Introduction to Algebraic Topology (Part-I)
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Lecture 27
Geometric Realization

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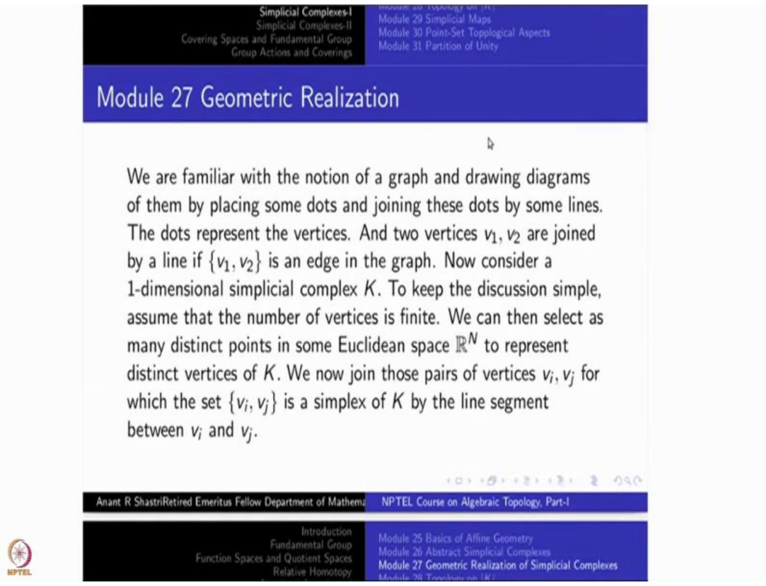

Today's topic is Geometric Realization of an abstract simplicial complex. Before I begin this one, let me take care of a few more examples that I have listed already, but could not cover.

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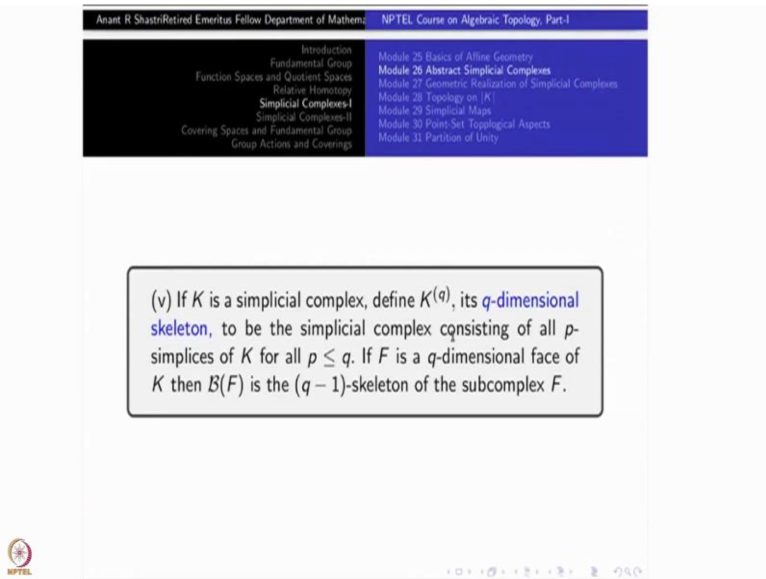

This example we are seeing, this we have seen, this is also an important example, if you start with a simplicial complex look at all faces of dimension less than or equal to one particular number, say q, less than or equal to q. Automatically, all the faces of whatever you have taken will be also there. Therefore, this will be a subcomplex, this subcomplex is called the q-dimensional skeleton of K and the notation is $K^{(q)}$.

For example, when you take F as a q-simplex, its (q-1)-skeleton is the previous example, BF, the boundary of F. Only F itself will be omitted and all its proper subsets will be there because F is the only q- simplex and that will be omitted because I am taking (q-1)- skeleton. So, this example generalises the previous example that we had here.

If you have any simplicial complex K, the zero-dimensional skeleton will be just the set of vertices. The 1-dimensional skeleton could also be set of vertices, if there are no one edges, if there are no 1-faces. That means the original simplicial complex itself is just a set of vertices.

Student: Sir.

Professor: So, you have to understand this notation carefully.

Student: Sir, here it should be p taken less than to q, right?

Professor: No, q-dimensional skeleton will include q . What you are saying is (q-1)- skeleton, q skeleton will include q, also. Zero-dimensional skeleton will include zero-dimensional simplex, that means only vertices will be there, 1-dimensional skeleton will have all the so-called edges, the 2-dimensional skeleton will have triangles and of course, edges and vertices.

But suppose, there are no triangles at all to begin with. Then the 2-dimensional skeleton, 3-dimensional skeleton, 4-dimensional skeleton or whatever you take, will be just the 1-dimensional skeleton only. In particular, a q- dimensional skeleton is a sub complex of a q plus 1-dimensional skeleton and so on. I am not saying anything more than that-- whatever immediately implies from the definition here.

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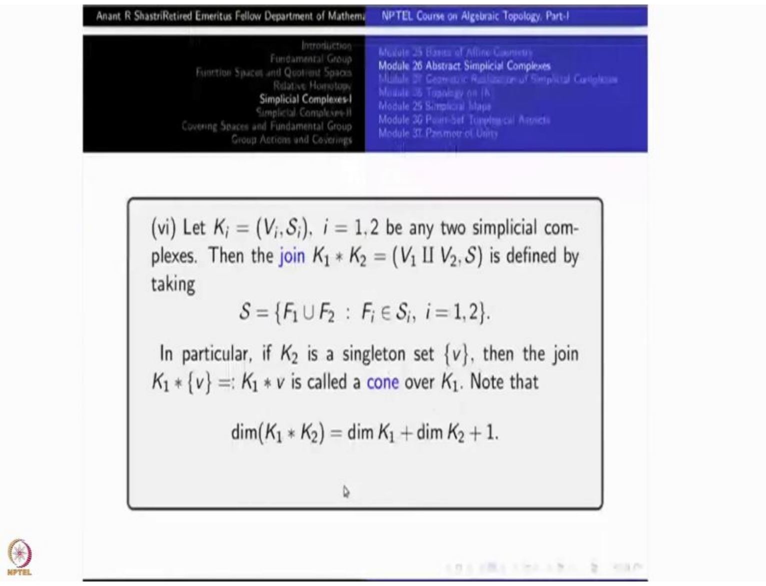

Here is something, constructing a new simplicial complex out of the old or just taking subcomplex and so on. So, take K_1 and K_2, two simplicial complexes whose vertex sets are V_1 and V2 and simplices are S1 and S2. We want to define the join of K_1 and K_2. This join later on will become a topological join when you take geometric realizations, this is not just concocted something.

But in principle, this is much simpler than the topological join. So, join of K_1 and K_2 is defined over the set of vertices which is disjoint union of V_1 and V_2 and nothing more, you have to take all the elements of V_1 and all the elements of V_2 as a disjoint union, then what is the set of simplices, that is what I have to define. The set of simplices will be also taken similarly: every simplex in K_1 family S1 and every simplex in K_2 and take their union.

So, $F_1 \cup F_2$, where F_i runs over S_i, including the empty set, remember S_i includes empty set also. In particular, all F_1 belonging to S_1, they will be there, all F_2 belonging to S_2 will also be there; then their union, disjoint union because these are treated as disjoint, V_1 and V_2 are treated as disjoint sets; this will be also a disjoint union. So, this is the simplicial complex on $K_1 * K_2$, the join of K_1 with K_2.

Now, by the very definition $K_1 * K_2$, it is equal to $K_2 * K_1$. So, this operation itself is commutative operation because V_2 disjoint union V_1 or V_1 disjoint union V_2, they are the same, so here also. So, that is an obvious thing. Now, if K_2 is empty then what do you get? You will get the vertex set just V_1 here and here you will get S_1, so then you will just get K_1 , $K_1 * \emptyset = K_1$.

(This is one of the reasons why the topological join X star Y is also defined to be X when Y is empty, if you recall, why I have taken that previously.)

In any case, if K_2 is empty then this is just K_1 or if K_1 is empty then it is just K_2. Now, if K_2 is a singleton set, then this becomes very interesting. Suppose, this singleton set V that means, one vertex and there are only two simplices namely, the empty set and the singleton, $K_2 = (\{*\}, \{\emptyset, \{*\}\})$. So, here is the singleto simplicial complex.

Then the joint will be what, you can just simply write instead of brackets and all that. It is just the cone over K1, so this is definition, the cone over K_1 is a special case. So, let us look at, what are the simplices in $K_1 * K_2$. All simplices in K_1 will be there, this extra vertex will be there. And for every simplex here, you put this extra vertex also, if it is a k-simplex here you will get a k plus 1 simplex by putting extra vertex.

All those things will be there. So, that is the cone over K_1. So, this is a special case of the joint. Similarly, I could have defined the suspension also here, the suspension will be sam as taking double cone, that means I have to take K_2 to be just two vertices and no edges, nothing more than just two vertices, $K_2 = (\{u_1, u_2\}, \{\emptyset, \{u_1\}, \{u_2\}\})$. And then perform this operation, $K_1 * K_2 = S(K_1)$, that will be a suspension of K_1.

So, when I want this, I will recall it, there is no need to worry about there. One of the important things which is very obvious here is that the dimension of $K_1 * K_2$ is dimension of K_1 plus dimension of K_2 plus 1, provided these right-hand side is defined. Of course, if any one of them is infinite, then the left hand side also will be infinite. And, in that sense, this equality makes sense always. So, let us see how to look at the top dimensional face here, top dimensional simplex here? Say that it has, it will have dimension of K_1 plus one element. Similarly, from K_2, the top dimensional face will have dimension of K_2 plus one element. If you take the union that will have this plus this plus 1 plus 1. So, this plus this plus 2 elements in it. Therefore, its dimension will be this plus this plus 1, So, that will be there already, so dimension of the left-hand side has to be at least that much. But it is also equal because if there is any simplex here, it will be the disjoint union of two things here, then its dimension has to be dimension of this plus dimension of this plus 1.

So, there will be corresponding simplex is here at least of that dimension. So, if you take for example, a vertex here and a vertex there, they are zero-dimensional but the join F1 union F2 that will become 1-dimensional 0 plus 0 plus 1. So, each simplex wise you can verify this identity.

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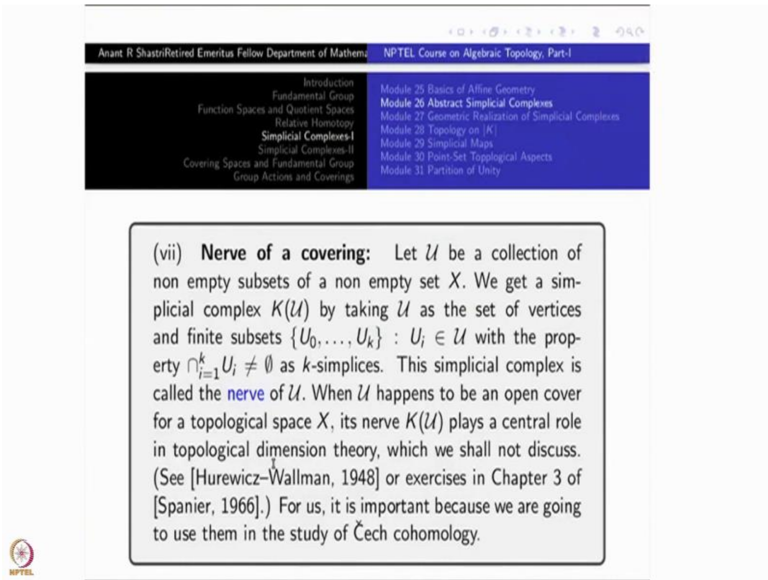

There is one interesting example here. Which is of importance in many other kinds of mathematics, not just within simplicial complexes. So, within topology of course, this is called the Nerve of a covering. As the name says, it has something to do with coverings of topological spaces. A little more generally, let $\mathcal{U}$ be a non empty collection of non-empty subsets, of a non-empty set X, let us take everything non empty.

So, $\mathcal{U}$ has subsets of X, each of them is non empty and $\mathcal{U}$ has at least one element in it, one member. You get a simplicial complex $K(\mathcal{U})$ by taking $\mathcal{U}$ as the set of vertices, these curly U becomes the set of vertices and all finite sequences $\{U_1, \dots U_k\}$, where each $U_i \in \mathcal{U}$, not all of them, but with the property that their intersection is non empty, $\cap_{i=1}^k U_i \neq \emptyset$. So, if you take the empty set here, you would have been in trouble that is why you would like to have first of all, each U_i non-empty then the intersection must be also non empty that is the condition, then only you will declare this as a simplex.

For example, you take members U_1, U_2. They are vertices, will there be an edge betwen them provided $U_1 \cap U_2 \neq \emptyset$. If it is empty then you will not put any edge there. So, if you take a sub collection here that will be also non empty. Therefore, I mean sub collection intersection will non-empty. Therefore, these automatically a simplicial complex. This simplicial complex is called Nerve of $\mathcal{U}$.

When $\mathcal{U}$ happens to be an open cover for a topological space X, it becomes important. (This is defined for any collection of X.) Now, suppose $\mathcal{U}$ is an open cover for a topological space X. Its nerve $K(\mathcal{U})$ plays a central role in topological dimension theory. The dimension of a nerve of a covering is what is used there. Then, ultimately, for an appropriate covering the dimension of the nerve of the covering will become the dimension of the manifold itself. So, this is there in dimension theory, I cannot explain it anymore, if you want, you can see this in the very fantastic book, [Hurewicz-Wallman], I have also included some exercises from Chapter 3 of Spanier, you can have a look at them also.

For us, there is another important thing which can be used, but again it is not in this course, namely in the study of Cech cohomology of a space, what is called Cech cohomology. There also it is used. So, this example is very useful, but, in this course, we are not going to cover this. Similarly, the simplicial complexes have been used in many other branches of mathematics-- especially in combinatorial algebra and combinatorics itself. And as I have told you, the computer scientists use it quite a bit. So, I will give you one more example which is somewhat dual to this construction, but this is more combinatorial. There is no topology in the construction of $K := K(k,n)$.

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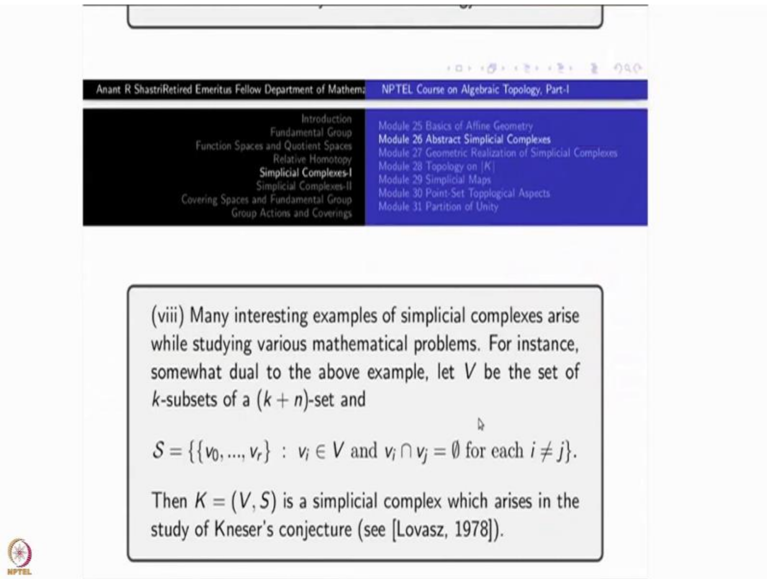

You start with V to be the set of k-subsets of a (k+n)-set X. To start with any set X, set is fixed set, but now, the set of vertices for K consists of all k-subsets of X. X should have something more than k elements no? So, let us say it has (k+n) elements. Then you take all k- subsets of X. For example $k+n=2+1=3$, $X=\{1,2,3\}$ and then you are only taking all the 2-subsets $\{1,2\},\{1,3\},\{2,3\}$. So if $k+n=3+7=10$ we are taking say k equal to 3 and k+n is 10, $X=\{1,2\ldots,10\}$ then what we are taking?$\{1,2,3\},\{2,3,4\}\cdots$ like that all the 3-subsets you are taking.

So, that is the set of vertices of this new simplicial complex that I am going to define. Then what are the simplices? Simplices will be subsets $\{V_0, V_1, \ldots, V_r\}$, where Vi's are inside V, (each V_i is a k-subset of X now, remember that), we will take r plus 1 of them with the condition that their pairwise intersection is empty. (for r=0 there is no condition.) If you have taken $V_0 = \{1,2,3\}$ then you cannot take $\{1,2,4\}$ or $\{3,4,5\}$ together with it.

So, that is why I am told to that this is somewhat dual to the previous example. There, the two subsets must intersect. Here, we are taking $V_i \cap V_j = \emptyset, i \neq j.$ Pairwise disjoint, k-subsets of a (k+n)-set. If you take a subfamily here that will also satisfy the same property therefore, it will be included. Therefore, K is a simplicial complex.

So, this thing arises in Kneser's conjecture in combinatorics. And if you are interested in that kind of mathematics, you may se this Lovasz paper, it is a very fine paper, you can read that, 1978. So let me go to now to geometric realization.

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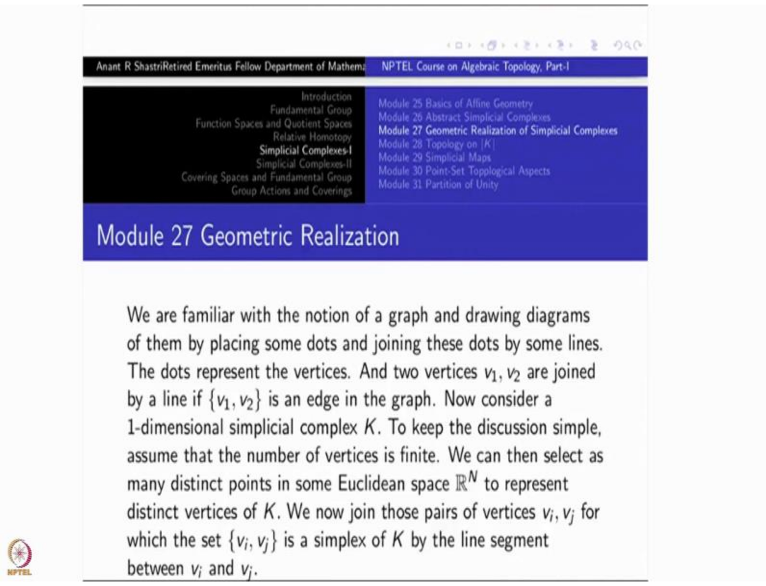

This begins with the basic idea of graphs. How do you do graphs? But you have to be bit careful with whatever you have learnt in graphs relearn, it here properly. So, graphs. You are familiar in drawing them on a piece of paper. When you do that, you get a subspace of $\mathbb{R}^2$. In graph theory, the topology of subspaces is not used so much. It is the combinatorial aspect, namely, whether a vertex is incident at an edge and what edge is used to join two of the verices--- that kind of the interrelation between edge and vertices is given a lot of importance there.

But you get a topological space there, namely, start with two vertices v_1 and v_2, join by a line, if $\{v_1, v_2\}$ is an edge in the graph, that is what you do.

Now, considered as a 1-dimensional simplicial complex, 1-dimensional means what? there are vertices and there are edges. So, 1-dimensional simplices are there and nothing more than that, we just have to concentrate on that. To keep the discussion simple, assume that it is a finite also, namely number of vertices is finite. You can then select as many distinct points of the Euclidean space, (try to do this in $\mathbb{R}^2$, may not be possible,) let us say, representing distinct vertices of the simplicial complex K, we can now join those pairs of vertices for which $\{v_i, v_j\}$ is an edge, is a 1-face of this K, that means $\{v_i, v_j\}$ should appear inside S, then only we will join them.

The only thing that we want to ensure is that we would like to have an independent status for the entire edge $\{v_i, v_j\}$. Suppose, have another egde you $\{v_k, v_l\}$, then these two edges may intersect in the interior of one of those points. We do not want that, that is not allowed. even when you are trying to draw a graph. So, that poses a problem. For not all graphs are embeddable in $\mathbb{R}^2$. You see, you may be knoowing such a result.

However, you can go to $\mathbb{R}^3$ or $\mathbb{R}^4$. Then it is possible to choose points such that when we draw edges, they do not intersect, the line segments should not intersect. When you complete all these line segments what you get is a geometric object, a subspace of $\mathbb{R}^3$, so there is a topology on that, so this is just a heuristic idea. We want to make this one more rigorous. So, let us see how, let us go step by step.

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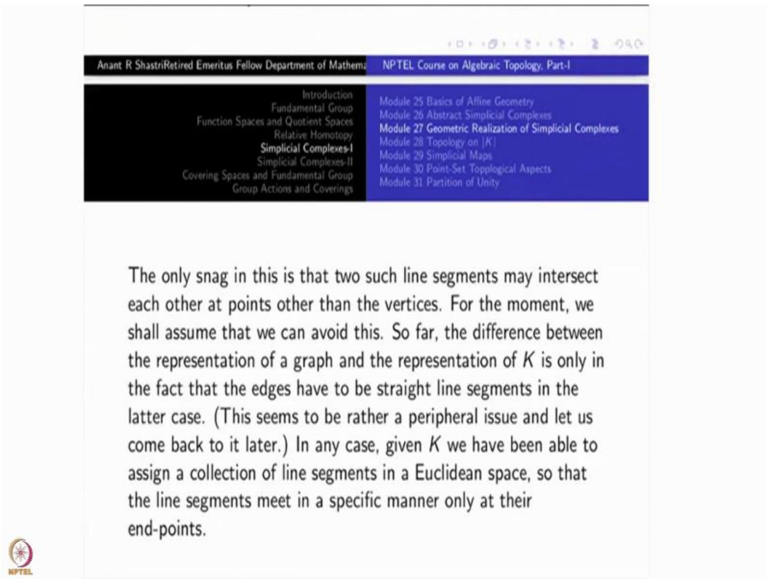

The only snag that I told you is that two-line segments, if we are chosen vertices arbitrarily, they may intersect, if they intersect only at the vertices then there is no problem, but if they intersect in between that point of intersection should be also declared as a vertex to be fair to graph theories, but it is not a vertex in our simplicial complex, there is no such vertex. That is why we do not want them to be intersecting.

So, difference between representing graph and representation of simplicial complex is only in that fact that we have used only straight lines inside $\mathbb{R}^n$ for edges. So, this is not such a great issue, if they intersect you can slightly bend an edge if necessary so that they do not intersect and get away with that. Namely, take a curve instead of a straight line. (But inside $\mathbb{R}^2$, even if you take a curve it may not be possible. For example, you know that the complete graph K_5 on five vertices cannot be drawn inside $\mathbb{R}^2$.)

So, that problem will still be there. So, in any case, given a finite $K,$ we have been able to assign a collection of segments in Euclidean space by increasing the dimension of the Eulcidean spaces, say maybe go to $\mathbb{R}^n,$ n large enough. After all, I have started a with a finite complex K. Suppose, the vertex set has n elements, then I can definitely do it in $\mathbb{R}^{n+1}$ absolutely no problem. Indeed in $\mathbb{R}^n$ itself we can do.

You have done it already. Namely, I can take the standard the standard n-simplex. What is it? The standard basis elements $e_0, e_1, \cdots$ so on, n of them can be taken. Then every simplicial complex with these as vertices is a subcompelx of the standard n-simplex. So, this brings back, it brings us back immediately to the geometric n-simplex that we have introduced.

In a sense, whatever I have told heuristically has already solved this problem, at least for the case when K is finite. So, you learn to do it, independent of choice of elements and so on, some kind of universality should be brought in. So, let us go again slowly a little bit.

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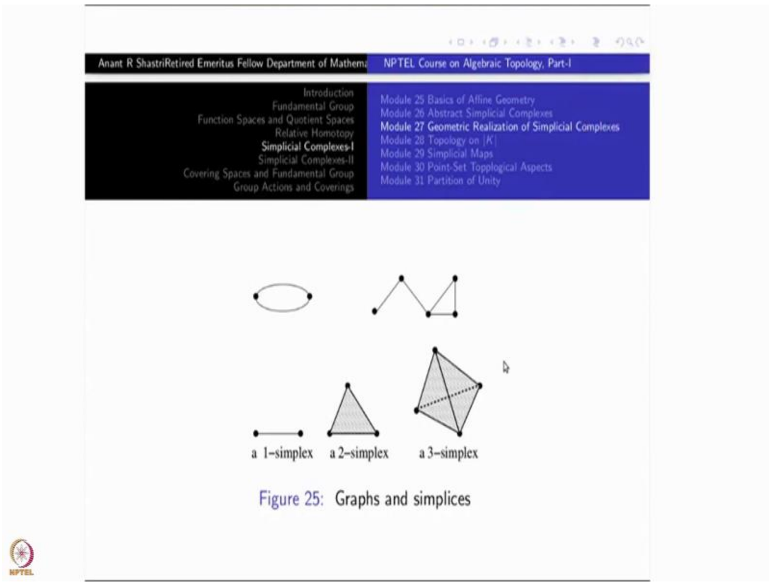

So, here are some examples of what is the representation of a graph and simplex and so on. The first example here is an example of a graph, but it is not a simplicial complex, why? If it is a simplicial complex, these two round edges must have been there, but given two vertices you can have only one single edge there viz., $\{v_1, v_2\}$. This is not a graph, this is a graph fine for some people, but this is not a simplicial complex.

The next one is a simplicial complex, you say 1-dimensional simplicial complex with 3 plus 2, 5 vertices 1, 2, 3, 4, 5, edges. This is a 1-simplex and that is a 2-simplex and this is the 3-simplex, that is a tetrahedron. So, in general a simplicial complex will be built out of this, this, this and more higher dimensional things and so on.

We want to define, we have already defined abstract simplicial complex independent of any pictures or any geometry, but we want to come back to the geometry and this process of coming back there must be some canonicals, there will not be any ambiguity, there should not be any choice, you choose something, he choose something and so on, that is the kind of thing we do not want.

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So, let us see how, what are the problems. If K has a 2-dimensional simplices also, say like v $\{v_1, v_2, v_3\}$ is such a simplex. We then take care that the three vertices are not chosen on a straight line. So, this then enables us to fill up the triangle formed by three vertices, we do this to all 2-simplices. But again, the 2-simplices, several 2-simplices should not intersect at all, that also has to be ensured, we have to ensure that two distinct triangles do not overlap.

This gives us a subspace of Euclidean space which is the union of number of triangles, lines etcetera. So, that is what I have already shown you here.

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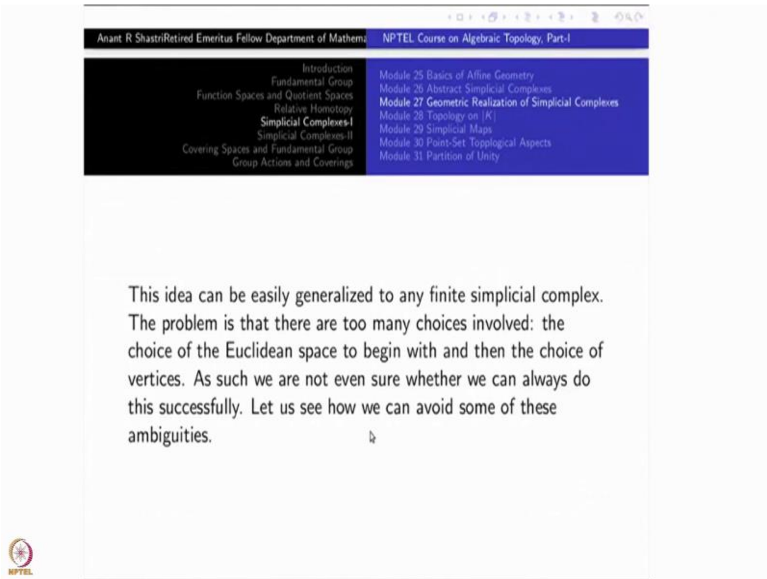

So far whatever idea you have, it is possible to avoid some of these ambiguities, but there will be some choice. After all, why you have chosen your definition you may say, but my definition has been chosen so that it brings a unification amongst all such ideas.

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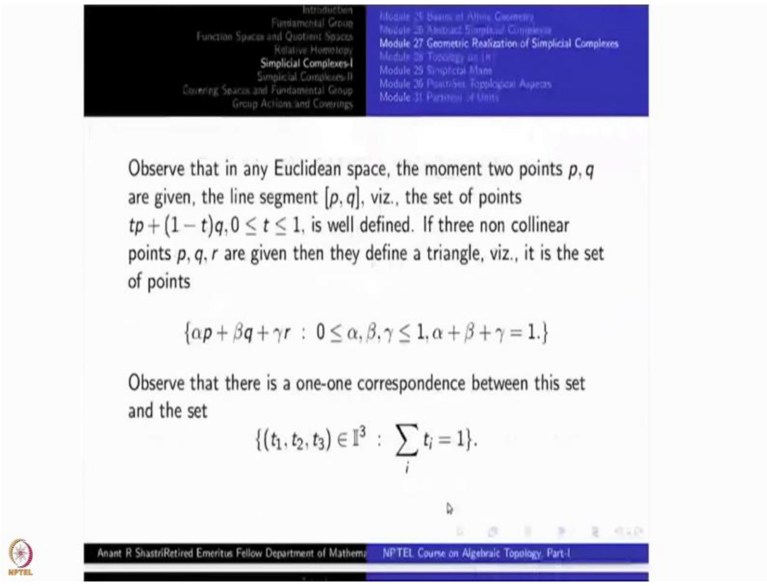

We go back to the affine structure of $\mathbb{R}^n$. In Euclidean space, you forget about the origin. Just think of this as the geometric object. Moment two points are given the line segment is well defined. It is consisting of t times p plus 1 minus t times q, the convex hull of these two points. If three non-collinear points are given, they define a triangle. You can say three points always define a triangle. But if they are collinear, it will be a degenerate triangle, I do not want that. So, what is the set of all points inside triangle? Alpha p plus beta q plus gamma r, alpha beta gamma are between 0 and 1 and sum total must be 1.

So, I am just recalling whatever you did in affine geometry. Observe that there is a one-one correspondence between the set of points t1, t2, and t3 inside $\mathbb{I}^3$ because these are all elements of $\mathbb{I} = [0, 1]$ and so three of them will be in $\mathbb{I}^3$, but also summation $t_1 + t_2 + t_3 = 1$. So, this itself is our standard 2-simplex, the join of e1, e2, and e3, the convex of e1, e2, and e3.

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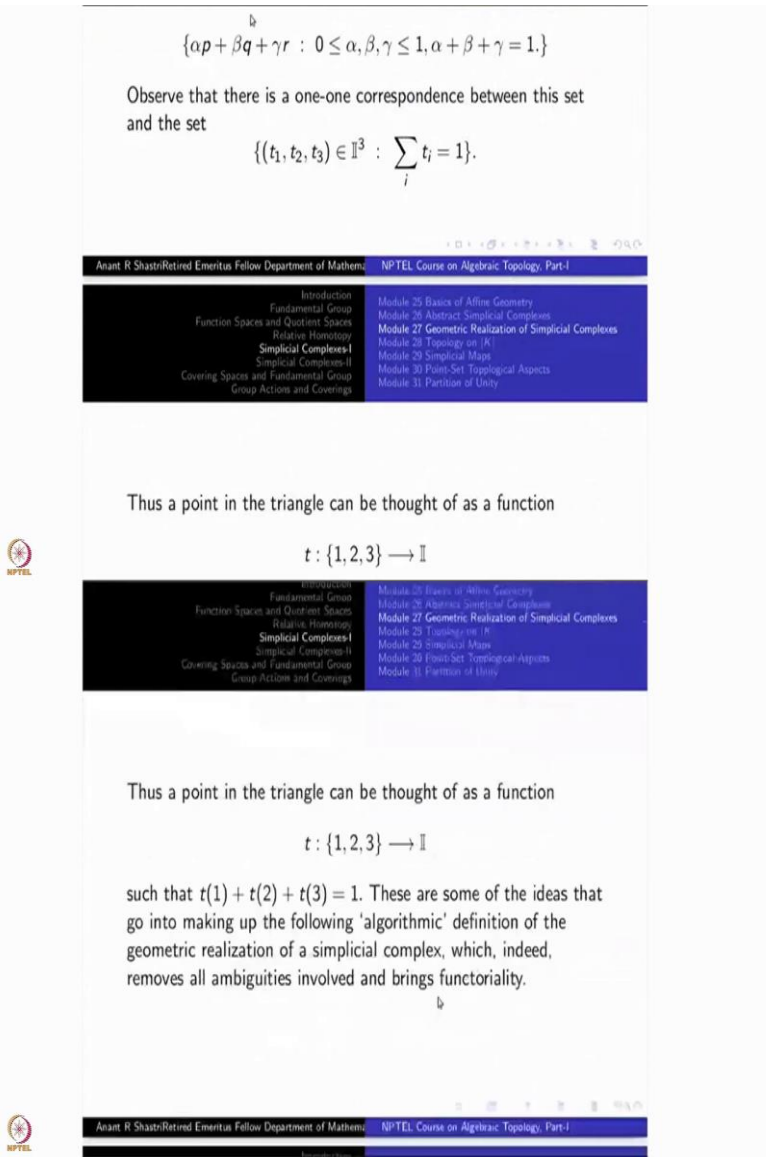

Thus, a point in the triangle can be thought of as a function now, this t function the t1, t2, t3, there are three coordinates of the function. So, it is a function from $\{1, 2, 3\} \to \mathbb{I}$. So, t is a function 1, 2, 3, into I, t1, t2, t3 combine it with the vertex is here p, q, r, or v1, v2, v3 you get a point of the simplicial complex the triangle here, t1 plus t2 plus t3 must be equal to 1 that is the condition, it is not any function. These are some of the ideas which go back to algorithmic definition of geometrical realization which will bring, remove all ambiguities in all and bring some to reality.

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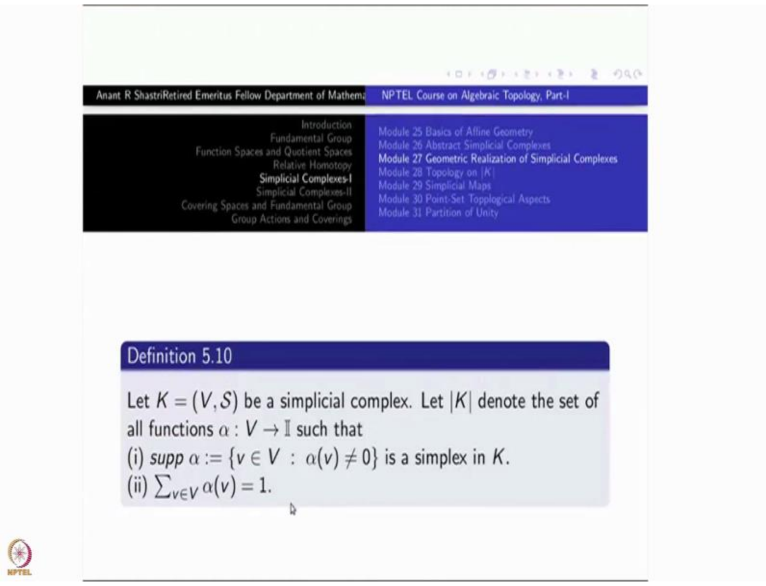

So, I have given you enough motivation for this definition. So, let me just give you a definition, K is a simplicial complex V is a set of vertices and S is the set of simplices then $|K|$, will denote the set of all functions on the vertex set taking values in the interval [0, 1] such that this notation, support alpha means what? All points of $v \in V$ wherein $\alpha(v) \neq 0$. Some points we may go to 0 under α.

So, this itself is a subset of V; it should be a simplex in K. That means, this set, the support of alpha must be inside S. This is a condition on alpha, I am not taking all functions, this must be one condition. The second condition is that the sum total of all $\alpha(v)$ be must be 1 . So, as observed previously, the support must be a simplex, the sum total must be 1.

Support is a simplex just means that support is finite, I am not assuming V is finite now, the support is a simplex automatically implies this is a finite set when you take the sum total is a finite sum because for all other $v \in V, \alpha(v) = 0$. If it is not 0, it will be inside this one, the support of alpha.

So, second condition makes sense because this left-hand side is a finite sum. It will be equal to 1, this is the condition. Look at all these, that set is denote by $|K|$. So, one obvious thing is that this is a subset of $\mathbb{I}^V$, satisfying this condition, so it is a close subset of $\mathbb{I}^V$. $\mathbb{I}^V$ I raised to V means what?

Sagnik Biswas: Hello sir.

Professor: $\mathbb{I} \times \mathbb{I} \times \cdots$V times.

Sagnik Biswas: Hello sir.

Professor: Yeah?

Sagnik Biswas: Yeah, these alpha, are they supposed to be some kind of coordinate function type thing?

Professor: I already told you that,

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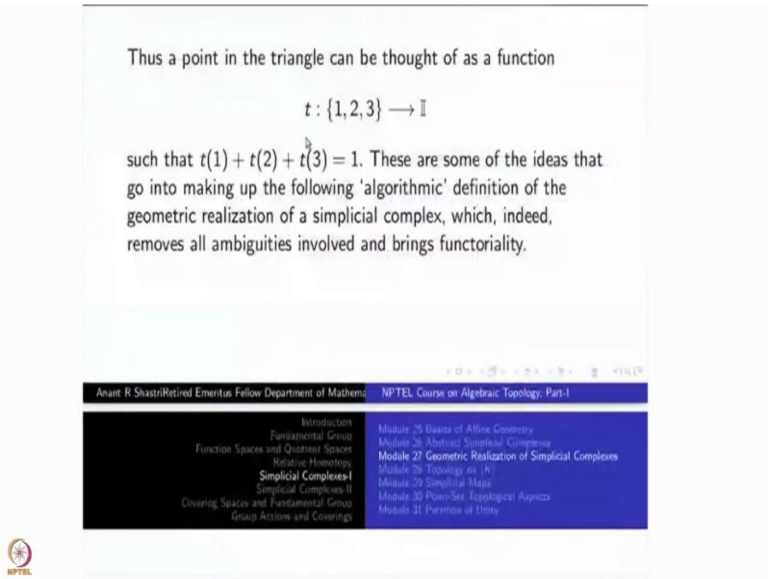

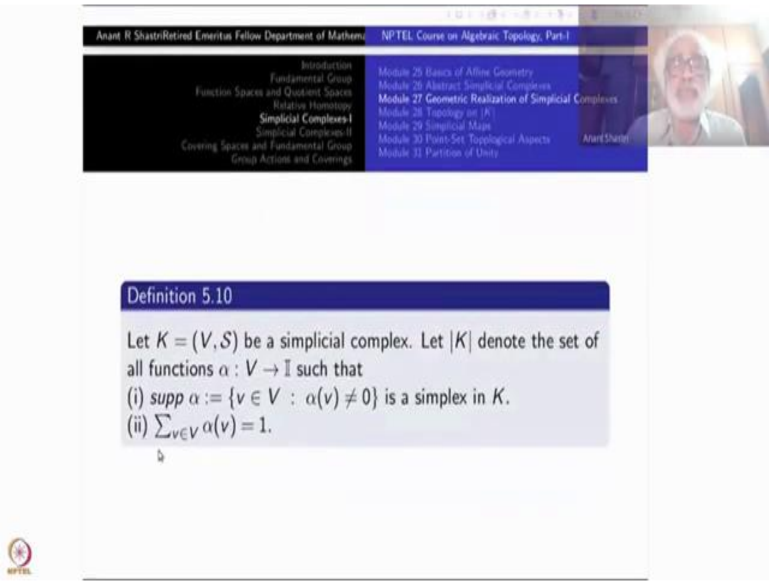

I have already told you here t1, t2, t3, right?

Sagnik Biswas: Okay.

Professor: Any function from V to I is an element of $\mathbb{I} \times \mathbb{I} \times \cdots$V times.

Sagnik Biswas: Okay.

Professor: So, you should know that product is same thing as functions. Cartesian product is the same thing as functions where on the indexing set, what is the value of the function? It is the i-{th} coordinate.

Sagnik Biswas: Yes.

Professor: Okay. So, this elementary point set, I think you know this much point-set-topology okay? Here.

Sagnik Biswas: Okay.

Professor: I have an extra condition. The coordinates must sum-up to 1 and each point has finitely many coordinate, only finitely coordinates are not 0, so that is the meaning.

Sagnik Biswas: Okay.

Professor: Okay?

Sagnik Biswas: Okay.

Professor: So, this is the way we are looking at this now, a simplex, a triangle, an edge, an edge is a function, elements of an edge is a function taking values defined on the vertex set, taking values inside 0 1, sum total must be 1. So, that is the meaning of points between this edge. So, this way we are not using any kind of picture, but we are still using the algebra and topology and everything of the interval is [0, 1], those things are still there. You should understand that.

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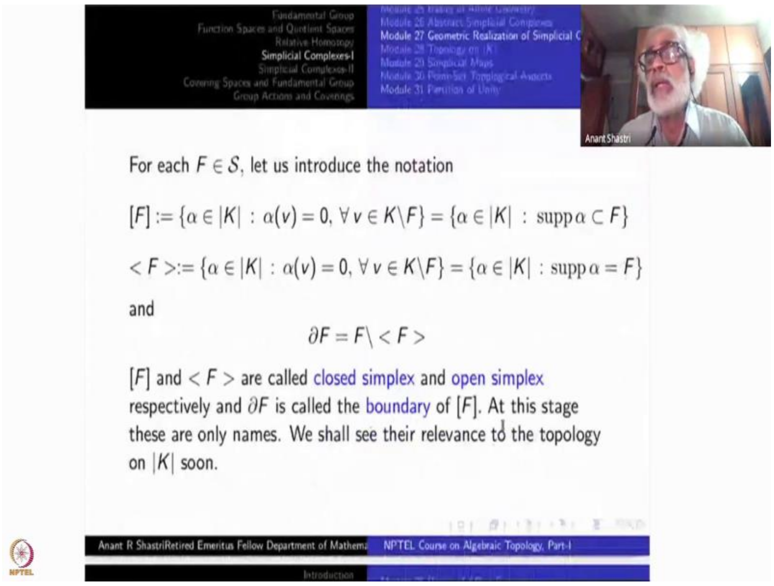

For each face F,a face of the simplex, let us introduce these two notations which will be used again and again, the mod F , $|F|$ or bracket F, $[F]$ (I have put this temporarily, just for the sake of clarity, later on, I will make it mod F, just not to confuse with this mod K.) So,

$[F] =$ set of all α belonging to $|K|$ such that $\alpha(v)$ is zero for every v outside F. That is mod F. That means, the support of α must be contained in F, may not be equal to all of F.

If it is equal to all of F, then this is the open simplex F, $\langle F \rangle$. This is the definition, One is open F, the other is closed F. So, $\langle F \rangle =$ the set of α in $|K|$, so that $\alpha(v) \neq 0$ iff $v \in F$. There is some problem here in the slide- the first part is not correct. Support of alpha equal to F. That is correct. Something here, the support of alpha equal to F this is correct.

And another expression for the boundary of F; it just means that, bracket F minus open F, it means those which are contained in here but at least one of the things must be 0 already that means it is proper, so only those things are boundary of F. For every F it must be non-zero is this one, maybe at least one of them extra it maybe 0, other than V minus F. ∂F is the set of all $\alpha \in [F]$ such that α vanishes on at least one of the vertices of F also. Such things are in the boundary of F. This $[F]$ and this $\langle F \rangle$ are called closed-simplex and an open-simplex respectively, corresponding to the abstract simplex F. We have a closed simplex and an open simplex respectively and ∂F is called the boundary of F. At this stage, these are the only names, we shall see the relevance of the names, closed, open, etcetera. When you see what is the topology which I want to take. So, I will stop here and continue the next module.