

Introduction to Algebraic Topology (Part-I)
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Lecture 26
Abstract Simplicial Complex

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<ul style="list-style-type: none"> Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings 	<ul style="list-style-type: none"> Module 25 Basics of Affine Geometry Module 26 Abstract Simplicial Complexes Module 27 Geometric Realization of Simplicial Complexes Module 28 Topology on \mathbb{R}^k Module 29 Simplicial Maps Module 30 Point-Set Topological Aspects Module 31 Barycentric Subdivision Simplicial Approximation Sperner Lemma Invariance of Domain Links and Stars Miscellaneous Exercises to Chapter 5
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Module 26 Abstract Simplicial Complexes

We shall now take up the study of spaces which are built-up from simplices. We are all familiar with the concept of a polygonal curve in a Euclidean space.

One of the great advantages of such spaces is that, to a large extent, the topological study of these spaces becomes elementary combinatorics. This is reflected right in the abstract definition with which we begin. In this section, we shall introduce the concept of abstract simplicial complex. The reader should wait till the next section for the underlying geometric motivations.

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So, today's topic is directly abstract simplicial complex. Last time we introduced some Affine Geometry, let me before going to abstract simplicial complexes, one more or one or two more concepts from Affine geometry let us cover that and then come to the study of these abstract simplicial complexes.

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<ul style="list-style-type: none"> Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings 	<ul style="list-style-type: none"> Module 28 Topology on \mathbb{R}^k Module 29 Simplicial Maps Module 30 Point-Set Topological Aspects Module 31 Barycentric Subdivision Simplicial Approximation Sperner Lemma Invariance of Domain Links and Stars Miscellaneous Exercises to Chapter 5
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Definition 5.4

Let x_1, x_2, \dots, x_n be points in \mathbb{R}^d . By a **convex combination** of these points we mean a finite linear combination $\sum \lambda_i x_i$ where $\sum \lambda_i = 1$, and each $\lambda_i \geq 0$. The **convex hull** of a set M is the set of all convex combinations of points in M and is denoted by $\text{conv } M$. Observe that $\text{conv } M \subset \text{aff } M$. A subset A of \mathbb{R}^d is said to be **convex** if $\text{conv } A = A$.

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So, here is, this theorem was done, here is the definition. Convex combination of these points, n points inside \mathbb{R}^d it is an Affine combination $\sum \lambda_i x_i$ where $\sum \lambda_i = 1$, and one more condition, each λ_i must be positive or non negative also, automatically each λ_i will be less than or equal to 1 because sum total is 1. Such a sum is called convex combination.

Just like $0 \leq t \leq 1$, and I will be taking t times p plus $1 - t$ times q . There it is convex combination of only two elements. But here you could take any finite set and you can talk about convex combination. The convex hull is the set of all convex combinations. Take any set M . Then you can take convex hull of that. It is called $\text{conv}(M)$ that is a notation, it is automatically a subset of $\text{aff}(M)$, Affine M . Here, all linear combinations are taken without the condition $\sum \lambda_i = 1$.

So, a subset A of \mathbb{R}^d is said to be convex, if convex hull of A itself is equal to A . This is similar to saying that a set is closed if it is equal to its closure. What is the meaning of that? If you take any convex combination of finite many points of A , it is again inside A . This is similar to the property of $\text{aff } M$.

So, a line segment is a convex set not an affine subspace. Similarly, the inside of a triangle, full, not just the boundary triangle, inside of a triangle is a convex set, Tetrahedron is convex set, these are our standard convex sets. There are many more. Even a square is a convex set. So, see that we are concentrating on a particular kind of convex sets soon.

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The slide contains a table of contents for the NPTEL course on Algebraic Topology, Part-I. The table lists modules from 25 to 31, including topics like Affine Geometry, Simplicial Complexes, Topology on \mathbb{R}^n , and Simplicial Approximation. Below the table of contents is Definition 5.5, which defines a geometric n -simplex A as the convex hull of any $n+1$ affinely independent points $\{v_1, \dots, v_{n+1}\}$ in \mathbb{R}^d . The vertices v_i are called the vertices of the simplex, and n is the dimension of A . Examples given are: a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron, etc.

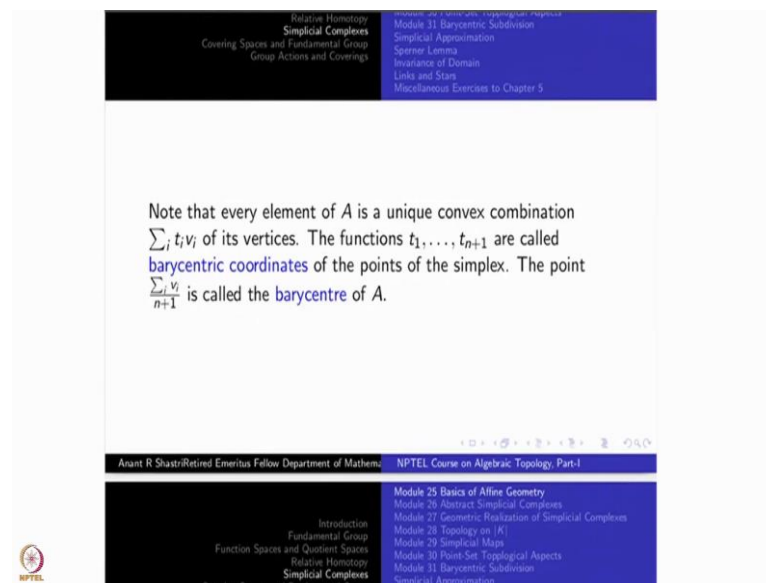
Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings	Module 25 Basics of Affine Geometry Module 26 Abstract Simplicial Complexes Module 27 Geometric Realization of Simplicial Complexes Module 28 Topology on \mathbb{R}^n Module 29 Simplicial Maps Module 30 Point-Set Topological Aspects Module 31 Barycentric Subdivision Simplicial Approximation Sperner Lemma Invariance of Domain Links and Stars Miscellaneous Exercises to Chapter 5
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Definition 5.5
 By a geometric n -simplex A , we mean the convex hull of any $n+1$ affinely independent points $\{v_1, \dots, v_{n+1}\}$ in \mathbb{R}^d . The elements v_i are called the vertices of the simplex. The number n is called the dimension of A .
 Thus a 0-simplex is nothing but a point and 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc.

So, this is what is called a geometric n-simplex, geometric n-simplex by the very name you may anticipate that there is going to be something n-simplex also without the tag 'geometric'. So, what is geometric n-simplex? It means convex hull of any n plus 1 affinely independent points v_0, v_1, \dots, v_n in \mathbb{R}^d .

The elements of v_i are called the vertices of the simplex. The number n is called the dimension of this A. The set A is the convex hull of v_0, v_1, \dots, v_n . The points v_0, v_1, \dots, v_n must be affinely independent. Then we call A a geometric n-simplex. A 0-simplex is nothing but a single point. Take one single point; what is the affine combination? λx where summation λ must be 1 now, so it is just $\lambda = 1$, so it is the singleton set $\{x\}$. So, its convex combination is also a single point. A 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is tetrahedron, these are a few names, afterwards we do not know what to say, the 4-simplex I do not know what name I should give. So, we just say a 4-simplex.

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Note that every element of A is a unique combination of these vertices, unique convex combination, the uniqueness follows because of affine independence of v_0, v_1, \dots, v_n .

If $\sum_i t_i v_i = \sum_i s_i v_i$ in two different ways then what you will get is $\sum_i (s_i - t_i) v_i = 0$

with at least one of the $s_i - t_i$ will not be equal to 0, and $\sum_i (s_i - t_i) = \sum_i s_i - \sum_i t_i = 1 - 1 = 0$.

So, that will give you that v_i 's are not affinely independent. Therefore, uniqueness follows. This is exactly similar to whenever you have a linearly independent set then every element in the linear span of these sets is a unique linear

combination of the n elements there, that is called a base is there, here these v_i 's are called vertices.

$$\sum_i v_i$$

There is one very special point by symmetry, what is it? It is $\frac{\sum_i v_i}{n+1}$, the summation v_i divided by $n+1$, there are $n+1$ points here. So, I am dividing by $n+1$ after taking summation, this is a convex combination and this point is called the barycentre of A , you may call it centroid also, there are lots of centres in plain geometry, circumcentre, incentre and so on various centres. Here we have just the barycentre. This barycentre is called, some people call it, centroid also, no problem.

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Theorem 5.2
Let A, B be any two geometric simplices. There exists an affine isomorphism $f : A \rightarrow B$ such that $f(A) = B$ iff $\dim A = \dim B$.

Proof: Suppose $\dim A = \dim B$. Choose some labelling $A = \{a_1, \dots, a_k\}, B = \{b_1, \dots, b_k\}$ where $k = \dim A + 1$. Take $f(a_i) = b_i$ and extend linearly. Conversely, if $f : A \rightarrow B$ is an affine isomorphism, then $\{f(a_i)\}$ are affinely independent in B and $\{f^{-1}(b_i)\}$ are affinely independent in A . Hence the dimensions must be the same.

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Here is an easy theorem, let A and B be any two geometry n -simplices. There exists an affine isomorphism of A to B , such that fA equal to B and A and B are isomorphic in that sense affine isomorphism. You put one condtion: dimension of A equal to dimension of B , just the dimension will tell you that they are affinely isomorphic.

So, this is the starting point, maybe you can say that the topology here is reduced to combinatorics, to finite geometry. So, this is the starting point of that, you are receiving more and more of this kind. Let us see, how to prove this. Suppose, dimensions are same. That means number of vertices are the same. So choose some labelling a_1, \dots, a_k and b_1, \dots, b_k . Here k is dimension of A plus 1, one more that the dimension, that is what we have seen the dimension by definition.

Take $f(a_i) = b_i$. It is one one correspondence. (Also, you could choose any other one-to-one correspondence.) Extend it linearly over $\text{conv}(A)$. Extend it linearly means what now? affine

linearly, $f\left(\sum_i \lambda_i a_i\right) = \sum_i \lambda_i b_i$, that is all. Just like in linear algebra. Once you have defined it over basis elements, it is uniquely defined. Exactly as in linear algebra, here also it is

uniquely defined because each $\sum_i \lambda_i a_i$ has unique expression, so that will give you an isomorphism.

Conversely, if f from A to B is an isomorphism, then a_1, \dots, a_k are affinely independent, implies f of a_i should be affinely independent. Similarly, if b_1, \dots, b_l are affinely independent then $f^{-1}(b_i)$'s are affinely independent. Therefore, definitely k and l must be same and hence dimension of A is equal to dimension of B . So, this is what we wanted to do in our meeting last time, we could not do it then, so I have done it now.

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Module 26 Abstract Simplicial Complexes

We shall now take up the study of spaces which are built-up from simplices. We are all familiar with the concept of a polygonal curve in a Euclidean space.

One of the great advantages of such spaces is that, to a large extent, the topological study of these spaces becomes elementary combinatorics. This is reflected right in the abstract definition with which we begin. In this section, we shall introduce the concept of abstract simplicial complex. The reader should wait till the next section for the underlying geometric motivations.

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Now, we will directly take up the study of abstract simplicial complexes which does not look like any topology at all, you can see now. So, you have to hold your horses for a while to see the topology behind them.

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The slide is a presentation slide from an NPTEL course. At the top, there is a navigation bar with a table of contents. The current slide is titled 'Definition 5.6'. The definition states: 'By a simplicial complex K we mean a pair (V, S) , where V is a set and S is a collection of finite subsets of V such that (i) $\forall v \in V, \{v\} \in S$; (ii) $F \in S$, and $F' \subset F \implies F' \in S$.' Below the definition, there is a mouse cursor. At the bottom, there is another navigation bar with a table of contents.

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Definition 5.6

By a **simplicial complex** K we mean a pair (V, S) , where V is a set and S is a collection of finite subsets of V such that

- (i) $\forall v \in V, \{v\} \in S$;
- (ii) $F \in S$, and $F' \subset F \implies F' \in S$.

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	Module 28: Topology on \mathbb{R}^n
	Module 29: Simplicial Maps

So, I have decided to direct start with the definition of abstract simplicial complex. It will consist of a set denoted by V and called as vertex set and another set denoted by S called the set of simplices, and is a collection of finite subsets of V . I will write down this one later on. S is a collection of finite subsets of V , it satisfies two more conditions, namely: (i) for every $v \in V$, $\{v\}$, the singleton is inside S .

Remember, elements of S are subsets of V . So, I should not write v inside of S but I should write $\{v\}$, singleton v inside S . (ii) Whenever, F is a subset belonging to S , all subsets of F will be also inside S . In other words, S is close undertaking subsets. So, these are the only two conditions that will make a simplicial complex. So, let us go on to some more definitions and properties.

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Elements of V are called vertices of K , those of S are called simplices of K . If F is inside S and number of points, namely cardinality of F is equal to q plus 1, then F is called a q -simplex and you can also say dimension of F is q . So, this is a definition of dimension, get the number of points minus 1.

Thus, the vertices of K are also 0-simplices of K . You have to take your vertex v , put it inside a bracket to make it a singleton $v, \{v\}$, it will become a 0-simplex. So, soon we will not keep doing this. We will have this liberty of not writing the bracket that is what, like we write sometimes $V \setminus 0$ to mean $V \setminus \{0\}$. Some set theoretic notation takes that kind of liberty, so only that liberty is there that is all. Thus, vertices of K are 0-simplices.

So, as a vertex, it an element of V but as a 0-simplex, it is a singleton $v, \{v\}$ which is inside S . That is the difference between logically. But you can say that, a vertex is also 0-face.

So, $F' \subset F$ and F itself is an n -face, then we say that F' is a face of F . Here it may be equality also. For example, if you take one vertex here, a singleton that singleton v will be a face.

So, all subsets of a simplex are called faces. Often a simplex is displayed by enumerating its 0-faces, enumerating 0-faces means what? Just writing down the set $\{v_0, \dots, v_n\}$, what are the elements of that set, just write down. Automatically all the subsets are there is there you have to think about it, that is all. This is just like writing when you have topology you just write down some basic open sets only then the topology is understood.

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The image shows a video lecture slide. At the top, a navigation bar lists: Group Actions and Coverings, Invariance of Domain, Links and Stars, and Miscellaneous Exercises in Chapter 5. The main content area contains the text: "Observe that we have allowed the empty subset also as a simplex in every simplicial complex. This is rather unusual in topology, but quite a convenient convention in a combinatorial set-up. We define the dimension of the empty face to be -1 . If V is a finite set, then K is called a finite simplicial complex." Below the text is a mouse cursor. At the bottom, a footer bar identifies the presenter as Anant R Shastri, Retired Emeritus Fellow, Department of Mathematics, and the course as NPTEL Course on Algebraic Topology, Part-I. A table of contents is visible at the bottom right, listing modules from 25 to 31, with Module 26, Abstract Simplicial Complexes, highlighted in blue. The presenter's video feed is in the top right corner.

Observe that we have allowed the empty subset also as a simplex. I never said F is not empty. Empty set is also allowed. What is the dimension of an empty set? There are no elements there, so cardinality is 0. Then I have subtracted one. So, the dimension is, by definition, -1 . Finally, whenever the vertex set V itself is finite then automatically S will be also finite because it is only a collection of finite subsets of a finite set V .

Therefore, that will also finite. Then K itself is called a finite simplicial complex, vertex set must be finite. So, these are just some names. I will keep reminding you again and again, so soon you will get used to it.

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	Manifolds and Manifolds
	Manifolds and Manifolds

The **dimension**, $\dim K$, of a simplicial complex is defined to be equal to the supremum of n such that K has a n -simplex. Thus, if K is finite then $\dim K < \infty$. A simplex in K is said to be **maximal** if it is not contained in another larger simplex. If the dimension of K is $n < \infty$ then there are simplices in K of dimension n and all simplices of dimension n are maximal. However, not all maximal simplices need be of dimension n . If this happens such a simplicial K is called a **pure** simplicial complex. This concept is very useful in combinatorial algebra.

Now, I can assign a dimension to the whole of K , K is a simplicial complex. So, what is this dimension? It is the supremum of all n where n is what, some simplex or dimension of some simplex indicated. Suppose, there is a simplex of dimension n then dimension of K will be at least n , you have taken the supremum. This supremum could be infinite. If there is a simplex of dimension n for every n , then dimension of K will be infinite. It is finite means, say k , what that means, there is at least one simplex of that dimension and all other simplexes are of dimension less than or equal to that \dim .

Note that finiteness of dimension of K does not mean that K is finite, but if K itself is finite, then the dimension has to be automatically finite. If the dimension of K is a finite, say n , then there are simplices of dimension n and all simplices of dimension are maximal. Maximal means what? You cannot have another larger simplex containing that. For then the dimension of K would have increased.

So, these simplices will be maximal which means all other simplices are subsets of simplices of dimension $k \leq n$. Not all maximal simplices need to be of the same dimension, remember, I can have a singleton 0 , singleton say 1 here and then a simplex somewhere both will be maximal, dimension will be 1 here, but the singleton 0 standing away that will be of dimension 0 .

So, whenever everything is contained in an n -simplex, such a simplex is called pure. These things are very very important in algebra and combinatorial mathematics, but for us this will not play much role. So, you can soon forget about this one, perhaps I am never going to use this word 'pure.'

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Definition 5.7

By a **simplicial map** $\varphi : K_1 \rightarrow K_2$ from one simplicial complex to another, we mean a set theoretic function on the vertex sets, $\varphi : V_1 \rightarrow V_2$, such that for each simplex F in K_1 , $\varphi(F)$ is a simplex in K_2 .

It is easily verified that the usual set-theoretic composition of two simplicial maps is again a simplicial map.

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Simplicial maps ϕ : you see, now, we have introduced the definition of simplicial complex. Now, we want to introduce the notion of what is the map between them, like topological spaces as a continuous map, if there are groups there is homomorphism and so on. So, you should have relations also what kind of functions are allowed here. So, take a Simplicial map from K_1 to K_2 . K_1 means what. So, K_1 itself has a structure: K_1 means it has a V_1 , a vertex set and S_1 , a set of simplices.

So, the function ϕ corresponds to actually a function from vertex set V_1 to vertex set V_2 with an additional hypothesis that for each simplex F inside K_1 , $\phi(F)$ makes sense, because ϕ is a function on V_1 to V_2 , $\phi(F)$ makes sense, this is a subset of V_2 , it must be inside K_2 means what? it is the simplex in K_2 , it must be inside S_2 , F belongs to S_1 means $\phi(F)$ belongs to S_2 , so that is the condition.

Image of every simplex must be a simplex, it need not be of the same dimension, number of elements in F maybe say 5, number of elements in $\phi(F)$ maybe 4, it cannot be more than 5. We know that, we know, it may be 4, it may be 3, it maybe 2 also, but ϕ being function its image will not be empty. If F itself is empty, $\phi(F)$ empty, it will be empty there is no problem.

So, usual set theoretic composition of functions ϕ from V_1 to V_2 , another one, ψ from V_2 to V_3 then $\psi \circ \phi$ makes sense. Automatically it will be simplicial because $\psi \circ \phi(F)$ it is simplex must be a simplex inside S_3 . So, composition of simplicial maps is a simplicial map, identity from K to K namely vertex to vertex is a simplicial map.

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The slide is a presentation slide with a dark blue header and footer. The header contains a table of contents with the following items: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, **Simplicial Complexes**, Covering Spaces and Fundamental Group, Group Actions and Coverings, Module 25: Basics of Affine Geometry, **Module 26: Abstract Simplicial Complexes**, Module 27: Geometric Realization of Simplicial Complexes, Module 28: Topology on \mathbb{R}^n , Module 29: Simplicial Maps, Module 30: Point-Set Topological Aspects, Module 31: Barycentric Subdivision, Simplicial Approximation, Sperner Lemma, Connectedness of Domains, Links and Stars, and Miscellaneous Exercises to Chapter 5. The main content area is white with a blue box for the definition. The footer contains the NPTEL logo, the name 'Anant R. Shastri (Retired Emeritus Fellow, Department of Mathematics)', and the course title 'NPTEL Course on Algebraic Topology, Part-I'.

Definition 5.8
By a **simplicial isomorphism** we mean a simplicial map with a simplicial inverse.

Note that a simplicial isomorphism is a bijection on the vertex set. However, the converse is not true, viz., a simplicial map which is a bijection on the vertex set need not be a simplicial isomorphism (exercise).

What is the meaning of a simplicial isomorphism? Simplicial isomorphism is first of all a bijection on the vertex sets, that means there is an inverse function on the vertex sets. The inverse must be also a simplicial map, a bijection of vertex sets need not be an isomorphism even if it is a simplicial map, the inverse may not be simplicial.

This is strangely typical of topology-- a continuous bijection may not be a homeomorphism, the inverse may not be continuous. Whereas, in linear algebra, if we have a linear bijection, the inverse is automatically linear. In group theory if you have a linear, if you have a group homomorphism which is a bijection, the inverse is automatically a group homomorphism. So, though this is like combinatorics which is like algebra, it is closer to topology in its nature, inverse has to be, you have to demand the inverse is also a simplicial map to be an isomorphism.

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The slide content is as follows:

<ul style="list-style-type: none"> Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings 	<ul style="list-style-type: none"> Module 26: Algebraic Topology, Part-I Module 27: Geometric Realization of Simplicial Complexes Module 28: Topology on \mathbb{R}^n Module 29: Simplicial Maps Module 30: Point-Set Topological Aspects
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Definition 5.9

By a subcomplex $K' = (V', S')$ of a simplicial complex K , denoted by $K' \subset K$, we mean a simplicial complex K' , such that $V' \subset V$ and $S' \subset S$.

Note that in this situation, the inclusion map $K' \subset K$ is a simplicial map.

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So some more terminologies. What is the meaning of a subcomplex $K' = (V', S')$ of a simplicial complex K ? we will just denote it by $K' \subset K$. So, what is the meaning of this? The vertex set V' must be a subset of V and S' must be a subset of S ; K' itself must be a simplicial complex on its own first of all and these inclusions must be valid, then only you call it a subcomplex.

What is the meaning of this? The vertex set V' must be a subset of the original set V . Whenever, you have a simplex inside K' , it must be simplex inside K also, that is the meaning of that. All the vertices of V' inside V . You may for example, take all the vertices of V equal to V' and allow some of the faces in S to be missing here, that is way of taking a subcomplex into subcomplex. Some of them are missing but, whatever you have taken they are inside S . You do not have to take, you do not take a new simplex, so that is a subcomplex.

Note that, in this situation the inclusion map $K' \subset K$ is a simplicial map because every simplex in K' is in K and the map is inclusion, ϕ is inclusion map here. So, I recall K' itself is a simplicial complex by simplicial complex, you know we need assert that K' is a simplicial complex.

What is the meaning of this? Conditions (i) and (ii) must be satisfied; number two is most important, whenever F belongs to K' and $F' \subset F$ then F' must be also in K' automatically. So, you cannot say that you keep one subset here and miss a subset of that set, that should not happen.

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	Links and Stars
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Examples

(i) Given a set V , let S be the set of all finite subsets of V . Then (V, S) is a simplicial complex. This simplicial complex contains as subcomplexes, all simplicial complexes whose vertex set is a subset of V . The isomorphism type of this simplicial complex depends only on the cardinality of V .

So, let us run through a few of the examples now, I may not be able to do all of them. Take any set V then take the collection of all finite subsets of that, S is all the collection of finite subsets, that is a simplicial complex. This corresponds to like we take, in topology what we do, we take a set and then take all subsets, it is the discrete topology. You are taking all subsets here which are finite of course because we are not allowed to take infinite subsets, all finite subsets we will take that is a simplicial complex.

This simplicial complex contains as a subcomplex, all the simplicial complexes whose vertex set is a subset of V . You cannot have anything better than that. That is why I told you that this is like a discrete space. In logic it is discrete. But you will see that in topology, later on, it is the opposite. The isomorphism type of this simplicial complex depends only on the cardinality of V . I just take any set W , if the cardinalities are the same, then taking all subsets here taking all subsets there, there will be a corresponding isomorphism. All that I have to do is write down a bijection of the vertices that is all. Take any bijection of vertices; it will be a simplicial isomorphism.

Vinay Sipani (Student): The dimension would be 1 plus cardinality of V , right?

Professor: Dimension of S of this K ?

Vinay Sipani (Student): Yes.

Professor: Dimension of this K could be infinite depends upon, what is V ? Suppose V is an infinite set. Then I can go on taking finite subsets larger and larger. If V is a finite set suppose there are only n elements. Then the dimension will be n minus 1. If V infinite, the dimension of this simplicial complex will be infinite.

Vinay Sipani (Student): Yes, sir.

Professor: Each simplex is finite, finite dimension, but dimension of K is becomes infinite.

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(ii) Consider the special case where V itself is finite with $n + 1$ elements. We get an important simplicial complex whose simplexes are all subsets of V . More specifically, if this set V consists of the $n + 1$ basic unit vectors $\{e_1, \dots, e_{n+1}\}$ in $\mathbb{R}^{n+1} \subset \mathbb{R}^\infty$, we then denote the corresponding simplicial complex by the symbol Δ_n . Recall that we had already used this symbol to denote the convex hull of these $n + 1$ points. This over-use of the same symbol is deliberate. These simplicial complexes are the building blocks of all other simplicial complexes.

Let us go to our second example. Consider a special case where V itself is finite that is what we are discussing with $n + 1$ elements. This is a very important one for us. Now, what am I taking, you are taking $n + 1$ element set, then you are taking all subsets, empty set, then all the singletons, then all the doubletons, all the tripletons and so on you are taking, okay? Yes or no?

Students: Yes, sir.

Professor: This is now characteristic of our what we are calling that standard n - simplex, what we do? Take the basic unit vectors $e_1, e_2 \dots e_{n+1}$ inside \mathbb{R}^{n+1} . We denote this corresponding simplicial complex by Δ_n . Now, I am giving you another specific case instead of arbitrary set V , I am taking V to be $\{e_1, \dots, e_{n+1}\}$. Then take all subsets of V . That simplicial complex we are denoting by Δ_n -- without the bars. Convex combination, remember convex hull of the set was denoted by $|\Delta_n|$. Soon you will understand what the difference between these two symbols. So, this overuse of the same symbol is deliberate because they are very closely related. So, you have to hold your horse till we come to the point--what is the difference between them, what I want to say is again I repeat take any simplicial complex with this property namely $n + 1$ point and then all subsets. They are all isomorphic to this Δ_n , because our previous thing here.

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(ii) Consider the special case where V itself is finite with $n + 1$ elements. We get an important simplicial complex whose simplices are all subsets of V . More specifically, if this set V consists of the $n + 1$ basic unit vectors $\{e_1, \dots, e_{n+1}\}$

This remark is applicable here also, take any bijection of vertex sets-- that will give you an isomorphism. So, up to isomorphism, the simplicial complex Δ_n is unique-- when you have fixed n .

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(iii) Any simplicial complex of finite dimension can be described by declaring all the maximal simplices in it, viz., take the collection of all subsets of all of these maximal simplices. This is especially effective while describing a finite simplicial complex; we need to merely list all its maximal simplices.

Any simplicial complex of finite dimension can describe by declaring all maximal simplices, if it is infinite dimension there may not any maximal simplex. So, that is it. You can do it only for finite dimension case. Look at what are all the maximal simplices, you will make a list of them, rest of them you can know. What are the other simplices? they are subsets of one of the members in this list. That is all.

In fact, all subsets of whatever list you have made, they must be there. So, this is an easy way of programming a computer for a simplicial complex, you just declare what is the rule for

simplicial complexes it understands then you give the list of maximum simplices, over. So, this is especially effective for describing finite simplicial complexes, we need to make a list of all maximal simplices.

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If F is a simplex, in a simplicial complex, then the set of all faces of F , just only take the faces of F including F itself, that itself would be a subcomplex, that subcomplex also we will denote by F . This is just like when you have a subset of a topological space X , then you write a subspace also by that same symbol, but what is a subspace? It consists of all those open sets inside X intersected with A , you intersect it at A that is the collection of open subsets of so far A , you do not keep on writing that.

So, there is like that denoted by F itself. The set of all proper faces of F which means F excluded that will be also subcomplex, just the biggest one F is not there but rest of them are there, that is subcomplex again, that will be denoted by $B(F)$, B corresponding to the boundary.

For example, suppose I take a 3-simplex, (all 3-simplexes are tetrahedrons,) and then I take the boundary namely the tetrahedron itself is not taken, all subsets are taken, what do I get? All the four triangles will come, they form the boundary of the tetrahedron, if you take a simplex, one simplex what will be its boundary, it will consist of just two singleton's which are the boundary points.

Vinay Sipani: So, it is like removing the interior.

Professor: Yeah, now, but we should not speak of that yet because we have not given any topology.

Vinay Sipani: Okay, sir.

Professor: Yeah, we are removing the whole simplex, simplex $\{v_0, v_1\}$; the boundary is a subcomplex that consist of $\{\{v_0\}, \{v_1\}\}$. okay?

Vinay Sipani: Yes.

Professor: I am just justifying the name for boundary of F. For this you can look at the copy of this one namely standard simplex, what is the standard 1-simplex? It was the line joining e_1 and e_2 . So, there it has a topology. We have not given topology for all the simplicial complexes. That is why I have brought this Δ_n and $|\Delta_n|$. The motivation is geometric but definitions are all abstract.

Vinay Sipani: Yes, sir.

Professor: If the dimension of F is n, then F is a carbon copy of Δ_n . I am just repeating the same thing. That F is isomorphic to Δ_n . This should already justify to some extent, the claim that simplicial complexes $\Delta_1, \Delta_2, \Delta_3 \dots$ and so on are the building blocks of simplicial complexes because each simplex there is just a copy of Δ_n , that n will depend upon dimension, that is all. I think I will stop here. Next time. We will do some more examples. Thank you.