Introduction to Algebraic Topology (Part-I)
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Lecture 28
Topology on |K|

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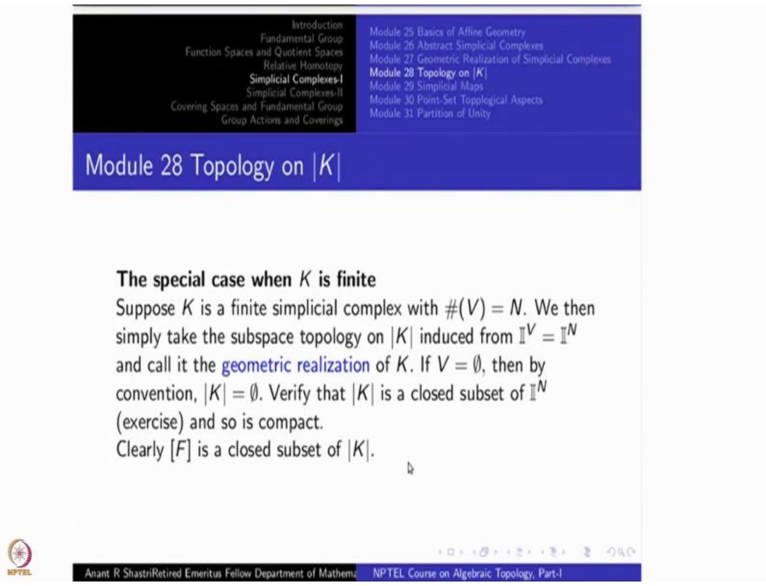

So, last time we have introduce the definition of $|K|$ when K is a simplicial complex. So, today we study the topology on mod K. Before that, let me just recall the definition and a few notations.

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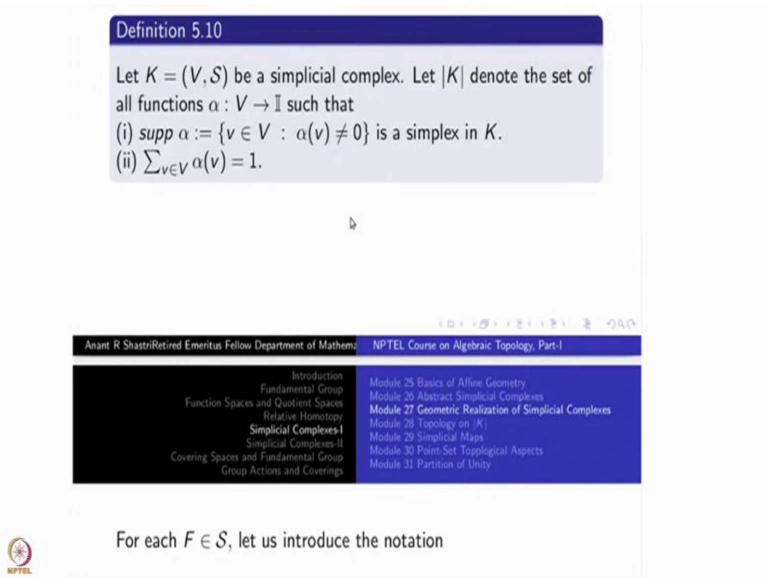

So, we took $|K|$ as a subset of the product set $\mathbb{I} \times \mathbb{I} \times \cdots$ taken V-number of times, number of vertices in V, whatever that many copies of $\mathbb{I}$ is taken, which is the same thing a taking functions from V to $\mathbb{I}$. Additional condition that the support of alpha, viz, set of points where alpha is not 0, must be a member of S and the sum total $\sum_v \alpha(v) = 1$.

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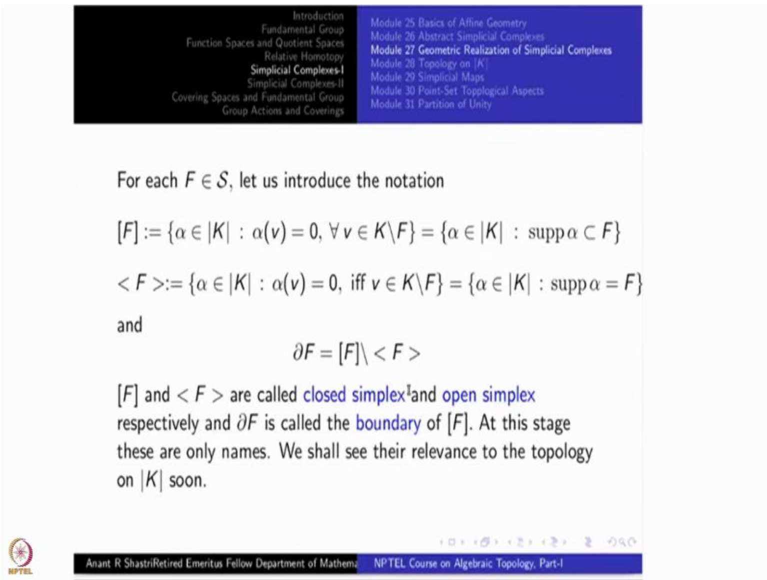

After that we introduce two notations. For each simplex F, the closed simplex $[F]$, bracket F, close bracket, this is the set of all α for which the support is contained inside F. This same thing as saying for every v inside $V \setminus F$, $\alpha(v) = 0$. The second one is the open simplex $\langle F \rangle$ which is the set of all $\alpha \in |K|$ such that $\alpha(v) = 0$ if and only if $v \in V \setminus F$. That means that, $\alpha(v) \neq 0$ for every $v \in F$ which is the same thing as saying $supp\,\alpha = F$. Then boundary of F, ∂F was defined as the closed simplex minus the open simplex. Let us come to the topology now.

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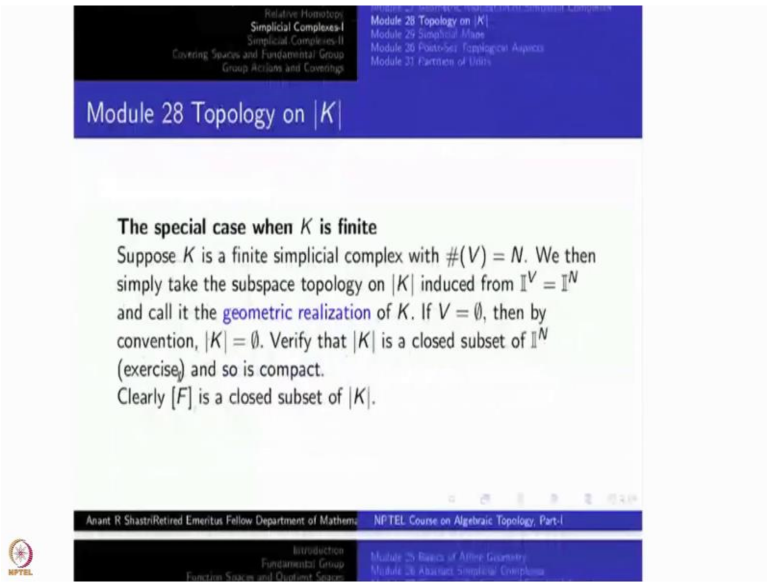

I will first treat the finite case because it is simple, that is all and it gives you the motivation. K is finite, same thing as V is finite. Then $|K|$ subset of $\mathbb{I}$ cross $\mathbb{I}$ cross... V number of times as many as elements of V. So, it is a finite cube $\mathbb{I}$ cross $\mathbb{I}$ cross ...V the finite cube. So, we can just simply take the subspace topology from this $\mathbb{I}^V$ which is nothing but $\mathbb{I}^N$, N is the cardinality of V. So, this is a familiar object for us, you see, we are inside $\mathbb{R}^N$.

So, take the induced topology namely, the subspace topology. Then you call it the Geometric Realization of K, Geometric Realization of K and denoted by $|K|$. It is not just a set. It has a topology, what is this topology? At least in the case of K is finite it is the subspace of $\mathbb{I}^V$ defined in a particular way. Anyway, it is a subset of I power V already, you take the subspace topology. You already noticed that it is given by a closed condition, so $|K|$ is a closed subset of this one.

In particular, it is compact, being closed subsets of a compact space. It is compact and each $[F]$ is again a closed subset of $|K|$, that is also given by a closed condition. If conditions are all given on the coordinate functions equal to something, then the intersection of several such sets here, all of them closed. So, it becomes a closed set.

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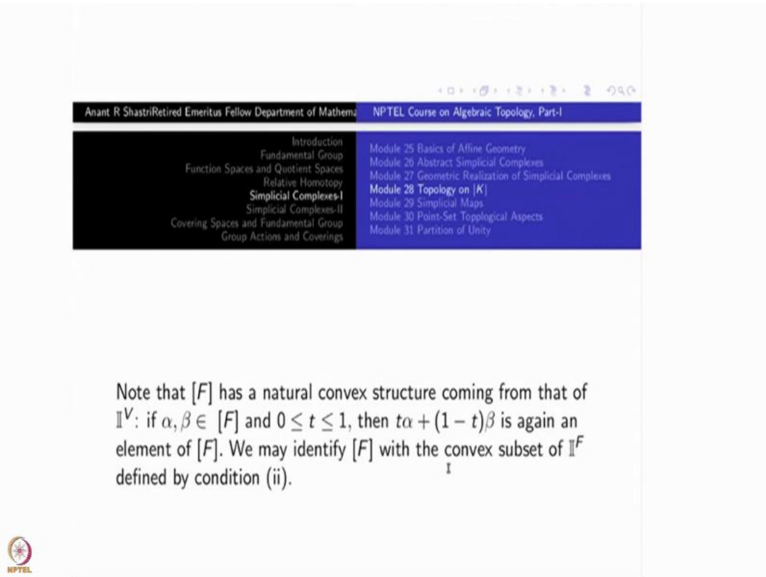

Note that closed simplex F has a natural convex structure coming from that of $\mathbb{I}^V$. $\mathbb{I}^V$ is a convex set in $\mathbb{R}^V$. So, that convex structure is there on $[F]$. Even the open simplex $\langle F \rangle$ itself is a convex subset. For, if alpha and beta are inside this closed (open) simplex, and t is between 0 and 1 and t times alpha plus 1 minus t times beta is again is an element of $[F]$(resp. $\langle F \rangle$). What is the meaning of this?

Take this one say this is $\gamma = t\alpha + (1-t)\beta$. $\gamma(v) \neq 0$ implies that $v \in F$ why? $\gamma(v) \neq 0$ then $\alpha(v) \neq 0$ or $\beta(v) \neq 0$ and in either case, $v \in F$. If v is a point outside F then both $\alpha(v) = 0 = \beta(v)$ and hence $\gamma(v) = 0$. Therefore the support of $\gamma \subset F$. That is the same as saying that γ is inside the closed simplex F.

You may identify $[F] = |F|$ with a convex subset of $\mathbb{I}^F$. You see that $\mathbb{I}^F \subset \mathbb{I}^V$ in a natural way. It is actually a quotient a retract of $\mathbb{I}^V$, because $F \subset V$, so given $\alpha \in \mathbb{I}^F$, you can extend it as a function $\alpha : V \to \mathbb{I}$ by putting $\alpha(v) = 0 \; \forall \; v \in V \setminus F$.

So, there is an identification of this one also. And of course, inside that you have to again take the condition to that $\sum_v \alpha(v) = 1$. So, what is the thing? I have just excluded all those coordinates for which these points is 0 anyway, so that is why I have taken only this part, that is all.

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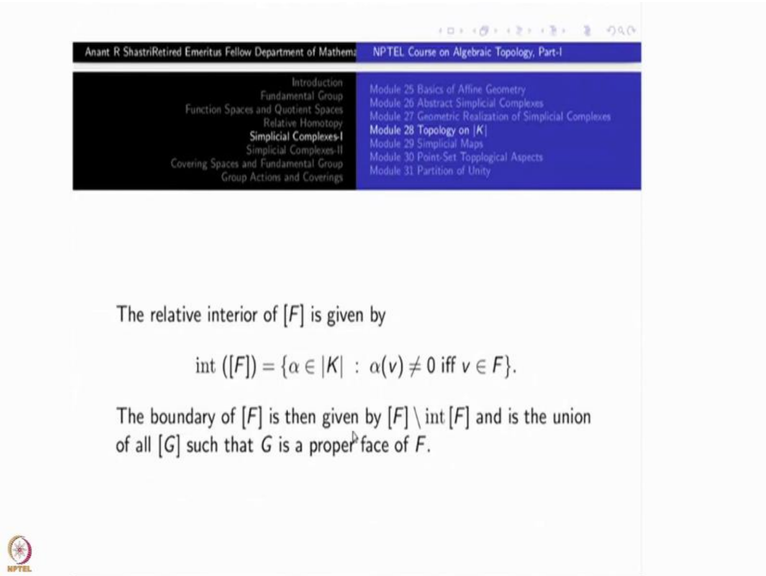

Now, comes the more interesting topology. The relative interior of $|F|$. The $relative$ interior of a subset inside a topological space is not exactly same the interior of of this in topological space X. The relative interior has to be define inside the space, it is a tricky thing. So, this can be done for geometric things, so that is what we are going to do.

For example, suppose you have a closed interval contained in $\mathbb{R}^2$. As subspace of $\mathbb{R}^2$, but it is actually inside $\mathbb{R}$. In $\mathbb{R}$, the closed interval has its interior which is the open interval, but as a subspace of $\mathbb{R}^2$, it has no interior. So, this open interval is the relative interior of the 1-simplex irrespective of where it is contained in. So, we are defining the word `irrespective' , what is that?

Interior of F, int(F) is precisely consists of those $\alpha \in |F|$, for which $\alpha(v) \neq 0 \quad forall\ v \in F$. This is another notation now, for a set which we have already defined, viz., it as open simplex F, $\langle F \rangle$.

Now, why this is called an open simplex F? It is the interior of $|F|$ in the usual sense. So, inside this $[F] = |F|$, open F is an open subset. The boundary of F by its very definition will be a closed subset consisting of those $\alpha \in |F|$ such that $\alpha(v) = 0$ for at least one of the vertices of F.

So, it is inside that proper subset union of subsets. Second point, again I am telling you. The closed simplex F is homoeomorphic to the geometric realization of F, where F is thought as a simplicial complex. Given a simplex, you take all its subsets that is a simplicial complex then what is the geometry realization of this? You have to take all functions from $F \rightarrow \mathbb{I}$, such that the sum of the vlaues is 1. That i how mod F can be identified with a subset of $\mathbb{I}^F$. The first condition on the support of α is automatically satisfied because every subset of F is already inside the simplicial complex.

So, this mod F, this bracket F is a subset of some simplicial complex $|K|$, the geometric realization K, can be identified with geometric realization of the simplicial complex F. Each map $\alpha : F \rightarrow \mathbb{I}$ can be identified with the extension $\hat{\alpha} : V \rightarrow \mathbb{I}$ F where $\hat{\alpha}(v) = 0$ for all v outside F and $\hat{\alpha}(v) = \alpha(v) \quad v \in F$.

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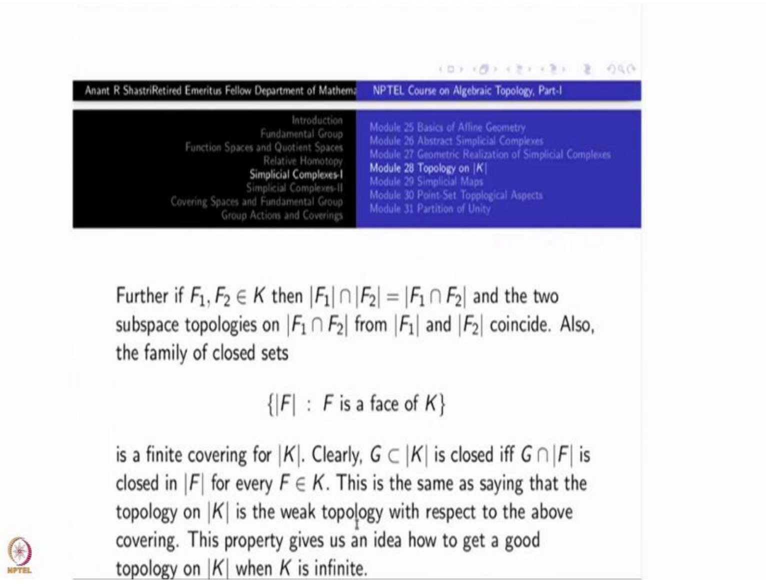

Now, inside $|K|$ what happens to the intersection of $|F_1|$ and $|F_2|$? Look at two simplices F_1 and F_2 in K. I am talking $|F_1| \cap |F_2|$ inside $|K|$. I will say that this precisely equal to $|F_1 \cap F_2|$. F1 intersection F2 if it is empty, then it should be empty. Those which have supports inside F1 and those which are supports inside F2 also, F1 and F2 are disjoint, there cannot be any such element, they will be totally disjoint.

Therefore, if $F_1 \cap F_2 = \emptyset$ it is Ok. But suppose it is non empty. Then it is a simplex in K, it is a face of both F_1 and F_2. Now I will say that $|F_1 \cap F_2| = |F_1| \cap |F_2|$. This is very straightforward set theoretic you see. Support of α is contained in $F_1 \cap F_2$ iff its contained in both F_1 and F_2. That is all.

Moreover, the two subspace topologies on the intersection coming from $|F_1|$ or coming from $|F_2|$, they coincide. This is very important thing we have talked. Because they are both equal to the subapce topology directly from $|K|$. Indeed that topology is homeomorphic to the topology on the standard simplex in the Euclidean space.

So, whatever tentatively I have put $[F]$ is now the same thing $|F|$, where F is a face of K. As soon as it is a face of K, $|F|$ this is a closed subset of $|K|$. There is a family of closed subset, everything in K must belong to one of them, it is by definition every element of mod K has support in one of the faces.

So, this is a cover for $|K|$ but it is not an open covering it is a closed covering. It is a finite because K itself is finite I have taken. Clearly a subset G of $|K|$ will be closed if and only if G intersection with $|F|$ is closed in $|F|$ for every F in K. This is the property of any finite closed covering for a topological space, it is an elementary property. How you define a continuous function on two closed sets continuous on each of them and on the intersection, they agree automatically is continuous on the whole space.

So, this is true for finite covers which are closed on each of them if you verify continuity and all the intersections the functions agree that function is defined on all of it then it will automatically continue. So, the same thing as saying that G intersection mod F is closed for every F if and only if G is closed. So, it says the standard wording here namely, it is a weak topology with respect to the above covering.

If you know, if you recall your point set topology, this property gives us an idea how to get a good topology on $|K|$ when K is infinite. This property is going to be our guiding principle now for defining a topology when K is infinite, so that is not finite. So, let me complete this one.

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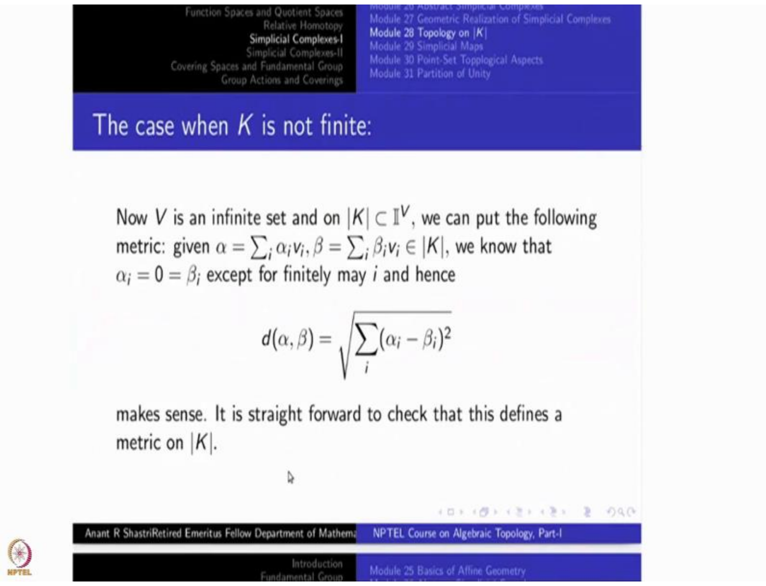

Let V be infinite set. $|K|$ is always a subset of $\mathbb{I}^V$, a closed subset of I raise to V, by definition. You can put the following metric now, first I will put a metric on this $|K|$ of course, for I raise to V, V is too big. So, I do not want to put a metric on that. What is it? Take any two elements $\alpha, \beta \in |K|, \alpha = \sum_i \alpha_i v_i, \beta = \sum_i \beta_i(v_i)$. Both finite sums. We have seen that α_i is $v_i - th$ coordinate of α. that is the meaning of this summation, and $\alpha_i \neq 0$only finitely many i. Similarly, $\sum_i \beta_i(v_i)$ is a finite combination like this. So, this is just a tentative notation, if you identify vi with an element of $|K|$, how do you do that? It is vi operating upon vi is 1 and vi operating on any other vj is 0. So, that is the element, these are the standard basis elements for I raise to V.

So, in that sense this summation makes sense, summation beta i vi belonging to $|K|$. We know that alpha i and beta is are 0 except for finitely many i, which is same thing as that these two are finite sums. Then, I can define distance between alpha and beta to be just the square root of the sum of the squares of the differences just like in the Euclidean space, take the coordinate functions, take their differences, take the square of that, take the sum, and then take the square of that, this makes sense because only for finitely many i's these are nonzero, I raise to V I cannot do that because this is too huge, then infinitely many coordinates may be nonzero.

So, for mod K you can do that because $|K|$ is contained inside, I raise to V for which, so this makes sense, direct verification just like in a direct sum of vector spaces this will be become a metric and we have a metric topology now, if its K is finite this is just the Euclidean metric in $\mathbb{R}^n$ because all those things I have to involve now.

So, whatever topology we have analysed for K finite that is metric topology only, from a Euclidean space, but we are using a different property of that space and then we will work to do this one, this metric topology on infinite thing is not a good thing for us. So, we will modify it.

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Let us denote this $|K|$ with the metric topology by $|K|_d$. The suffix d because we are not finally interested in this one, this is only an intermediate step. Clearly, it coincides with the Euclidean topology when V is finite. Therefore, topology induced on each $|F|$ from $|K|_d$ is the same as Euclidean topology that we have been taking namely of the standard simplexes and so on, that is what we have done so far.

For each $|F|$, the topology that we have defined using an isomorphism, using a homeomorphism of this one with a standard simplex. That topology is the same as the subspace topology from $|K|_d$.

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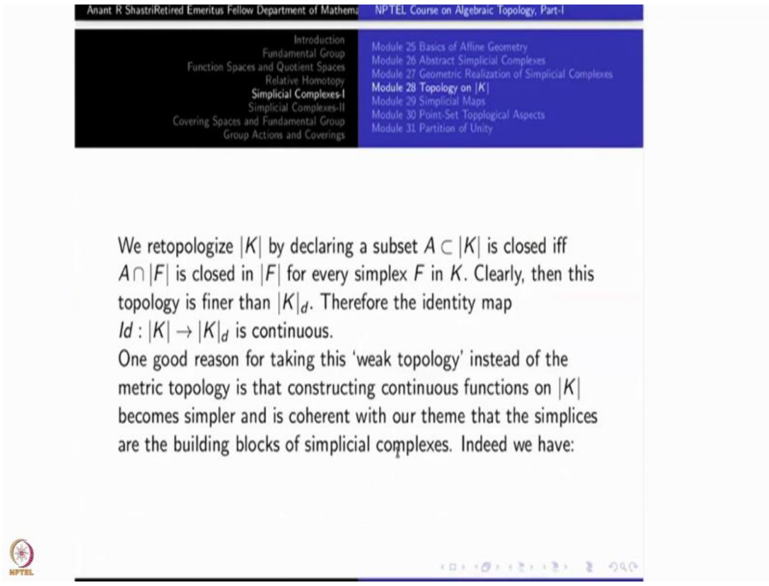

Now, I want to redefine the topology on the whole of $|K|$ by declaring a subset of A, a subset A of $|K|$ is closed if and only if, A intersection $|F|$ is closed in $|F|$, for every F in S. This property which we have verified when K is finite, this becomes the definite defining property for the topology in the general case, a subset is closed if and only if intersection with each $|F|$ is closed in $|F|$. So, this is the (weak) topology coherent with respect to the family of closed subsets $\{|F| \;:\; F \in S\}$.

Clearly, this topology is finer than $|K|_d$. If A is already closed inside $|K|_d$, then A intersection $|F|$ will be closed inside as well as $|F|$ because $|F|$ has the same subspace topology from $|K|_d$. So, all closed subsets of $|K|_d$ are inside this topology, weak topology. This is a funny name this is called weak topology for some different reason, but with respect to this $|K|_d$, it is stronger than the topology of $|K|_d$ because everything closed subset inside $|K|_d$ is already closed here. It is the finer than topology than $|K|_d$. A metric topology has certain properties. $|K|$ is even better. It has more open sets. The one good reason for taking this week topology, (I have put it in a bracket because this is not weaker than $|K|_d$ but it is actually finer) instead of the metric topology is that constructing continuous functions on $|K|$ becomes simpler and is coherent with our theme that the simplices are the building blocks of simplicial complexes, so this is the motivation I will restate it as a theorem.

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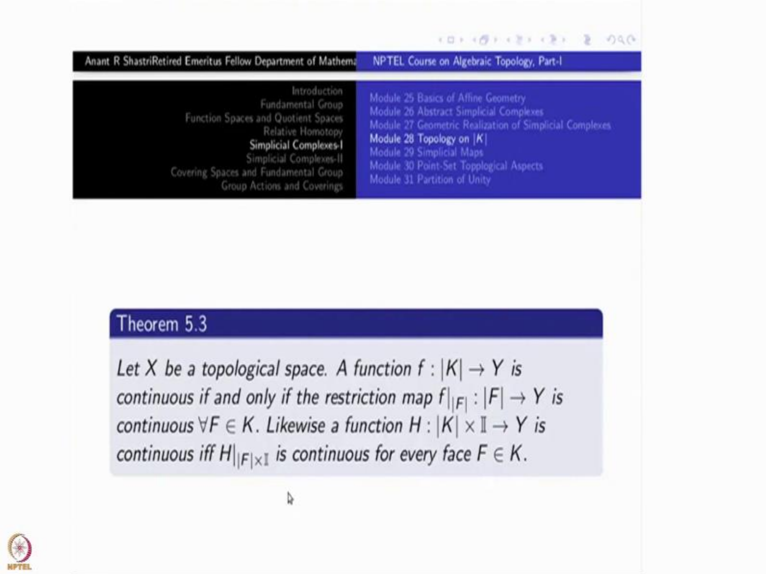

Take any topological space X, or this X should be Y. Then take any topological space Y, a function f from $|K|$ to Y is continuous if and only if restriction map of f to $|F|$ to Y is continuous for every F. What is F? F is s simplex in K. We write this one all the time instead of writing $K = (V, S)$. To be accurate, we must say, for every F inside S, the set of simplices.

Likewise, a function H from $|K| \times \mathbb{I} \to Y$ is continuous if and only if H restricted to $|F| \times \mathbb{I}$ is continuous for every face $F \in S$. The first statement is follows directly from the definition of the topology on $|K|$. Take a subset here which is closed, its inverse image is closed here is what I want to show. The inverse image here is nothing but F restricted to F inverse of that set. By definition, if this is closed, but if this is closed for every F then that is closed because that is the weak topology and conversely, this is fine.

But now same thing is claimed for the product, for this careful proof of this one has to be written down and that uses function space topology.

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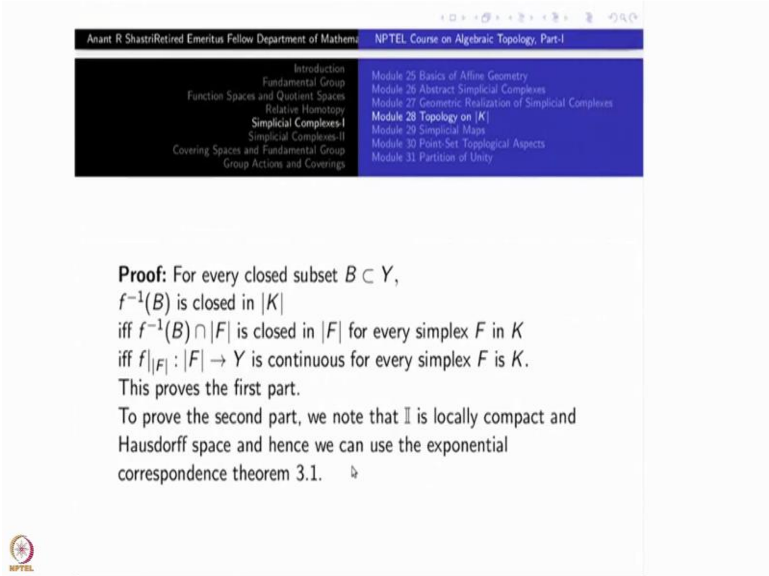

To prove the second part (first part I have done): We know that, $\mathbb{I}$ is locally compact. In fact $\mathbb{I}$ is compact an Hasudorff. Whenever we have a locally compact Hausdroff space, we can use that exponential correspondence theorem.

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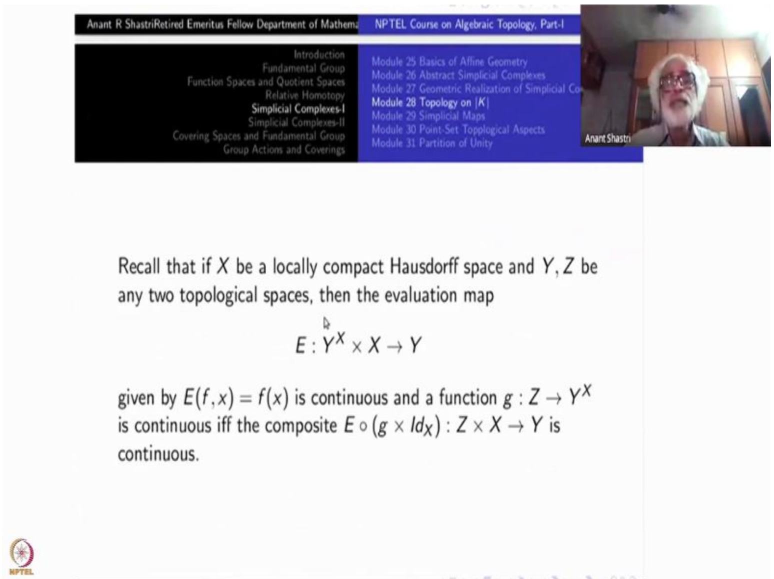

So, I am just recalling the exponential correspondence theorem. If X is locally compact, Y and Z are any topological spaces, the evaluation map E from $Y^X \times X$ to Y given by $E(f,x) = f(x)$ is continuous. And any function from any topological space Z to Y^X is continuous if and only if $E \circ (g \times Id_X)$ is continuous. These two theorems, this was part A, part B of that exponential correspondence theorem. So, I put X to be the closed interval [0, 1]. That is allowed because $X = \mathbb{I}$ is a locally compact.

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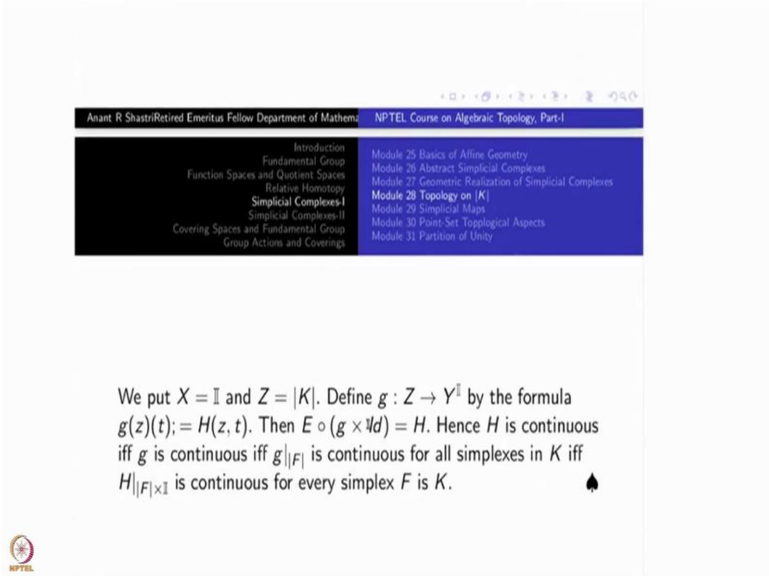

We put $X = \mathbb{I}$ and $Z = |K|$. Then you can define g from this $|K|$ to this $Y^{\mathbb{I}}$, Y is an arbitrary space, by the formula $g(z)(t) := H(z,t)$. Remember H is given function from $|K| \times \mathbb{I}$ to Y. So, first coordinate is in $|K|$, a second coordinate is in $\mathbb{I}$. Then take E composite g cross Id that will be precisely H, that is the definition. Hence, H is continuous if and only if g is continuous by exponential correspondence theorem.

Now, from the first part, g is continuous if and only if g restricted to each $|F|$ is continuous for every $F \in S$. Which is same thing as, now going back to exponential correspondence this g restricted to $|F|$ is continuous if and only if E composite g cross Id restricted to this one, which is same thing as H restricted to $|F| \times \mathbb{I}$, is continuous.

So, exponential correspondence is used, first go to the power here and then come back both sides here if and only if this part has to be use. We will stop here. Next time we study maps between simplicial complexes, what kind of maps we have to study there, okay. Thank you.