of S. This is definition for star-shaped sets.

Then all that I do is I have define an homotopy S cross $\mathbb{I}$ to S. In the beginning, in the starting it is identity map. At the end it is the single point s naught, the constant function s_0. So, I have s_0 here and the identity here, s goes to s naught is the first one. s goes to identity of s itself is a second map. I just join them: $ts_0 + (1-t)s$. When t is 0, it is s. Identity. s goes to s. When t is 1, no matter what s is, it is s_0, the single point.

So, obviously this is a continuous function. This is just a linear combination of you know scalar multiples and adding subtracting and so on inside a vector space. So, all these things that I am talking in $\mathbb{R}^n$. You can have $\mathbb{R}^n$, or $\mathbb{C}^n$ or any vector space. This will be true. So, this will give you homotopy of the identity map with the constant map. If identity map is homotopic constant map, our theorem 1.1 will tell you that the space is contractible.

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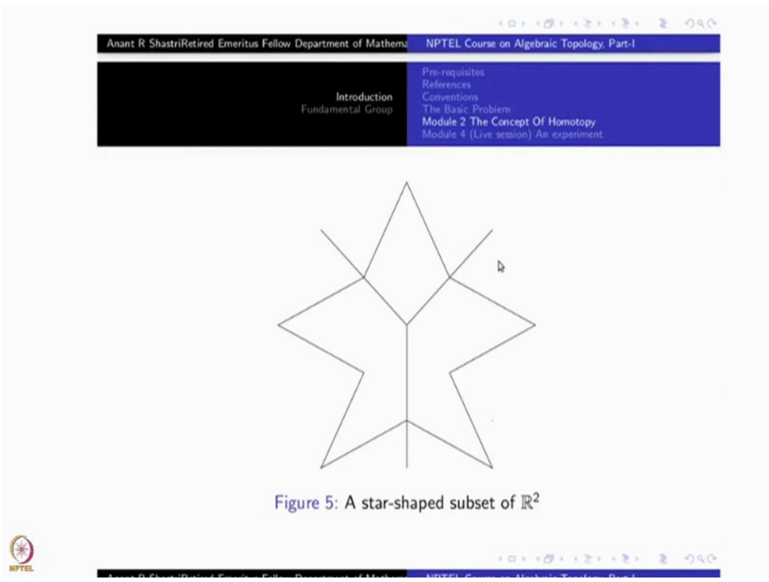

Figure 5: A star-shaped subset of $\mathbb{R}^2$

So, we have a picture of what is the star shaped set here. We see these 3 lines meeting at the point. That the point is s naught. To take any point inside this figure, you can join it to that point s naught by a line. A line segment completely inside of the picture. So, you can take a point here. This line segment is up till here.

To take a point here for example, because the point itself is not there. So, there is no question. The two points must be there. Any other point of the segment s t must be inside this one. So, that is called star-shaped set. This is a stat-shaped set inside $\mathbb{R}^2$.

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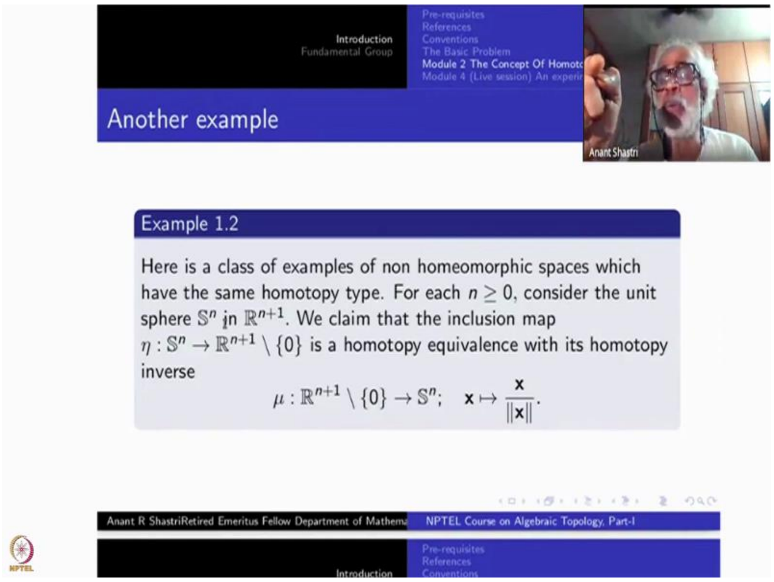

Let us have one very important example. This will give you a sequence of spaces, pairs of spaces. They are not homoeomorphic to each other, but they are homotopic to each other. This time I am not giving you things which are homotopic to constant map, constant homotopic to a single point. They are not contractible but the two spaces are of same homotopy type. So, what are they?

For each n greater or equal to 0, you take the unit sphere, $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. For example, when n is 0, What is this? This is $\mathbb{R}$. And this is $\mathbb{S}^0$. $\mathbb{S}^0$ is what? Whatever unit vectors inside $\mathbb{R}$; plus 1 and minus 1. In $\mathbb{R}^2$, what is a unit sphere? $\mathbb{S}^1$ is the unit circle; the set of all unit vectors inside $\mathbb{R}^2$. So, what we are doing is to throw away the 0 from $\mathbb{R}^{n+1}$; $\mathbb{S}^n$ is inside, included inside. This is the subspace of that. Unit vectors are never 0. So, this is a subspace. So, this η is the inclusion map.

We are going to show that this $\eta : \mathbb{S}^n \to \mathbb{R}^{n+1} \setminus \{0\}$ itself is a homotopy equivalence, with its homotopy inverse μ which is very nicely given, namely, take any vector divide by its norm. So, that is the map μ. This μ is homotopy inverse of the inclusion map.

One way is clear namely you take a unit vector. Think of this as a vector here, non-zero vector here. If you divide by the normal, you will get the same x, same vector because norm is already

1. So, that means mu composite eta is actually identity. What you have to show is eta composite mu is homotopic to the identity of $\mathbb{R}^{n+1} \setminus \{0\}$. This is what you have to show. So, that is the homotopy inverse.

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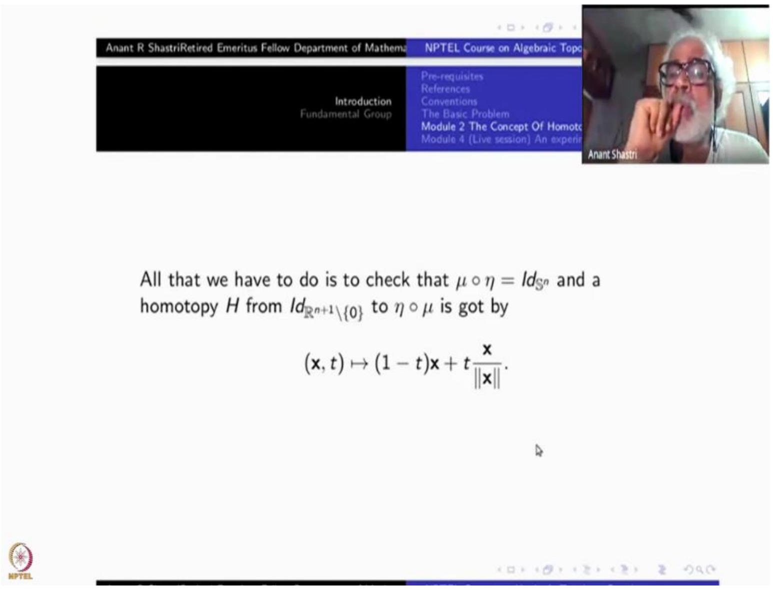

Whenever you can actually write the homotopy it is nothing but joining the two points. So, this is identity x. This is x by norm x. I am joining them, 1 minus t times x plus t times x by norm x. The point here is that the right-hand side will never be 0. Therefore, this entire thing is taking place inside Rn plus 1 minus 0. Why? Can you tell me why?

Because take any non-zero vector, to start with, x is non-zero vector. Then x by norm x is also a vector in the same direction. They are on the same ray emanating from 0, open 0. 0 is not there, open ray emanating from 0 and paaing through x. So, when you join these the whole line segment it, will be away from 0. So, this is the homotopy inverse of the inclusion. This example you have to understand very clearly, because there will be modifications of this one. This will keep coming again and again.

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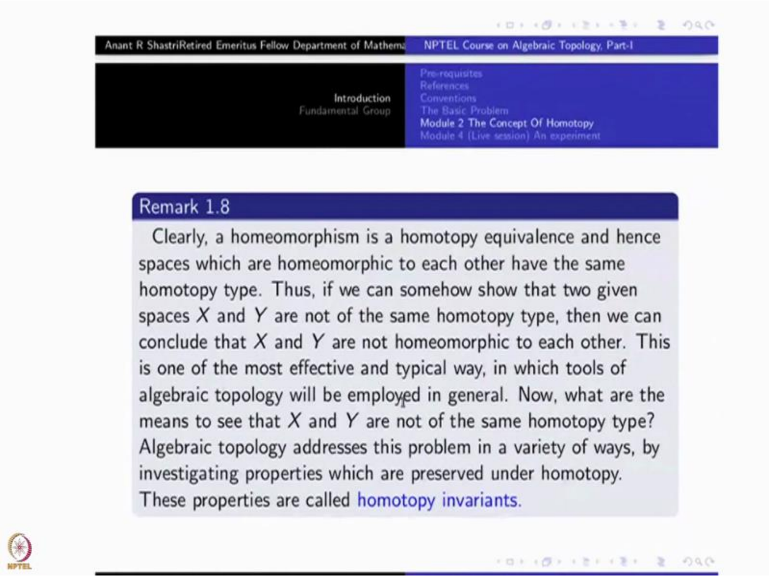

A homeomorphism is definitely a homotopy equivalence, because f composite g is actually equal to identity, g composite f equals to identity of the corresponding spaces. So, if you know that X and Y are not same homotopy type, then they cannot be homoeomorphic. So, this is one of the effective ways, how algebraic topology, how homotopy theory is employed.

So, how to determine whether two spaces have same homotopy type or not? This is our fundamental question. This is answered in various ways by cooking up different algebraic or algebraic -homotopy invariants. Just like we did in point set topology, if something is a T1 space, other is not a T1 space, they cannot be homoeomorphic. If something is T2 and other one is not a T2 space, they cannot be homoeomorphic. So, similarly you have to cook up various invariants---topological or homotopy type invariants.

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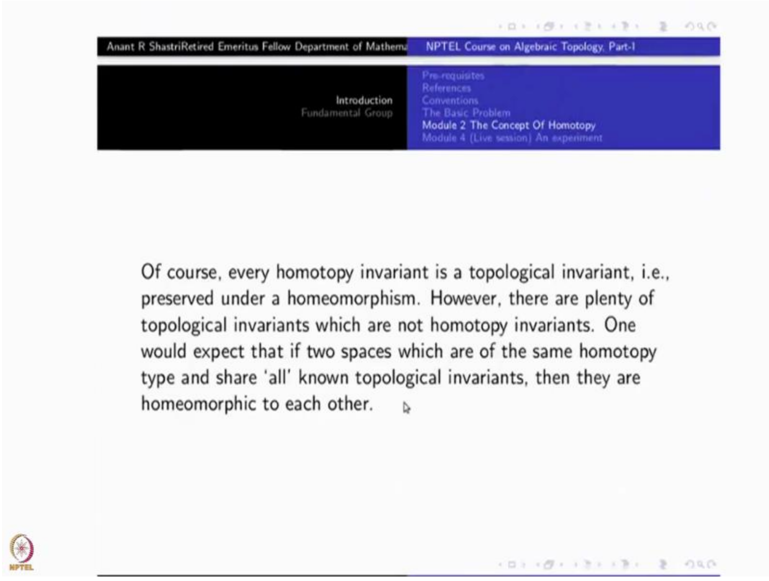

Every homotopy invariant is also a topological invariant. There is no problem. But there are many topological invariants which are not homotopy invariants. We have already seen one here, namely $\mathbb{S}^n$ and $\mathbb{R}^{n+1} \setminus \{0\}$. They have the same homotopy type. They are not homoeomorphic. It is very easy to see, how? $\mathbb{S}^n$ is compact and $\mathbb{R}^{n+1} \setminus \{0\}$ is not compact. Very easy to see. So, they are not homoeomorphic to each other but they are of same homotopy type.

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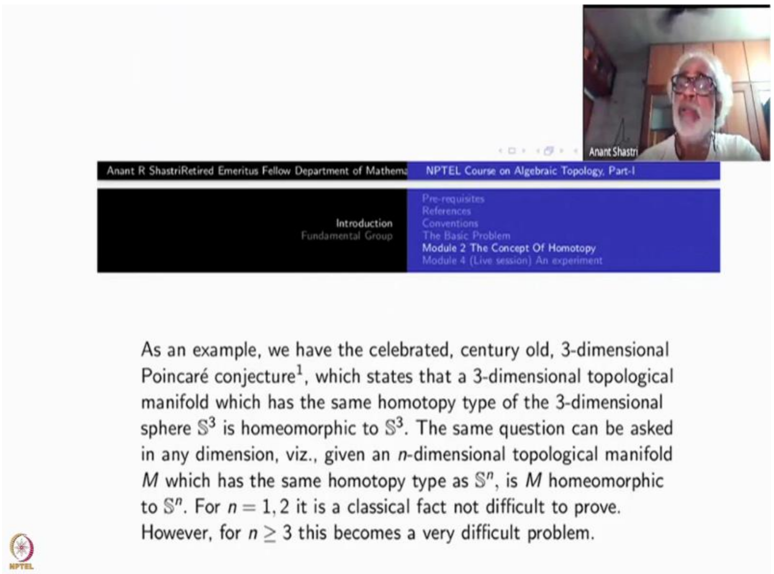

As an example, I will just discuss a few historical things here. There is the celebrated, century old, 3-dimensional Poincare conjecture, which states that something which `looks like' $\mathbb{S}^3$ up to homotopy type, almost like the three spheres up to homotopy type is actually homoeomorphic to $\mathbb{S}^3$. That was the Poincare conjecture, in dimension 3.

You can ask the same question in higher dimension also. You can ask in lower dimension also. Take something which is homotopy type of a circle. Will it be actually a circle? Of course, we have to be careful here namely what is the meaning of something like. So, that is the concept called `manifolds'.

You take a one-dimensional manifold which is homotopy type of a circle. It is not hard to show that it is actually a circle. This can be studied in point set topology also, but you might not have studied. So, if time permits, we will try to give you a proof of this one. Same question you can ask for $\mathbb{S}^2$.

And the answer is again, yes. But that is already a little bit difficult but we will try to answer it in the second part of this course. As soon as you come to 3-dimension this was a problem posed by Poincare which was answered only very recently namely 18 years back. But, n bigger than 3, this was already answered by many other people 50, 60, 70 years back.

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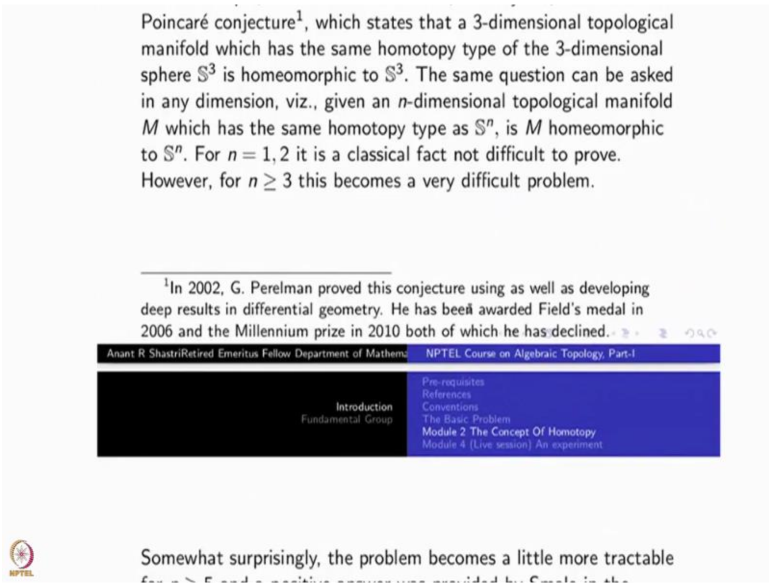

In 2002, G Perelman, a Russian mathematician solved this conjecture using very typical, very strange kind of differential geometry namely heat equations and so on. Unexpected somewhat. He was awarded Field's medal in 2006 and a Millennium prize 2010. Both of which he has declined. He has not accepted them. This fellow is much more stranger than our Ramanujan. He is a weird guy.

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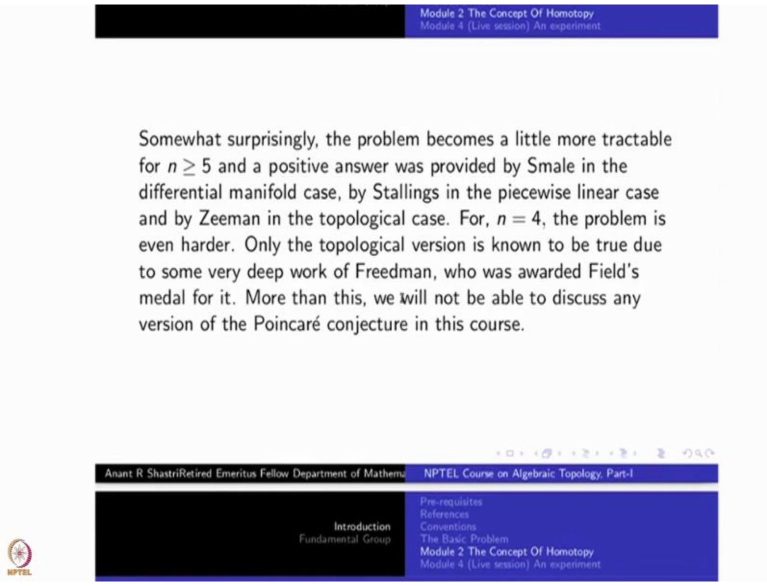

Somewhat surprisingly, the problem for n greater equal to 5 not 4 was solved by Smale in a differential topology case, Stallings in the piecewise linear case there are different versions when you go higher dimension, Zeeman in topological case. n equal to 4 two was another big problem. So, that was solved by in 1980 by Freedman, and he was awarded Field's medal for it.

So, this is all I can say about this problem. We cannot even go nearer to this problem in this course. We will be quite far away from that. However, next time I will tell you, there are some very, very great results which we can prove in this course. So, this I will tell you next time. Thank you.