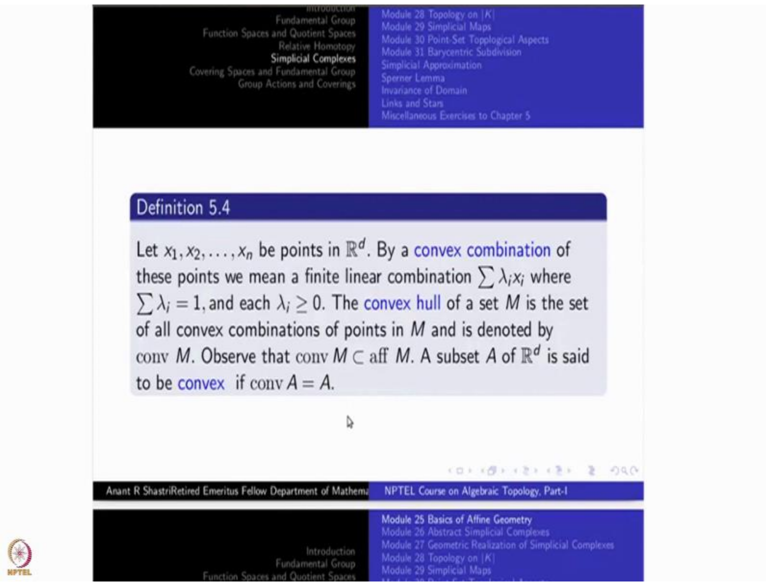
Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 26
Abstract Simplicial Complex

(Refer Slide Time: 00:16)

So, today's topic is directly abstract simplicial complex. Last time we introduced some Affine Geometry, let me before going to abstract simplicial complexes, one more or one or two more concepts from Affine geometry let us cover that and then come to the studyof these abstract simplicial complexes.

(Refer Slide Time: 01:02)



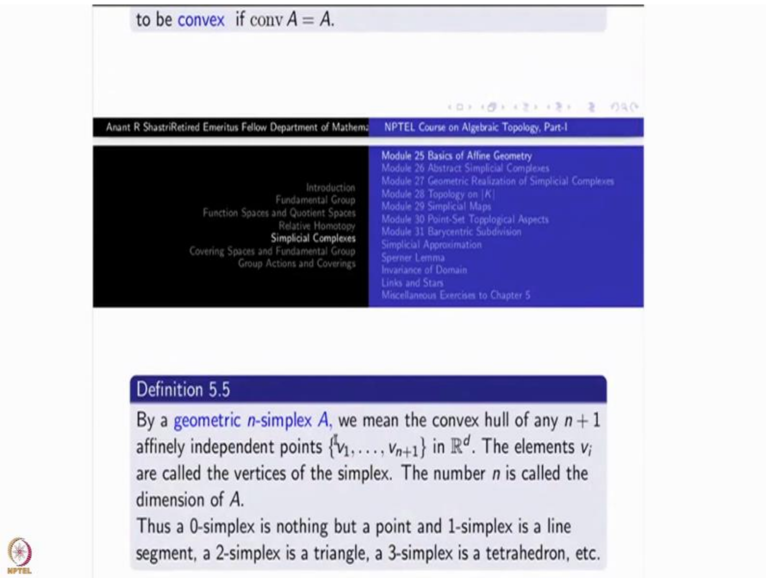
So, here is, this theorem was done, here is the definition. Convex combination of these points, n points inside $\mathbb{R}^d$ it is an Affine combination lambdai xi summation lambda_i equal to 1, and one more condition, each lambdai must be positive or non negative also, automatically each lambda i will be less than or equal to 1 because sum total is 1. Such a sum is called convex combination.

Just like 0 less than equal to t less than equal to 1, and I will be taking t times p plus 1 minus t times q. There it is convex combination of only two elements. But here you could take any finite set and you can talk about convex combination. The convex hull is the set of all convex combinations. Take any set M. Then you can take convex hull of that. It is called conv(M) that is a notation, it is automatically a subset of aff(M), Affine M. Here, all linear combinations are taken without the condition lambda i equal to 0.

So, a subset A of $\mathbb{R}^d$ is said to be convex, if convex hull of A itself is equal to A. This is similar to saying that a set is cosed if it is equal to its closure. What is the meaning of that? If you take any convex combination of finite many points of A, it is again inside A. This is similar to the property of aff M.

So, a line segment is a convex set not an affine subspace. Similarly, the inside of a triangle, full, not just the boundary triangle, inside of a triangle is a convex set, Tetrahedron is convex set, these are our standard convex sets. There are many more. Even a square is a convex set. So, see that we are concentrating on a particular kind of convex sets soon.

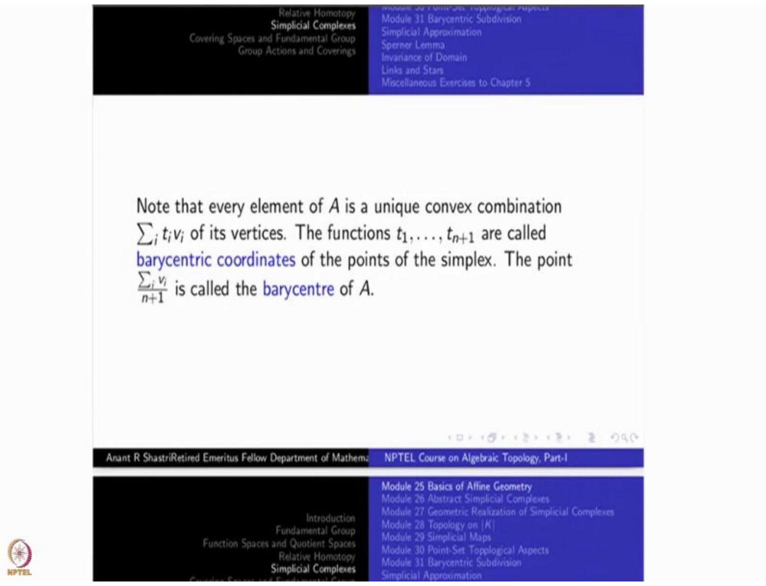
(Refer Slide Time: 03:32)



So, this is what is called a geometric n-simplex, geometric n-simplex by the very name you may anticipate that there is going to be something n-simplex also without the tag `geometric'. So, what is geometric n-simplex? It means convex hull of any n plus 1 affinely independent points $v_0, v_1 \dots, v_n$ in $\mathbb{R}^d$.

The elements of v_i are called the vertices of the simplex. The number n is called the dimension of this A. The set A is the convex hull of $v_0, v_1 \dots, v_n$. The points $v_0, v_1 \dots, v_n$ must be affinely independent. Then we call A a geometric n-simplex. A 0-simplex is nothing but a single point. Take one single point; what is the affine combination? Lambda times x where summation lambda must be 1 now, so it is just lambda is 1, so it is the singleton set $\{x\}$. So, its convex combination is also is a single point. A 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is tetrahedron, these are a few names, afterwards we do not know what to say, the 4-simplex I do not know what name I should give. So, we just say a 4-simplex.

(Refer Slide Time: 05:20)



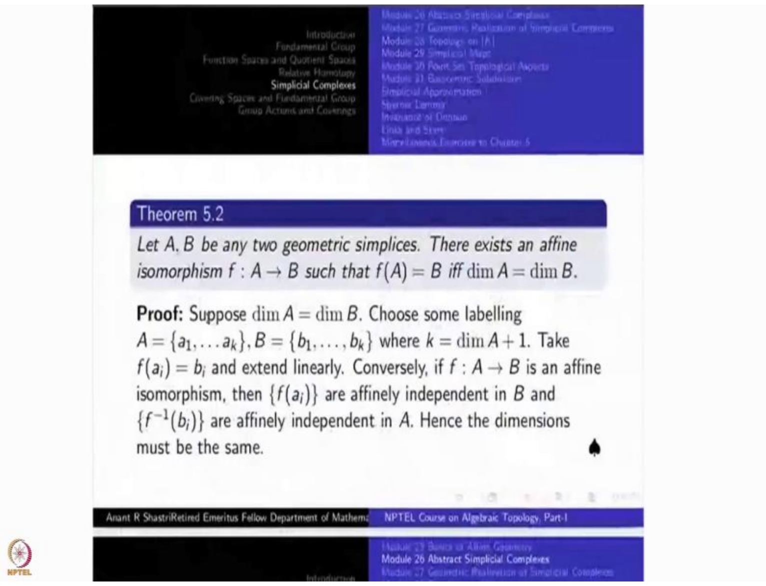
Note that every element of A is a unique combination of these vertices, unique convex combination, the uniqueness follows because of affine independence of $v_0, v_1 \dots, v_n$.

If x= $\sum_i t_i v_i = \sum_i s_i v_i$ in two different ways then what you will get is $\sum_i (s_i - t_i) v_i = 0$

with at least one of the si minus ti will not be equal to 0, and $\sum_i (s_i - t_i) = \sum_i s_i - \sum_i t_i = 1 - 1 = 0.$ So, that will give you that vi's are not affinely independent. Therefore, uniqueness follows. This is exactly similar to whenever you have a linearly independent set then every element in the linear span of these sets is a unique linear combination of the n elements there, that is called a base is there, here these vi's are called vertices.

There is one very special point by symmetry, what is it? It is $\frac{\sum_i v_i}{n+1}$, the summation vi divided by n plus 1, there are n plus 1 points here. So, I am dividing by n plus 1 after taking summation, this is a convex combination and this point is called the barycentre of A, you may call it centroid also, there are lots of centres in plain geometry, circumcentre, incentre and so on various centres. Here we have just the barycentre. This barycentre is called, some people call it, centroid also, no problem.

(Refer Slide Time: 07:38)



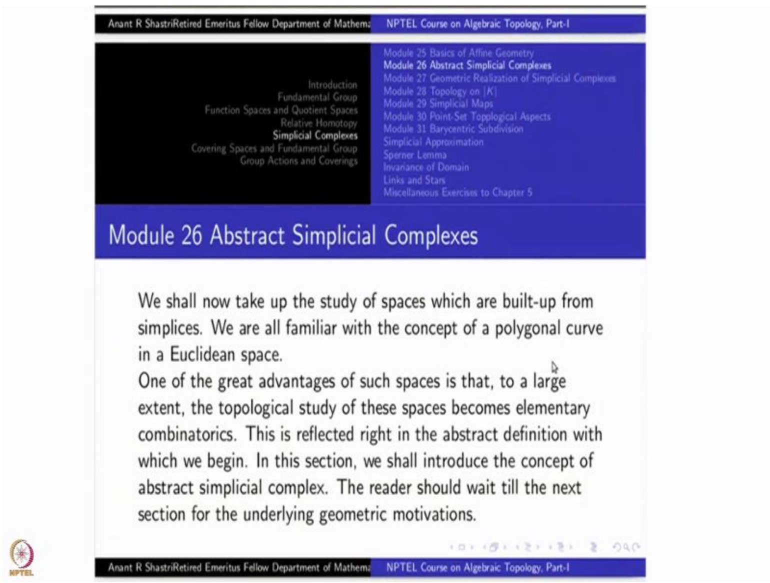
Here is an easy theorem, let A and B be any two geometry n-simplices. There exists an affine isomorphism of A to B, such that fA equal to B and A and B are isomorphic in that sense affine isomorphism. You put one condtion: dimension of A equal to dimension of B, just the dimension will tell you that they are affinely isomorphic.

So, this is the starting point, maybe you can say that the topology here is reduced to combinatorics, to finite geometry. So, this is the starting point of that, you are receiving more and more of this kind. Let us see, how to prove this. Suppose, dimensions are same. That means number of vertices are the same. So choose some labelling $a_1, \dots a_k$ and $b_1, \dots, b_k$. Here k is dimension of A plus 1, one more that the dimension, that is what we have seen the dimension by definition.

Take $f(a_i) = b_i$. It is one one correspondence. (Also, you could choose any other one-to-one correspondence.) Extend it linearly over conv(A). Extend it linearly means what now? affine linearly, $f(\sum_i \lambda_i a_i) = \sum_i \lambda_i b_i$, that is all. Just like in linear algebra. Once you have defined it over basis elements, it is uniquely defined. Exactly as in linear algebra, here also it is uniquely defined because each $\sum_i \lambda_i a_i$ has unique expression, so that will give you an isomorphism.

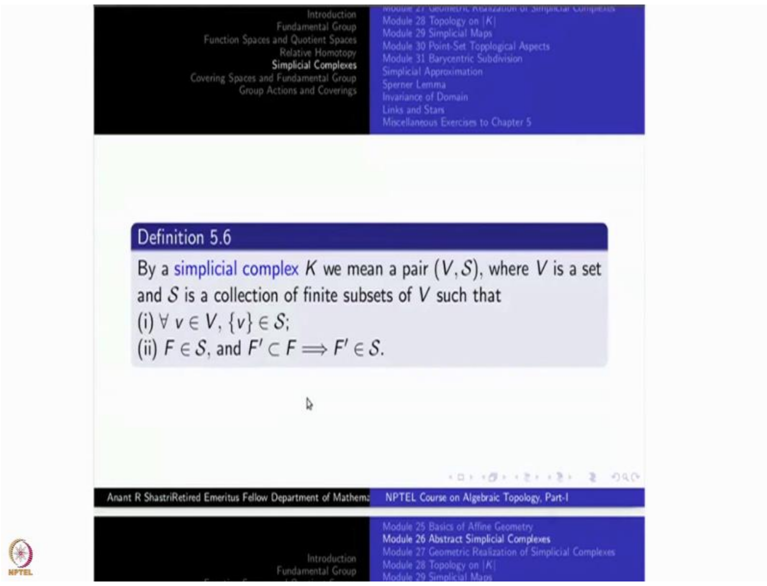
Conversely, if f from A to B is an isomorphism, then $a_1, \cdots a_k$ are affinely independent, implies f of ai should be affinely independent. Similarly, if $b_1, \cdots, b_l$ are affinely independent then $f^{-1}(b_i)$'s are affinely independent. Therefore, definitely k and l must be same and hence dimension of A is equal to dimension of B. So, this is what we wanted to do in our meeting last time, we could not do it then, so I have done it now.

(Refer Slide Time: 10:26)



Now, we will directly take up the study of abstract simplicial complexes which does not look like any topology at all, you can see now. So, you have to hold your horses for a while to see the topology behind them.

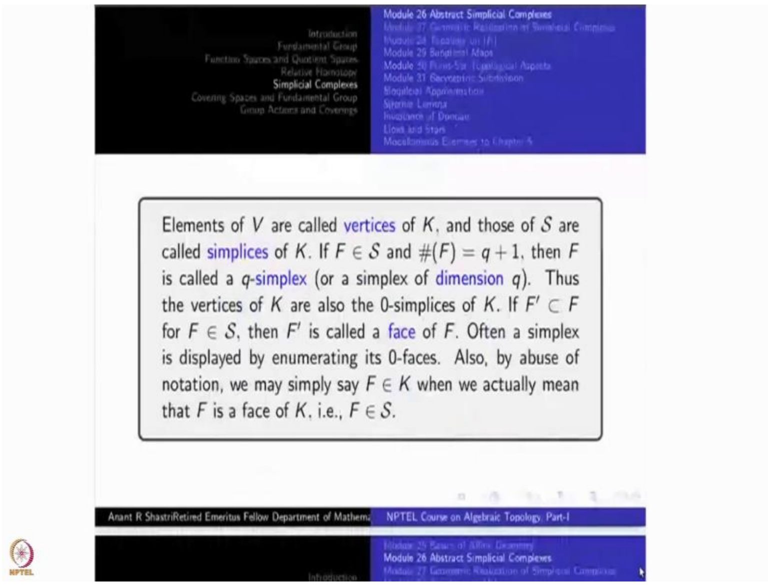
(Refer Slide Time: 10:56)



So, I have decided to direct start with the definition of abstract simplicial complex. It will consist of a set denoted by V and called as vertex set and another set denoted by S called the set of simplices, and is a collection of finite subsets of V. I will write down this one later on. S is a collection of finite subsets of V, it satisfies two more conditions, namely: (i) for every $v \in V$, $\{v\}$, the singleton is inside S.

Remember, elements of S are subsets of V. So, I should not write v inside of S but I should write $\{v\}$, singleton v inside S. (ii) Whenever, F is a subset belonging to S, all subsets of F will be also inside S. In other words, S is close undertaking subsets. So, these are the only two conditions that will make a simplicial complex. So, let us go on to some more definitions and properties.

(Refer Slide Time: 12:33)



Elements of V are called vertices of K, those of S are called simplices of K. If F is inside S and number of points, namely cardinality of F is equal to q plus 1, then F is called a q-simplex and you can also say dimension of F is q. So, this is a definition of dimension, get the number of points minus 1.

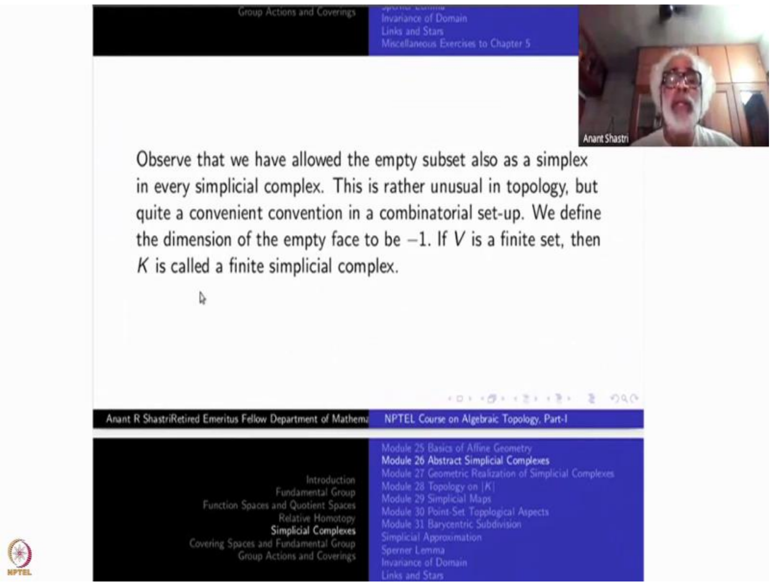
Thus, the vertices of K are also 0-simplices of K. You have to take your vertex v, put it inside a bracket to make it a singleton v, $\{v\}$, it will become a 0-simplex. So, soon we will not keep doing this. We will have this liberty of not writing the bracket that is what, like we write sometimes $V \setminus 0$ to mean $V \setminus \{0\}$. Some set theoretic notation takes that kind of liberty, so only that liberty is there that is all. Thus, vertices of K are 0-simplices.

So, as a vertex, it an element of V but as a 0-simplex, it is a singleton v, $\{v\}$ which is inside S. That is the difference between logically. But you can say that,a vertex is also 0-face.

So, $F' \subset F$ and F itself is an n-face, then we say that F' is a face of F. Here it may be equality also. For example, if you take one vertex here, a singleton that singleton v will be a face.

So, all subsets of a simplex are called faces. Often a simplex is displayed by enumerating its 0-faces, enumerating 0-faces means what? Just writing down the set $\{v_0, \cdots, v_n\}$, what are the elements of that set, just write down. Automatically all the subsets are there is there you have to think about it, that is all. This is just like writing when you have topology you just write down some basic open sets only then the topology is understood.

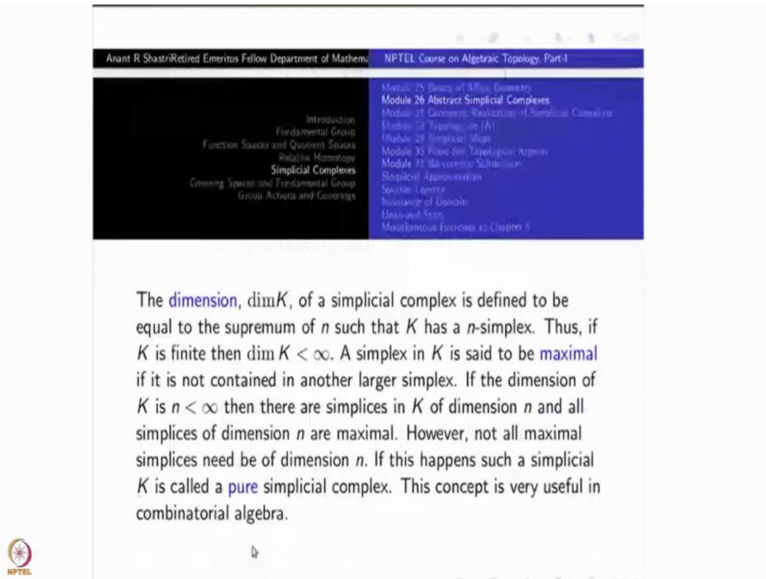
(Refer Slide Time: 15:21)



Observe that we have allowed the empty subset also as a simplex. I never said F is not empty. Empty set is also allowed. What is the dimension of an empty set? There are no elements there, so cardinality is 0. Then I have subtracted one. So, the dimension is, by definition, -1. Finally, whenever the vertex set V itself is finite then automatically S will be also finite because it is only a collection of finite subsets of a finite set V.

Therefore, that will also finite. Then K itself is called a finite simplicial complex, vertex set must be finite. So, these are just some names. I will keep reminding you again and again, so soon you will get used to it.

(Refer Slide Time: 16:29)



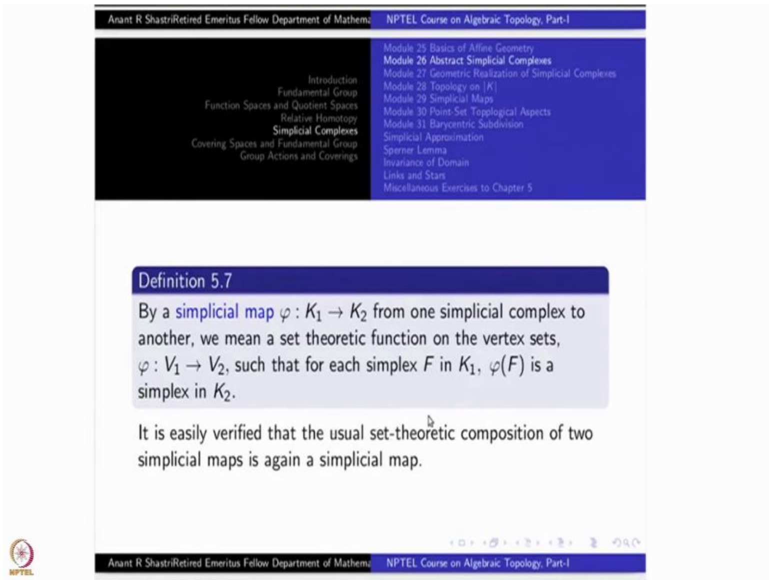
Now, I can assign a dimension to the whole of K, K is a simplicial complex. So, what is this dimension? It is the supremum of all n where n is what, some simplex or dimension of some simplex indicated. Suppose, there is a simplex of dimension n then dimension of K will be at least n, you have taken the supremum. This supremum could be infinite. If there is a simplex of dimension n for every n, then dimension of K will be infinite. It is finite means, say k, what that means, there is at least one simplex of that dimension and all other simplexes are of dimension less than or equal to that dim.

Note that finiteness of dimension of K does not mean that K is finite, but if K itself is finite, then the dimension has to be automatically finite. If the dimension of K is a finite, say n, then there are simplices of dimension n and all simplices of dimension are maximal. Maximal means what? You cannot have another larger simplex containing that. For then the dimension of K would have increased.

So, these simplices will be maximal which means all other simplices are subsets of simplices of dimension $k \leq n$. Not all maximal simplices need to be of the same dimension, remember, I can have a singleton 0, singleton say 1 here and then a simplex somewhere both will be maximal, dimension will be 1 here, but the singleton 0 standing away that will be of dimension 0.

So, whenever everything is contained in an n-simplex, such a simplex is called pure. These things are very very important in algebra and combinatorial mathematics, but for us this will not play much role. So, you can soon forget about this one, perhaps I am never going to use this word `pure.'

(Refer Slide Time: 19:28)



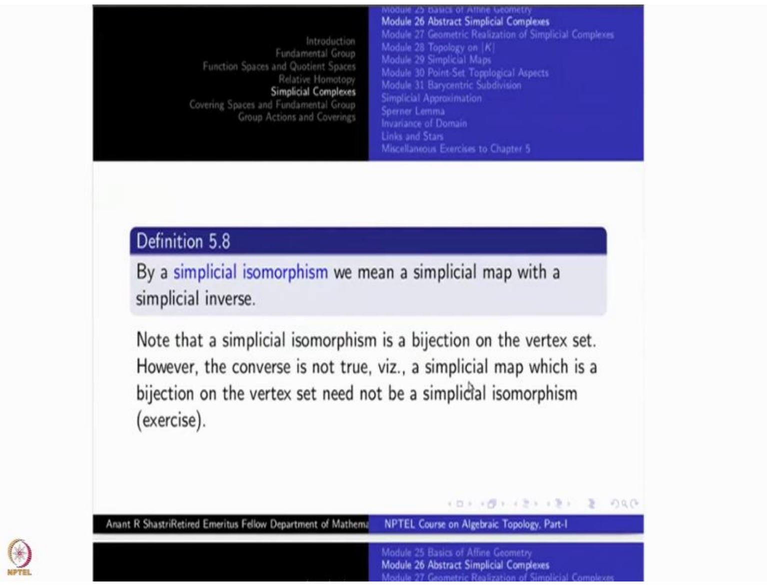
Simplicial maps ϕ : you see, now, we have introduced the definition of simplicial complex. Now, we want to introduce the notion of what is the map between them, like topological spaces as a continuous map, if there are groups there is homomorphism and so on. So, you should have relations also what kind of functions are allowed here. So, take a Simplicial map from K1 to K2. K1 means what. So, K1 itself has a structure: K1 means it has a V_1, a vertex set and S_1, a set of simplices.

So, the function ϕ corresponds to actually a function from vertex set V1 to vertex set V2 with an additional hypothesis that for each simplex F inside K1, $\phi(F)$ makes sense, because phi is a function on V1 to V2, $\phi(F)$ makes sense, this is a subset of V2, it must be inside K2 means what? it is the simplex in K2, it must be inside S2, F belongs to S1 means phi F belongs to S2, so that is the condition.

Image of every simplex must be a simplex, it need not be of the same dimension, number of elements in F maybe say 5, number of elements in $\phi(F)$ maybe 4, it cannot be more than 5. We know that, we know, it may be 4, it may be 3, it maybe 2 also, but ϕ being function its image will not be empty. If F itself is empty, phi F phi empty, it will be empty there is no problem.

So, usual set theoretic composition of functions phi from V1 to V2, another one, psi from V2 to V3 then psi composites phi makes sense. Automatically it will be simplicial because psi of phi F it is simplex must be a simplex inside S3. So, composition of simplicial maps is a simplicial map, identity from K to K namely vertex to vertex is a simplicial map.

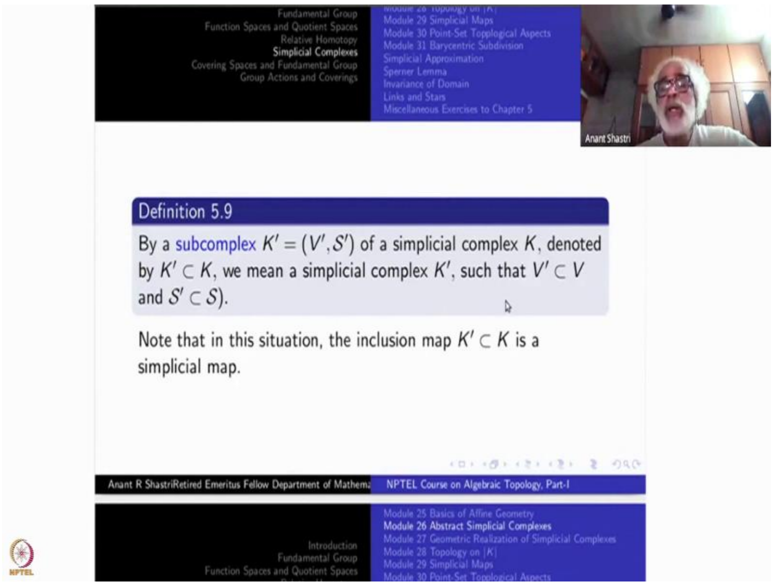
(Refer Slide Time: 22:30)



What is the meaning of a simplicial isomorphism? Simplicial isomorphism is first of all a bijection on the vertex sets, that means there is an inverse function on the vertex sets. The inverse must be also a simplicial map, a bijection of vertex sets need not be an isomorphism even if it is a simplicial map, the inverse may not be simplicial.

This is strangely typical of topology-- a continuous bijection may not be a homeomorphism, the inverse may not be continuous. Whereas, in linear algebra, if we have a linear bijection, the inverse is automatically linear. In group theory if you have a linear, if you have a group homomorphism which is a bijection, the inverse is automatically a group homomorphism. So, though this is like combinatorics which is like algebra, it is closer to topology in its nature, inverse has to be, you have to demand the inverse is also a simplicial map to be an isomorphism.

(Refer Slide Time: 24:21)



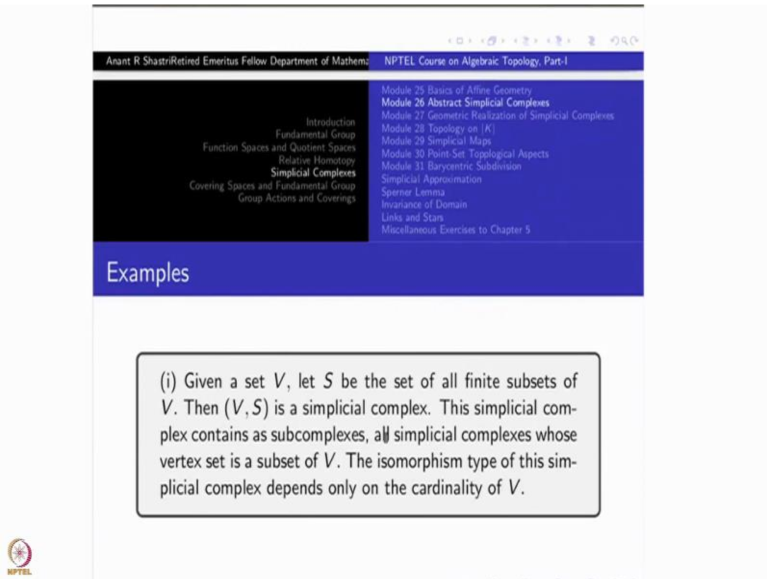
So some more terminologies. What is the meaning of a subcomplex $K' = (V', S')$ of a simplicial complex K? we will just denote it by $K' \subset K$. So, what is the meaning of this? The vertex set V' must be a subset of V and S' must be a subset of S; K' itself must be a simplicial complex on its own first of all and these inclusions must be valid, then only you call it a subcomplex.

What is the meaning of this? The vertex set V' must be a subset of the original set V. Whenever, you have a simplex inside K', it must be simplex inside K also, that is the meaning of that. All the vertices of V' inside V. You may for example, take all the vertices of V equal to V' and allow some of the faces in S to be missing here, that is way of taking a subcomplex into subcomplex. Some of them are missing but, whatever you have taken they are inside S. You do not have to take, you do not take a new simplex, so that is a subcomplex.

Note that, in this situation the inclusion map $K' \subset K$ is a simplicial map because every simplex in K' is in K and the map is inclusion, phi is inclusion map here. So, I recall K prime itself is a simplicial complex by simplicial complex, you know we need assert that K prime is a simplicial complex.

What is the meaning of this? Conditions (i) and (ii) must be satisfied; number two is most important, whenever F belongs to K' and $F' \subset F$ then F' must be also in K' automatically. So, you cannot say that you keep one subset here and miss a subset of that set, that should not happen.

(Refer Slide Time: 27:13)



So, let us run through a few of the examples now, I may not be able to do all of them. Take any set V then take the collection of all finite subsets of that, S is all the collection of finite subsets, that is a simplicial complex. This corresponds to like we take, in topology what we do, we take a set and then take all subsets, it is the discrete topology. You are taking all subsets here which are finite of course because we are not allowed to take infinite subsets, all finite subsets we will take that is a simplicial complex.

This simplicial complex contains as a subcomplex, all the simplicial complexes whose vertex set is a subset of V. You cannot have anything better than that. That is why I told you that this is like a discrete space. In logic it is discrete. But you will see that in topology, later on, it is the opposite. The isomorphism type of this simplicial complex depends only on the cardinality of V. I just take any set W, if the cardinalities are the same, then taking all subsets here taking all subsets there, there will be a corresponding isomorphism. All that I have to do is write down a bijection of the vertices that is all. Take any bijection of vertices; it will be a simplicial isomorphism.

Vinay Sipani (Student): The dimension would be 1 plus cardinality of V, right?

Professor: Dimension of S of this K?

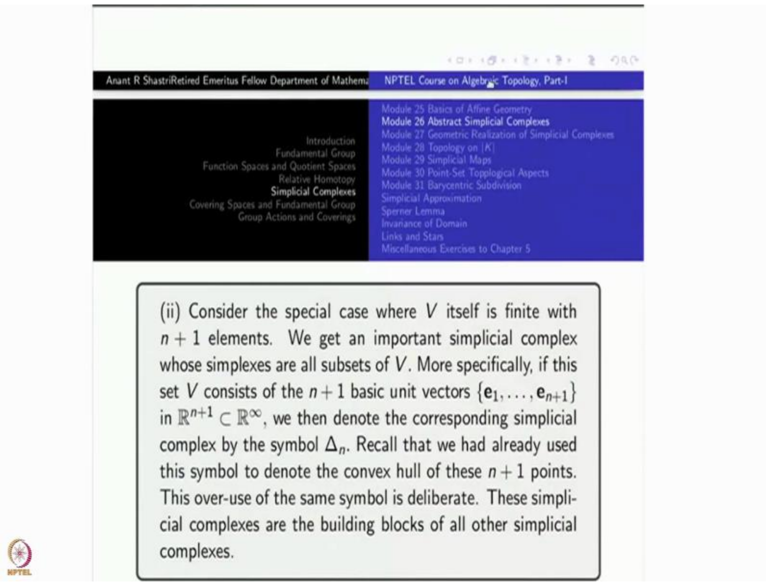
Vinay Sipani (Student): Yes.

Professor: Dimension of this K could be infinite depends upon, what is V? Suppose V is an infinite set. Then I can go on taking finite subsets larger and larger. If V is a finite set suppose there are only n elements. Then the dimension will be n minus 1. If V infinite, the dimension of this simplicial complex will be infinite.

Vinay Sipani (Student): Yes, sir.

Professor: Each simplex is finite, finite dimension, but dimension of K is becomes infinite.

(Refer Slide Time: 30:25)



Let us go to our second example. Consider a special case where V itself is finite that is what we are discussing with n plus 1 elements. This is a very important one for us. Now, what am I taking, you are taking n plus 1 element set, then you are taking all subsets, empty set, then all the singletons, then all the doubletons, all the tripletons and so on you are taking, okay? Yes or no?

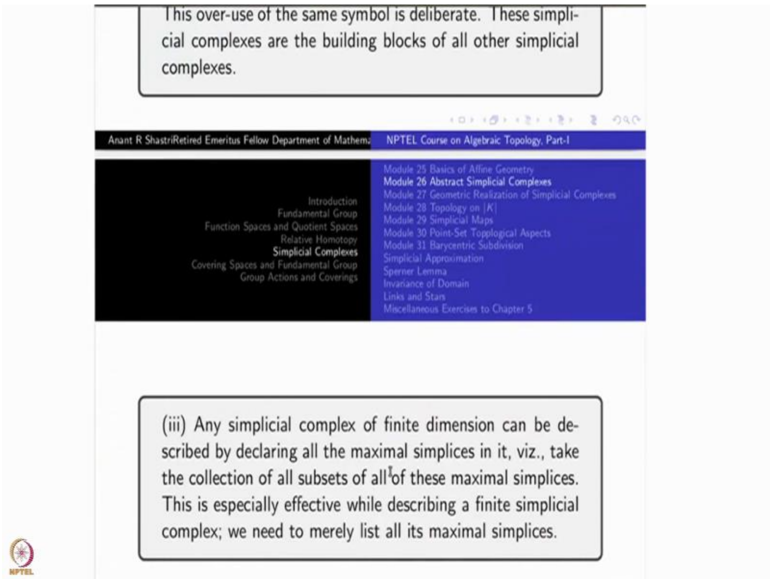
Students: Yes, sir.

Professor: This is now characteristic of our what we are calling that standard n- simplex, what we do? Take the basic unit vectors $e_1, e_2 \cdots e_{n+1}$ inside $\mathbb{R}^{n+1}$. We denote this corresponding simplicial complex by Δ_n. Now, I am giving you another specific case instead of arbitrary set V, I am taking V to be $\{e_1, \ldots, e_{n+1}\}$. Then take all subsets of V. That simplicial complex we are denoting by Δ_n-- without the bars. Convex combination, remember convex hull of the set was denoted by $|\Delta_n|$. Soon you will understand what the difference between these two symbols. So, this overuse of the same symbol is deliberate because they are very closely related. So, you have to hold your horse till we come to the point--what is the difference between them, what I want to say is again I repeat take any simplicial complex with this property namely n plus 1 point and then all subsets. They are all isomorphic to this Δ_n, because our previous thing here.

(Refer Slide Time: 32:45)

This remark is applicable here also, take any bijection of vertex sets-- that will give you an isomorphism. So, up to isomorphism, the simplicial complex Δ_n is unique-- when you have fixed n.

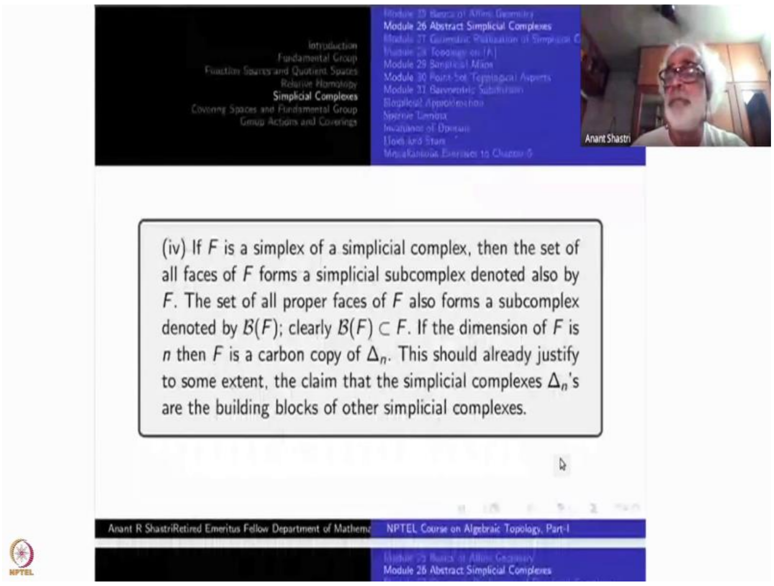
(Refer Slide Time: 33:11)



Any simplicial complex of finite dimension can describe by declaring all maximal simplices, if it is infinite dimension there may not any maximal simplex. So, that is it. You can do it only for finite dimension case. Look at what are all the maximal simplexes, you will make a list of them, rest of them you can know. What are the other simplices? they are subsets of one of the members in this list. That is all.

In fact, all subsets of whatever list you have made, they must be there. So, this is an easy way of programming a computer for a simplicial complex, you just declare what is the rule for simplicial complexes it understands then you give the list of maximum simplices, over. So, this is especially effective for describing finite simplicial complexes, we need to make a list of all maximal simplices.

(Refer Slide Time: 34:30)



If F is a simplex, in a simplicial complex, then the set of all faces of F, just only take the faces of F including F itself, that itself would be a subcomplex, that subcomplex also we will denote by F. This is just like when you have a subset of a topological space X, then you write a subspace also by that same symbol, but what is a subspace? It consists of all those open sets inside X intersected with A, you intersect it at A that is the collection of open subsets of so far A, you do not keep on writing that.

So, there is like that denoted by F itself. The set of all proper faces of F which means F excluded that will be also subcomplex, just the biggest one F is not there but rest of them are there, that is subcomplex again, that will be denoted by BF, B corresponding to the boundary.

For example, suppose I take a 3-simplex, (all 3-simplexes are tetrahedrons,) and then I take the boundary namely the tetrahedron itself is not taken, all subsets are taken, what do I get? All the four triangles will come, they form the boundary of the tetrahedron, if you take a simplex, one simplex what will be its boundary, it will consist of just two singleton's which are the boundary points.

Vinay Sipani: So, it is like removing the interior.

Professor: Yeah, now, but we should not speak of that yet because we have not given any topology.

Vinay Sipani: Okay, sir.

Professor: Yeah, we are removing the whole simplex, simplex $\{v_0, v_1\}$; the boundary is a subcomlex that consist of $\{\{v_0\}, \{v_1\}\}$. okay?

Vinay Sipani: Yes.

Professor: I am just justifying the name for boundary of F. For this you can look at the copy of this one namely standard simplex, what is the standard 1-simplex? It was the line joining e_1 and e_2. So, there it has a topology. We have not given topology for all the simplicial complexes. That is why I have brought this Δ_n and $|\Delta_n|$. The motivation is geometric but definitions are all abstract.

Vinay Sipani: Yes, sir.

Professor: If the dimension of F is n, then F is a carbon copy of Δ_n. I am just repeating the same thing. That F is isomorphic to Δ_n. This should already justify to some extent, the claim that simplicial complexes $\Delta_1, \Delta_2, \Delta_3 \cdots$ and so on are the building blocks of simplicial complexes because each simplex there is just a copy of Δ_n, that n will depend upon dimension, that is all. I think I will stop here. Next time. We will do some more examples. Thank you.