Introduction to Algebraic Topology (Part-I)
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Lecture 29
Simplicial maps

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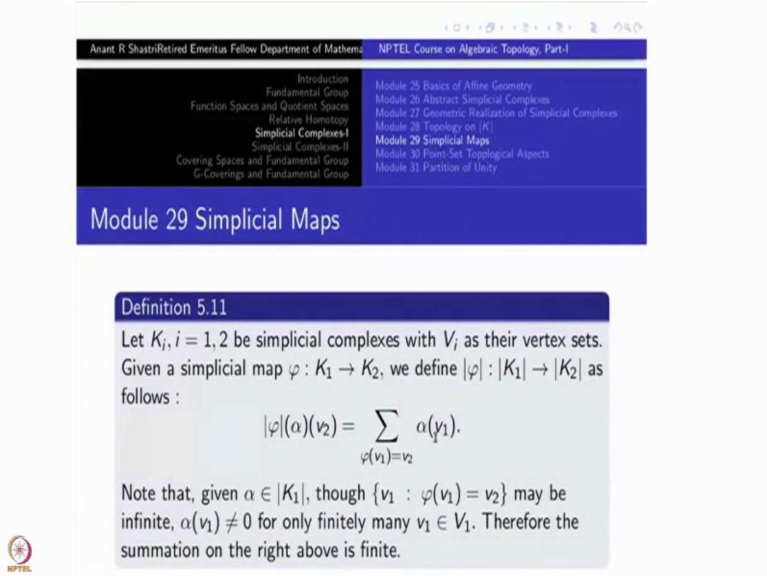

Having introduced abstract simplicial complexes and simplicial maps between them, we assigned a geometric realization, a topology corresponding to a simplicial complex. The next task is to convert the simplicial maps between two simplicial complexes into corresponding continuous functions from the geometric realization. So, today's topic is simplicial maps.

Start with two simplicial complexes $K_i = (V_i, S_i)$ with V_i as vertex sets. A simplicial map ϕ from K_1 to K_2 is nothing but a vertex function $\phi : V_1 \to V_2$ which takes the simplices of K_1 and simplices of K_2. So, once you have such a thing you want to define a $|\phi|$ from $|K_1|$ to $|K_2|$, the geometric realizations of K_1 and K_2, so you can call this $|\phi|$ also as the geometric realization of ϕ; it will be a continuous function. That is the aim.

But how do we take the `canonical' definition? Namely, if ϕ maps several of the vertices to the same vertex then at that vertex you take the sum of all the values of alpha which are mapped onto that vertex. Namely, we have to define $|\phi|(\alpha)$ (α is a function, remember, on the vertex set V_1, so, $|\phi|(\alpha)$ has be a function on the vertex set V_2.) So, let v_2 be a vertex of K_2, that means v_2 is inside V_2. Now $|\phi|(\alpha)(v_2)$ will be defined as the sum of all the $\alpha(v_1)$, where $v_1 \in V_1$ are mapped onto v_2 by ϕ.

There may be several points which are mapped onto v_2 by this by ϕ. So, take all of them and evaluate them at alpha and take the sum. So, the right-hand side here is a real number, a non-negative real number. This sum may be empty also, there may not be any v_1 mapped onto v_2, this sum may be non-empty, or the sum itself maybe 0 for each $\alpha(v_1)$ may be 0, so that is also possible. The sum total we have to see that it is less than or equal to 1. That is important. It is always non-negative, that is fine. If it is bigger than 1 it does not make sense. But this cannot be bigger than 1 because all the time it is some of the vertices evaluated by the same α, α evaluated only some of the vertices. Even if we evaluate α at all the vertices, namely, finitely many, the sum total is equal to 1, so this thing is always less than or equal to 1.

So, as a function it is well defined, it makes sense. One more thing we have to verify so that $|\phi|(\alpha) \in |K_2|$. Namely, in order that mod phi alpha must be a point of mod K2, its values at all the various points sum total must be equal to again 1. So, if I vary the points v_2 over V_2 here, then I am going to take all the points inside v1, which are mapped onto various points inside v2, but this sum will be taken over all the v_2's.

Therefore, all the v1s will be taken care of. Therefore, the summation will become 1, Therefore, this summation on this left-hand side is well defined, defines a function from $|K_1|$ to $|K_2|$. This summation is never infinite, by the very argument I have given because there are only values of alpha taken at various points, various vertices.

Moreover, one more thing we have to verify, look at all the points wherein $|\phi|(\alpha)$ is not 0, that is called support of mod phi alpha, this support of mod phi alpha must be, must be simplex of K_2. So, why that is true?

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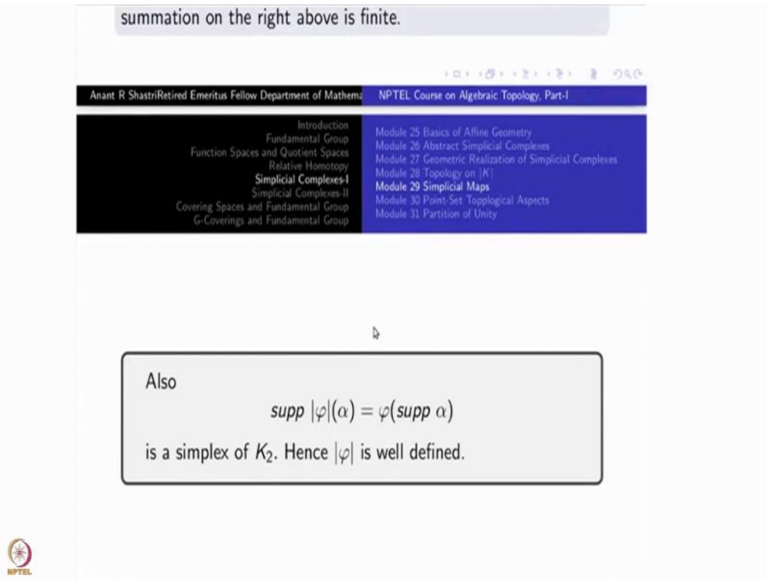

We need to have support of mod phi alpha as a member of S_2 but this is nothing but phi of support of alpha, look at all the points wherein alpha is not 0 only those things will contribute and nothing else, all of them will contribute of course, because $\alpha(v_1) \neq 0$ then this value will be counted in $|\phi|(\alpha)(v_2)$, where $v_2 = \phi(v_1)$. So, support of mod phi alpha is phi of support of alpha; $supp\ ||\phi|(\alpha) = \phi(supp\ \alpha)$.

This is a simplex, phi is a simplicial map, therefore, phi of support of alpha is a simplex. With all these things the verification that mod phi is well defined is over.

Essentially, you have to think of this as summing up the coordinates. So, if we are working with product topology, automatically this will be a continuous function but we are not working with product topology. What we are working with is this weak topology, we have to verify that $|\phi|$ is continuous. But verifying continuity is easier in the case of weak topology.

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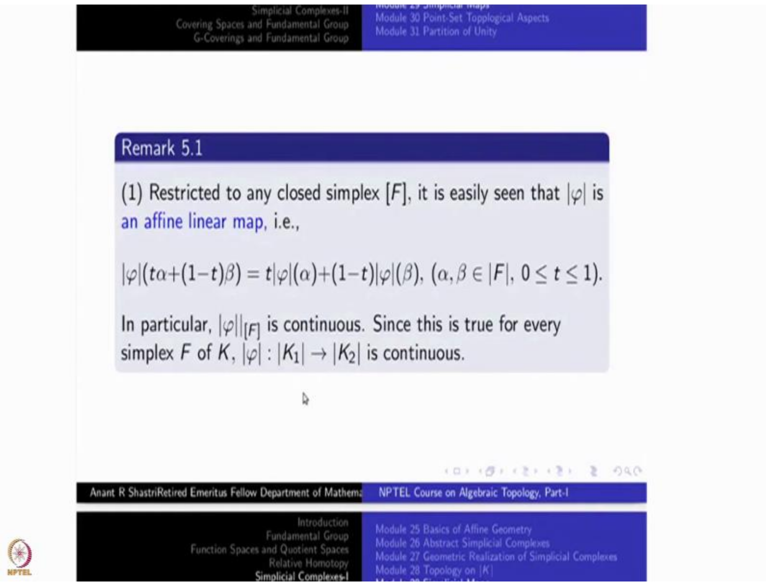

Restricted to any closed simplex $|F|$, $|\phi|$ is a linear map, affine linear map, when I say linear here, it is affine linear. Namely, $|\phi|(t\alpha+(1-t)\beta) = t|\phi|(\alpha)+(1-t)|\phi|(\beta)$. Here, alpha and beta are elements of this closed simplex. Therefore, this convex combination makes sense, that will also an element of closed simplex, mod phi of that is t times mod phi of alpha plus 1 minus t times mod phi of beta. Can you say why? Evaluate both sides at each vertex v_2 of K_2 and check that the two sides are the same, you can see or go back to this definition.

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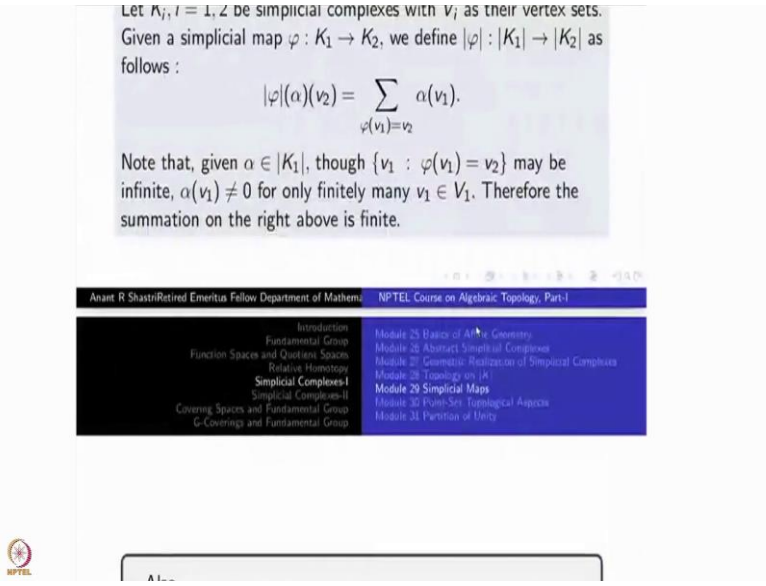

The left-hand side is $\sum_{\phi(v_1)=v_2} (t\alpha+(1-t)\beta)(v_1)$. operating upon vertices which are mapped But before taking the sum you can pull out this t and 1-t out and we get $t\alpha(v_1)+(1-t)\beta(v_1)$.

Therefore, upon taking the sum, we get the sum of two terms $t \sum_{\phi(v_1)=v_2} \alpha(v_1) + (1-t) \sum_{\phi(v_1)=v)2]\beta(v_1).}$ And this is nothing but the RHS.

So, if you restrict mod phi to closed simplex F it is continuous because it is affine linear. Since this is true for every mod F, where F range is overall the simplices of K_1, that is enough for continuity $|\phi|$ from mod K1 to mod K2. This is one of the criterions for continuity, this is the definition of the weak topology on mod K1. So, we are successfully defining mod of phi for each simplicial map phi from K_1 to K_2.

Thus on each chunk of simplexes, on a line, on a triangle, on tetrahedrons and so on, $|\phi|$ will be linear (i.e., affine linear, there is no origin for all of them). So, this is what is going to be, what we are going to say unifying the concept of linear approximation now. So, we have generalized just the linear map which is very rigid.

So, now, we can cut down one linear map in a small portion, another linear, another linear, combination of linear, so this is what is going to happen with this one. So, in some sense you can think of this mod phi as a linear map, all simplicial maps give back on mod K on the geometric realization something like a linear map. Especially, they will be continuous also.

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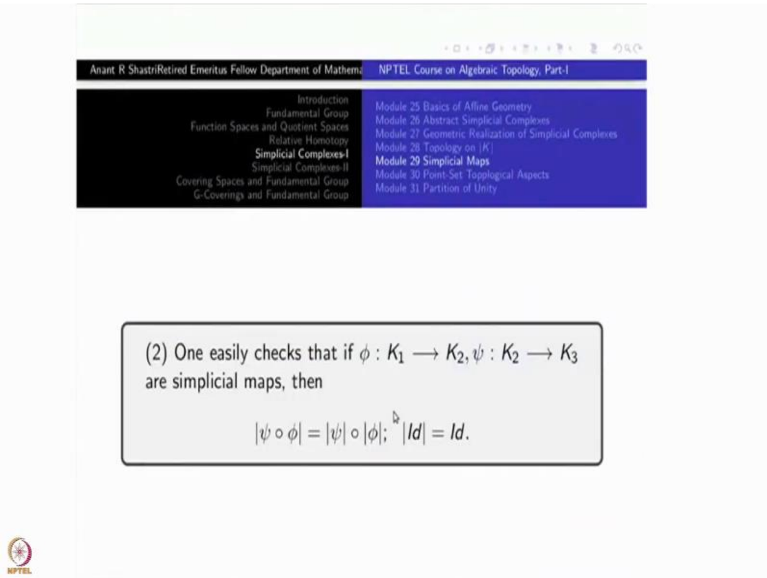

Another interesting thing here is that under composition this mod is compatible. Namely, first, you take ϕ and then follow it by ψ, these two are simplicial maps, you get $\psi \circ \phi$. Now you take the mod of that, it is eqaul to the composite of the two to moduli, viz., $|\psi| \circ |\phi|$. And modulus of identity is the identity. That is pretty clear because look at this definition here.

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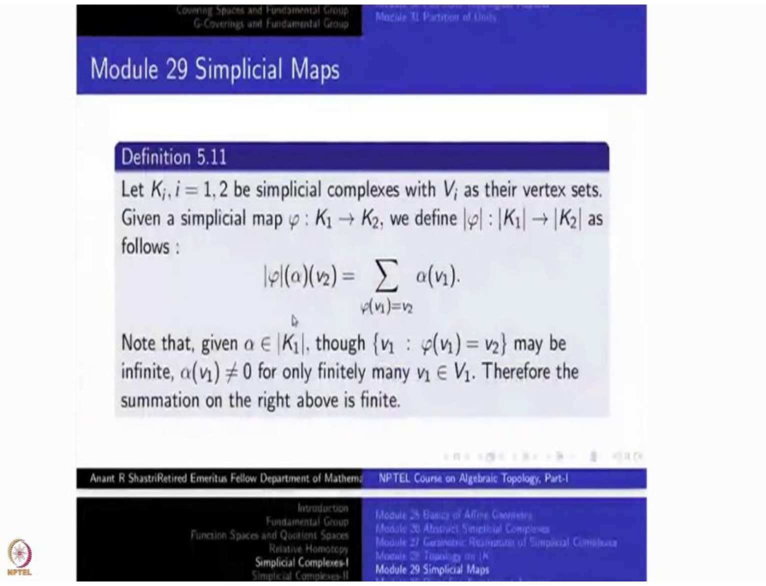

If $\phi : K \to K$ is the identity map then $\phi(v_1) = v_1$; there is no summation nothing. So, phi of alpha will be just alpha. Similarly, if you take another psi on this one, mod phi is done if you take another psi of that, by very definition, which psi of this one you have to take, psi of alpha, mod psi of alpha is again have taken the similar definition.

So, it will be a composition, so composition is also not difficult to verify. There are finite summations, you can interchange the summation also if you like, there is no need to do that, but you can say that the total summation is the same thing as the ones you to take the summation and other summation. So, composition is also difficult to verify.

Student: Hello sir, like for phi map, where simplex is mapped to a simplex, is it the same happening with mod phi too?

Professor: Mod phi is on geometric simplex, it is going inside geometry simplex, mod F you have to take bracket F closure of F, it will go to phi F closure, it will go inside of that that is all, that is what we already define. Support of phi alpha, this will tell you, this precisely tells you that.

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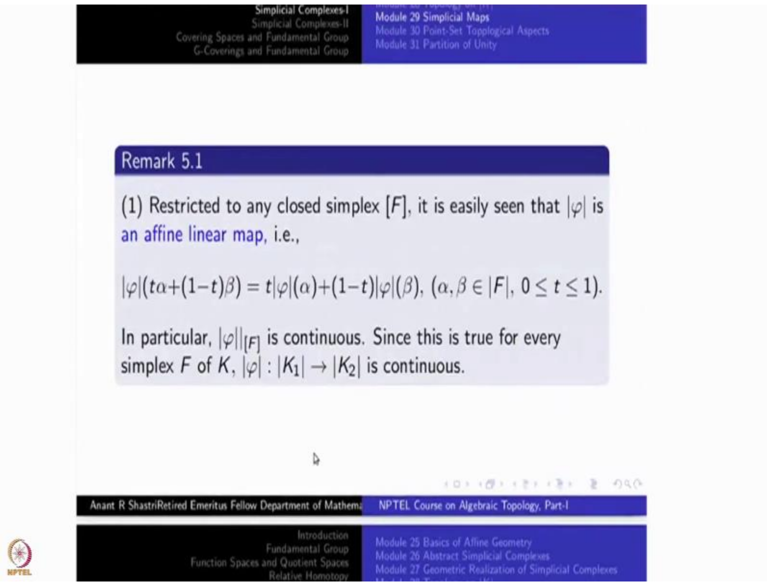

Restricted to each simplex it is inside, taking inside to that one. Support does not go out of that, phi of support this tells you the story.

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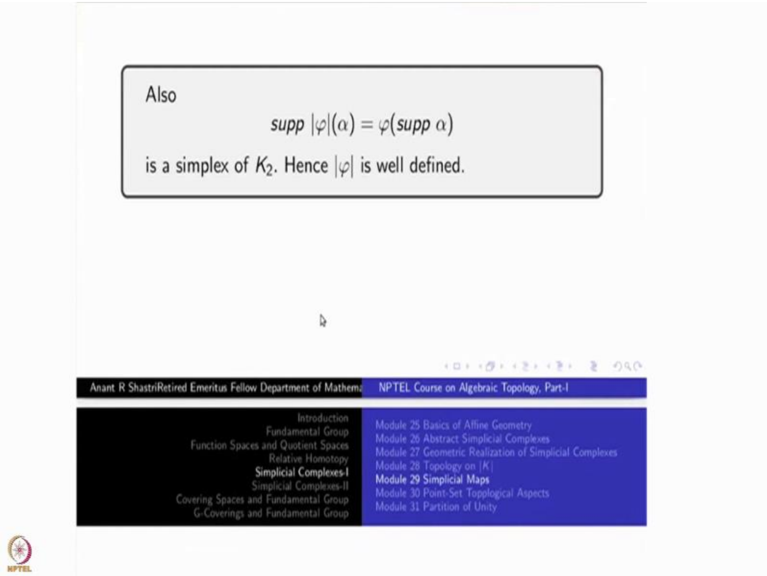

Support of mod phi alpha is phi of support of alpha, phi support of alpha if support of alpha is F, and phi of support of alpha is phi F that is a simplex. What you may have is here you may have an edge there may have just a vertex because both the vertices of the edge might have mapped to the same point.

Here you may have tetrahedron there you may have a triangle. If you have an edge here it will not the image will be a tetrahedron because there are only2 elements, function, saturated function will take only a smaller number of points, if at all, at most, that many these are all finite sets. Phi of any of those sets.

Vijay Sipani: Yes sir.

Professor: Will ever put that many elements but it is simplex in the other simplicial complex K_1 and K2. For, K_1 and K_2 may be the same also, that is why you can talk about the identity map, identity map is automatically a simplicial map.

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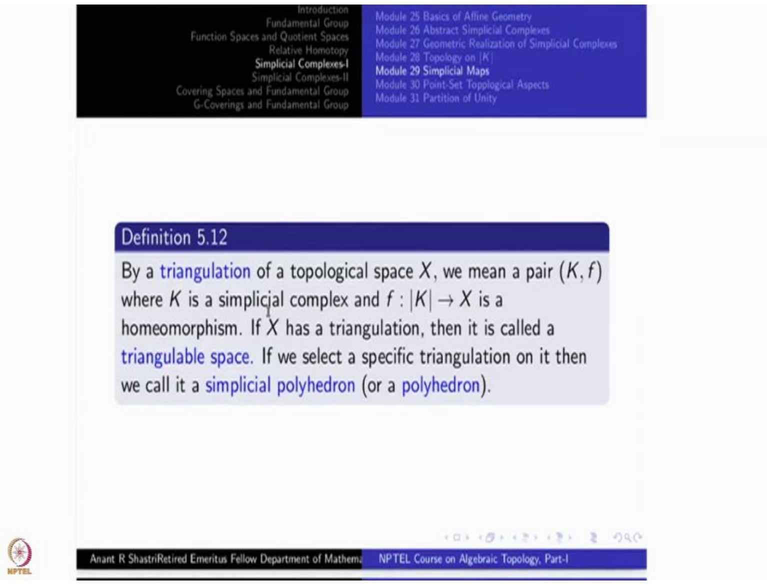

Now, we go ahead, what we do with simplicial maps and so on. By a triangulation of a topological space X, we mean a pair, a simplicial complex K and a homeomorphism f from mod K to X. You start with a topological space X it has no structure of a simplicial complex or anything but K is a simplicial complex and mod K is the topological space.

The function $f : |K| \to X$ is a homomorphism. So this is the definition of a triangulation. The word triangulation is borrowed from 2-dimension. When you take a surface, and cut it into a number of triangles. But now, we can use it for one-dimension, zero-dimension, n-dimension, fifty-dimension all of them. The word is only `triangulation' all the time, though you do not see triangles, if you go to three-dimension there will be tetrahedrons, and so on.

This notion has a parallel in measure theory—all the time you keep using the word `volume'. The two-dimensional volume is area, one-dimensional volume is length, but we do not have many words. So, four- dimension, five-dimension we are just taking volume it is just like that.

So, if we select a specific triangulation on it, then we call it a simplicial polyhedron, there may be several triangulations for a given topological space or there may not be any.

If there is one, then we say X is triangulable. If you fix one triangulation, then X is called a triangulated space or simplicial polyhedron, sometimes just polyhedron. So, do not confuse this polyhedron with convex polyhedron inside $\mathbb{R}^n$. A convex polyhedron is a special case of a polyhedron that is all.

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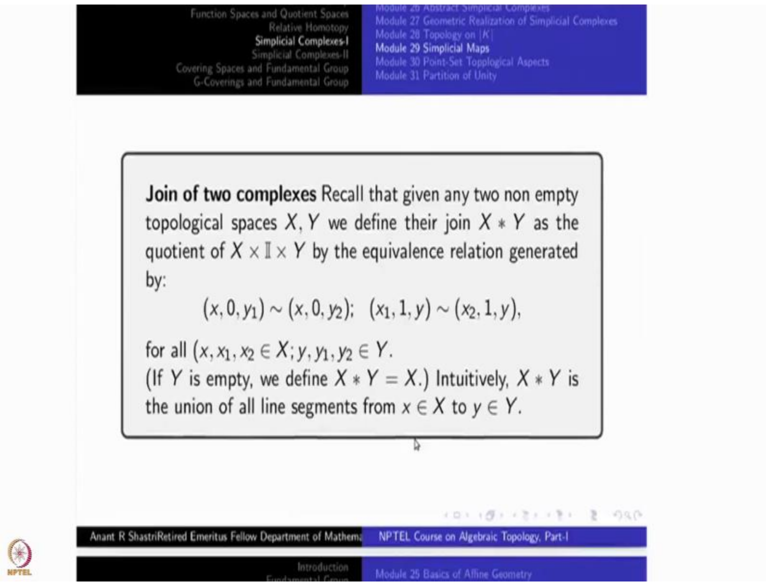

Now, we will have some examples here one by one. Recall that the join of two topological spaces X, Y is defined as the quotient of $X \times \mathbb{I} \times Y$, wherein x comma 0 comma arbitrary y1 was identified with x comma 0 comma y2, y1, y2 vary over Y. Similarly, on the other end, x1 comma 1 comma y was identified to x2 comma 1 comma y, x1, and x2 varying over X and y is kept as fixed. So, this was the definition of the typological join of X and Y and we have also defined the join of two simplicial complexes. So, these two are not entirely distinct. There is a close relation between them.

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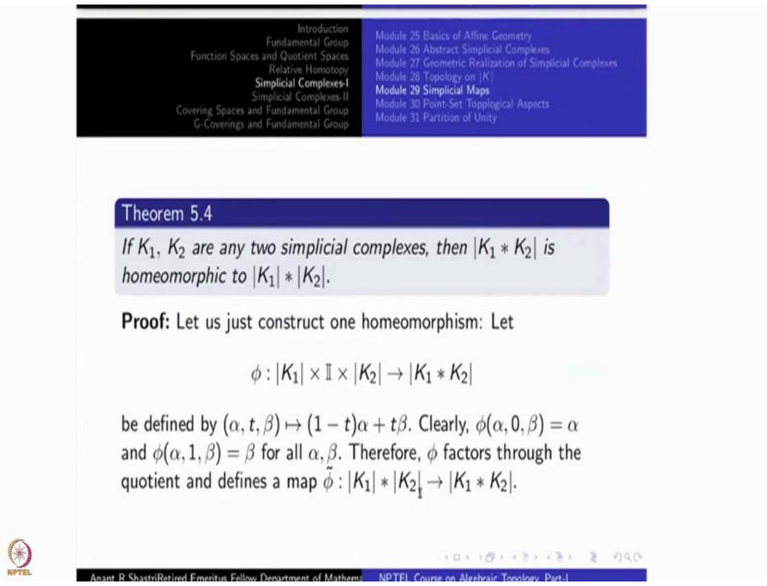

So, if K_1 and K_2 are any two simplicial complexes, then $|K_1 * K_2|$ is the join $|K_1|$ and $|K_2|$ that is homoeomorphic to $|K_1| * |K_2|$. $|K_1|$ is a polyhedron, $|K_2|$ is a polyhedron, and look at $|K_1| * |K_2|$, you do not know whether it is a polyhedron. Not only it is a polyhedron, it is triangulated by just $K_1 * K_2$. mod of K1 star K2 is homoeomorphic to this is the conclusion here.

Indeed, this homeomorphism is such that (remember that $|K_1|$ and $|K_2|$ are subspaces of their join and here K_1 and K_2 are subcomplexes of $K_1 * K_2$. Therefore, here also $|K_1|$and $|K_2|$ will be subspaces of $|K_1 * K_2|$. Under this homeomorphism, these subspaces are mapped to the corresponding subspaces identically. So, that will be easy, that will follow by just looking at the homeomorphism that you are constructing. So, what do we do?

We construct the map at the mother level, i.e, before taking the quotients. So, this is the left-hand side. Remember, what we have is $|K_1| * |K_2|$. I am defining map from $|K_1| * |K_2| \to |K_1 * K_2|$. So, this is a quotient space of $|K_1| \times \mathbb{I} \times |K_2|$. So instead of defining it on the join, I am defining a map on the mother itself then verifying that it factors down to the quotient. This is the way continuous functions are defined on quotient spaces.

So, how do we defend this one? An element of this, remember, can be thought of as a line from a point here and a point there, the line segment. So, a point alpha, a point beta, and a point $t \in \mathbb{I}$ between. So, this is a product, there is some identifications later on, so I am not going to use identification right now, so I directly taking the product. Assign it to 1 minus t times alpha plus t times beta which is finally supposed to be the line segment. That is why I am defining it like that: $\phi(\alpha, t, \beta) = (1-t)\alpha + t\beta$.

Now, alpha and beta, what are they? The 1 minus t times alpha plus t times beta should make sense here. So, how does it this make sense here? α (and β) are functions respectively on the vertex sets V_1 (and V_2) into $\mathbb{I}$. Remember how $K_1 * K_2$ is defined. The vertex set of $K_1 * K_2$ is the disjoint union of vertex sets V_1 and V_2. Therefore, alpha which is a function on V_1 (and β, a function on V_2) make sense as a function on $V_1 \coprod V_2$ by extending it by on V_2 (respectively on V_1); that is the meaning of this one. Next $(1-t)\alpha$ and $t\beta$ also make sense and so we can tale their sum, which happens to be a well defined function $V_1 \coprod V_2 \to \mathbb{I}$. Moreover the support of $\phi(\alpha, t, \beta)$ is contained in the union of the support of α and support of β which is simplex in $K_1 * K_2$.

So, first of all, you have to see that if I put t equal to 0 then beta is not at all coming into picture. Similarly, if t is 1 then alpha does not come into picture this will be just beta.

Therefore, by the very definition of the identifications, the function ϕ factors through the quotient map to define a continuous function continuous function $\hat{\phi} : |K_1| * |K_2| \to |K_1 * K_2|$. So, the definition is fine, now we have to verify that it is a bijection with a continuous inverse. The bijection is already taken care of by the very fact that by the expression here, what is the meaning of expression on this side?

Here you must remember that simplices is here are just disjoint union of simplices here and simplices there. So, modulus of those simplices are just nothing but points of this way and then you have to take 1 minus t times this plus 1 minus t times. The sum total has to be 1, that is why you have to take, cut it down by I say 1 minus t plus t then take this out. So, there is a map that lets us see why this is continuous.

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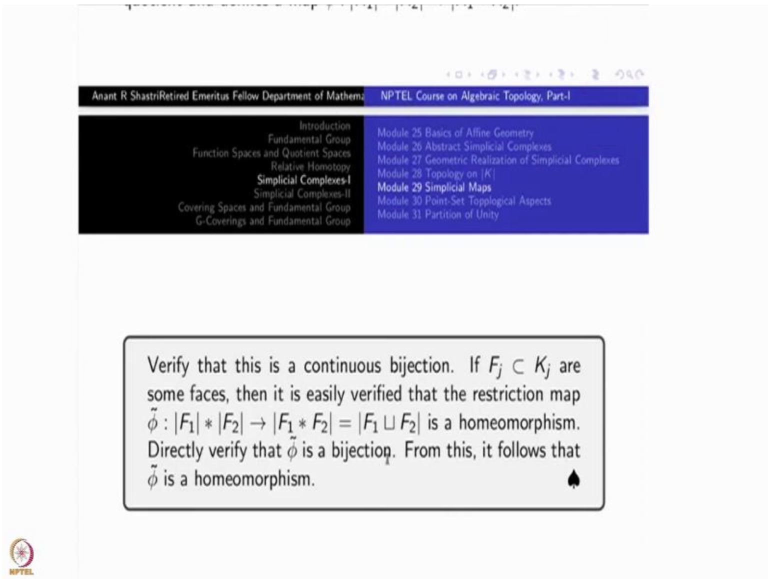

Suppose F_1 is inside K_1 and F_2 inside K_2 are some faces. Then restrict the entire ϕ to just these faces. The image never goes out of $|F_1 \cup F_2|$. Indeed, we get a map $\hat{\phi} : |F_1| * |F_2| \to |F_1 \cup F_2|$. It is straight forward to check that this is a homeomorphism.

Every element γ in $|K_1 * K_2|$ belongs to a unique $\langle F \rangle$, where F is a simplex of $K_1 * K_2$. Every such simplex F is a disjoint union $F_1 \cup F_2$, with $F_i \in S_i$, in a unique way. From these two obsrvations, the bijectively of $\hat{\phi}$ follows. The continuity of the inverse follows from that of the restriction $\hat{\phi} : |F_1| * |F_2| \to |F_1 \cup F_2|$.

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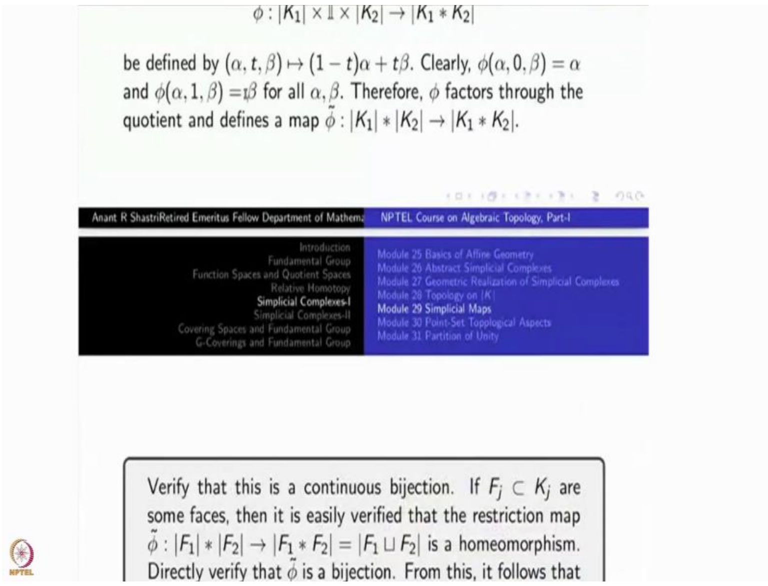

So, the restricted map is actually is an affine linear isomorphism.

Thus the simplicial joint corresponds to the topological joint when you take the geometric realization, so that is the meaning of this. In this correspondence is v canonical, the have obtained. You see, there is no other structure here it is just the structure of this intrinsic structure of these two things are used.

So, this will automatically give you associativity as well. K1 star K2 star K3, all of them are associated to K1 star K2 star (())(28:53). So, this is what we call a canonical homeomorphism. And the simplicial map if K1 to K2 they are simplicial map K2 to K3 then you go their correspondence simplicial map there is a commutative diagram under these homeomorphisms.

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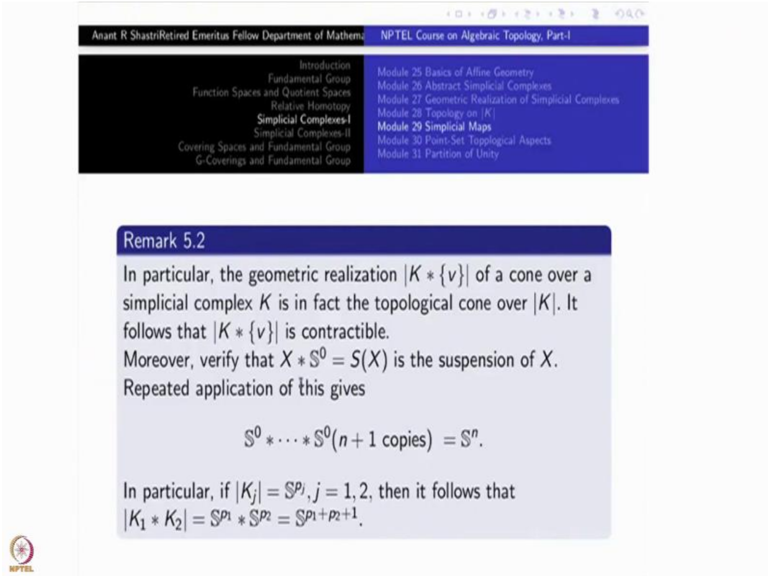

This is the generic thing. Let us now specialize. You can take K_1 to be any simplicial complex K, but take the second, a single point. Then by the definition $K * \{v\}$ is nothing but the cone over K. First take the cone over K and then take its geometric realization. This is the same as taking the geometric realization and then take the cone over $|K|$. So modulus of the cone over a simplicial complex K is in fact that the topological cone over $|K|$. Because we have , $|CK| = |K * \{v\}| \equiv |K| * \{v\} = C(|K|)$.

In particular $|K * \{v\}|$ is contractible. Similarly, if you take X star with the two point space $\mathbb{S}^0 = \{-1, 1\}$. You see that is the suspension we have defined. So, you can take $\mathbb{S}^0 * K$, here and then take the modulus that will correspond to suspension of $|K|$, this also this part we have already seen.

Repeated application of $\mathbb{S}^0 *$ gives you what? $|\mathbb{S}^0 * \mathbb{S}^0| \equiv \mathbb{S}^1$. So, if you tak n+1 copies that gives you $\mathbb{S}^n$. In particular, if you start with some simplicial complex K_i such that geometric realization of that is a sphere of some dimension n_i, let us say, then if you take the join of these two, then take the geometric realization it , that will do same thing as join of the two spheres correspondence spheres $\mathbb{S}^{n_i}$ which will be equal to $\mathbb{S}^{n_1+n_2+1}$. So this is what we have seen join of these two, so this is a special case of this one. So, you can use many special cases to derive various results from this joint. So, let us say some more examples, the simpler example now.

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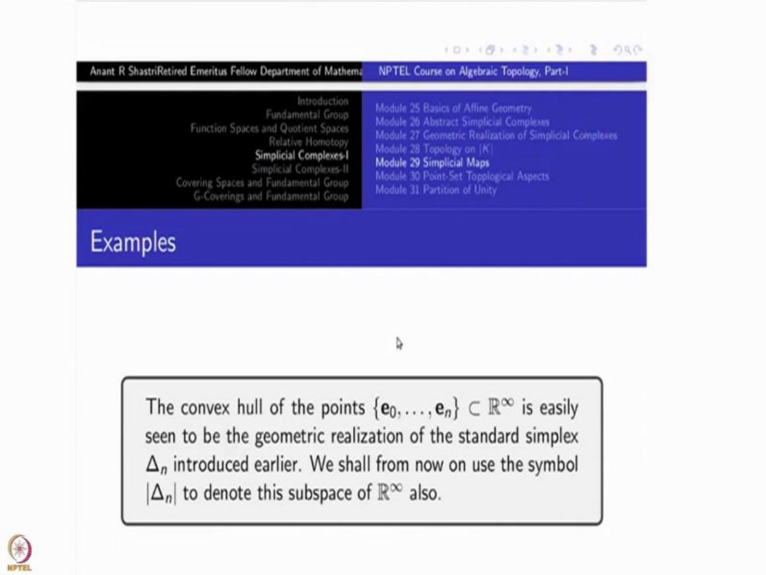

The convex hull of our set $\{e_0, e_1 \ldots, e_n\}$ inside $\mathbb{R}^\infty$ is easily seen to be isomorphic to the geometric realization of standards simplex Δ_n. That is what we have seen by definition. We shall now use this symbol now, $|\Delta_n|$ for both of them whether it is inside $\mathbb{R}^{n+1}, \mathbb{R}^{n+2} \cdots \mathbb{R}^\infty$ we will use the same symbol $|\Delta_n|$ for the geometry realization, and the standard n-simplex.

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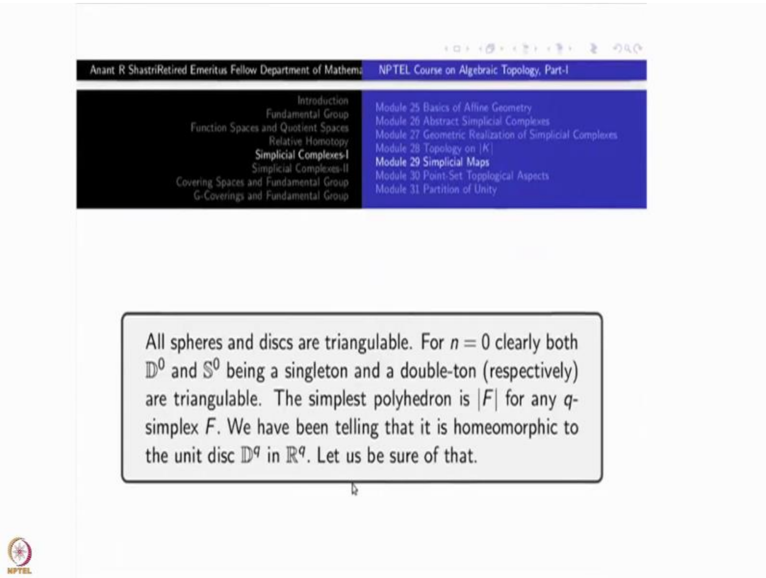

All spheres and discs are triangulable, it means what? There is a polyhedron, there is a simplicial complex, mod of the simplicial complex will be homeomorphic to these spaces. So, what are these, suppose n is 0, what is $\mathbb{D}^0$? $\mathbb{D}^0$ naught a single point. what is $\mathbb{S}^0$? $\mathbb{S}^0$ is the two point space. So there is nothing to prove there, so singleton, doubleton they are triangulable.

The simplest polyhedron is $|F|$ for any q-simplex F which is is homoeomorphic to $|\Delta_q|$, mod delta q that is what you have seen. So, we have been telling that this $|F|$ is homoeomorphic to the unit disc $\mathbb{D}^q \subset \mathbb{R}^q$. So, just to recall all these things, I will tell you how we have done it.

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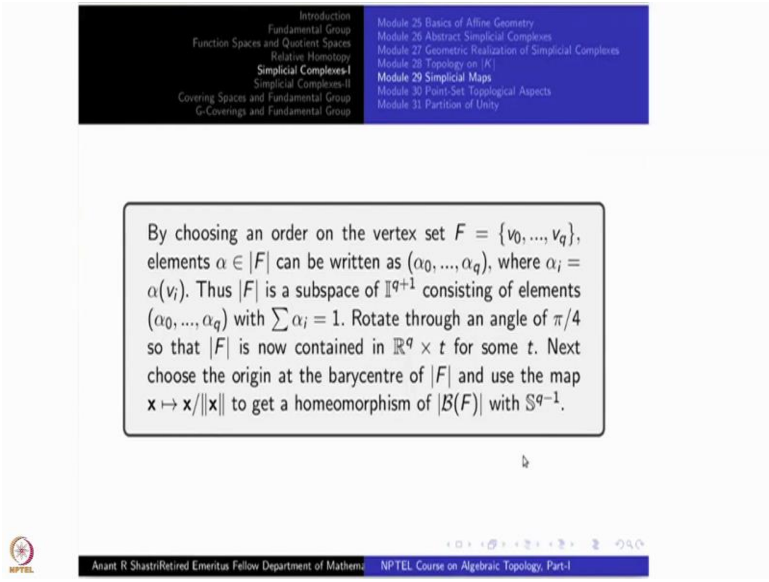

Let me start with a set of vertices $V = \{v_0, v_1, \ldots, v_q\}$ for F. An element α of $|F|$ is described by describing its coordinate functions, values of the coordinates $\alpha_0, \alpha_1, \ldots, \alpha_q$. $(\alpha_i = \alpha(v_i))$.

alpha is a function from this set V into $\mathbb{I}$, such that $\sum_i \alpha_i = 1.$ Thus, mod F is a subspace of $\mathbb{I}^{q+1}$ consisting of of elements $(\alpha_0, \alpha_1, \ldots, \alpha_q)$ such that $\sum_i \alpha_i = 1.$ You can see that it is a convex hull of, $\{e_0, e_1 \ldots, e_q\}$. It makes an angle $\frac{\pi}{4}$ with the x-axis or x0 -axis, whatever, so rotate it through an angle $\frac{\pi}{4}$ so that it will be contained inside some horizontal plane $\mathbb{R}^q \times \{t\}$.

Next, choose the origin at the barycentre of $|F|$, what is the barycentre of $|F|$? It is $\beta = \frac{v_0 + v_1 + \cdots + v_q}{q+1}$, you think of that as the origin. Then you take $x \mapsto \frac{x}{\|x\|}$. x going to to get a homeomorphism of $|\mathcal{B}(F)|$, the union of all the boundary faces onto the round sphere $\mathbb{S}^{q-1} \times \{t\}$.

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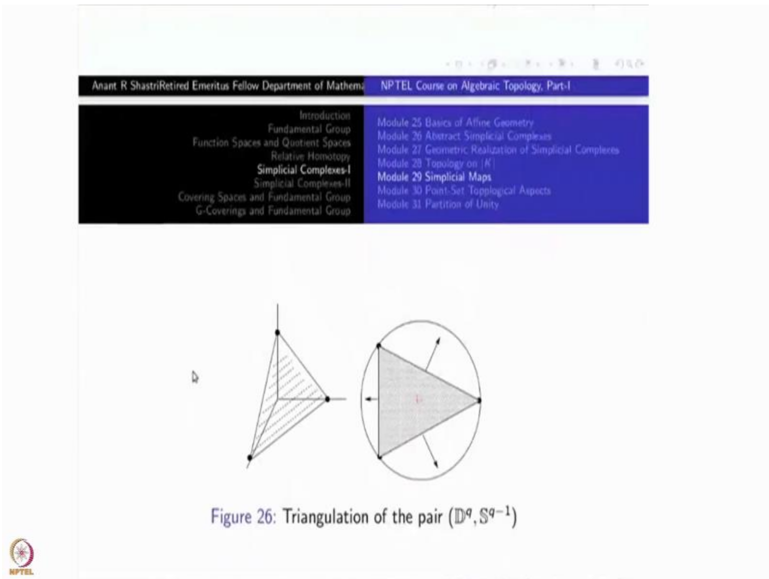

Figure 26: Triangulation of the pair $(\mathbb{D}^q, \mathbb{S}^{q-1})$

So, here is a picture this is a triangle, you can say, these are e_0, e_1,and e_2, then the convex hull of that is the triangle. The triangle is making 45 degrees with the x-axis. you rotate it and translate it to bring this point β to the x-y plane, the the entite triangle inside x-y plane . Rotation and translation because I do not want to change its geometry just for a while.

So, then this has become horizontal in the horizontal plane. you choose the orthocentre or barycentre as the origin then you take x by norm x; project all of the boundary of this triangle goes to this sphere homeomorphically, this is just like a projection map. Once you have a homeomorphism over the boundary to the $\mathbb{S}^1$ you can use the so-called cone construction to say that the full disk is homeomorphic to the full simplex.

So, I will describe that one in general. So, this is what is going to have triangulation as a pair, this is supposed to be the full disc $\mathbb{D}^q$ and the boundary. Simultaneously, the pair gets triangulated, triangulation means now what? You have a simplicial complex is the boundary of simplicial complex giving you when you take mod respectively the Dq and Sq minus 1. So, I will describe this cone construction once for all again and again this will come.

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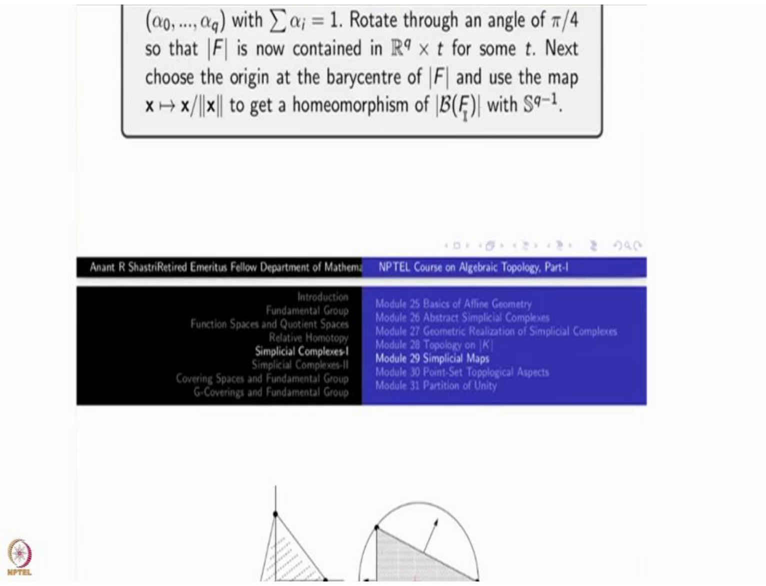

Remember the boundary of F is nothing but all the subsets except F itself. So, I want to say that, this is going to give you $\mathbb{S}^{q-1}$, so this part we have seen. Now, you take the cone over this, that is going to be the same thing as F is going to give you F back when you take modulus. If you take the cone over F, see for each subset of F you have to add an extra point.

So, that will give you the full set F, the maximum set F one point was missing, the cone will come there and it will be the full set. So, this is one way of looking at the boundary of BF, the cone over that one is homoeomorphic to again mod F. Modulus of BF is homoeomorphic, so this is what we have to use on both sides now. So, this is what I am saying.

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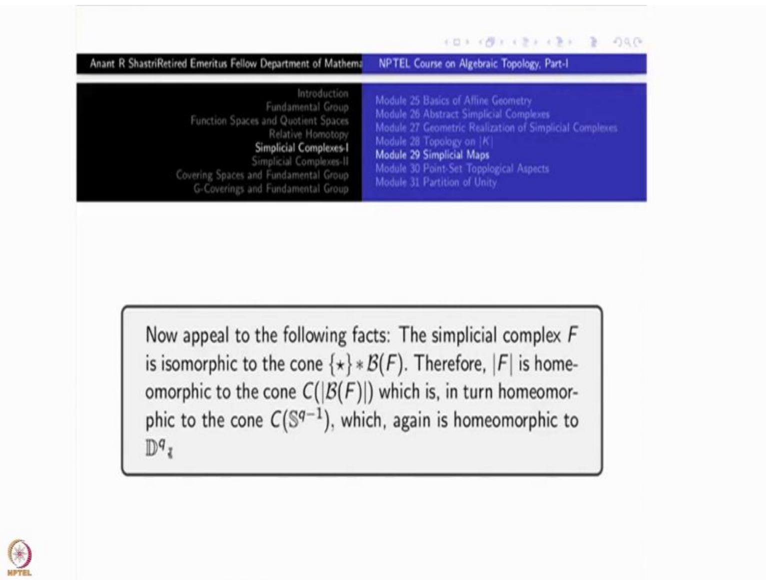

Now appeal to the following fact, simplicial complex F is isomorphic to start off, this is the star a single point is a cone, this is the join of the single point at B F. Therefore, mod F is homoeomorphic to the cone, the cone over BF. I take BF and takes modulus of that and then take the cone.

First, you take the modulus or first take the cone or then modulus this gives us the same thing, which is in term homoeomorphic the cone. Which again is homoeomorphic to the full disc, this was also what we have seen. So, a cone over any sphere is homoeomorphic to the disk. Let us stop here.