Introduction to Algebraic Topology (Part-I)
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Lecture 25
Basic Affine Geometry

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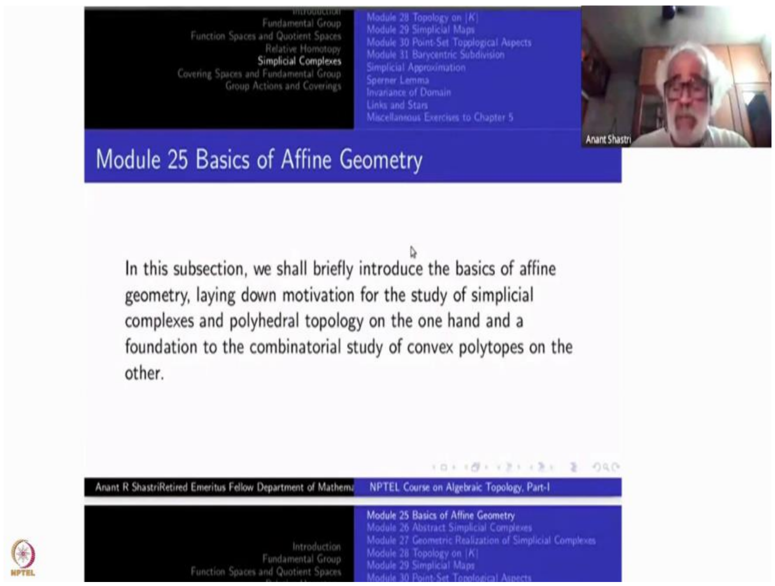

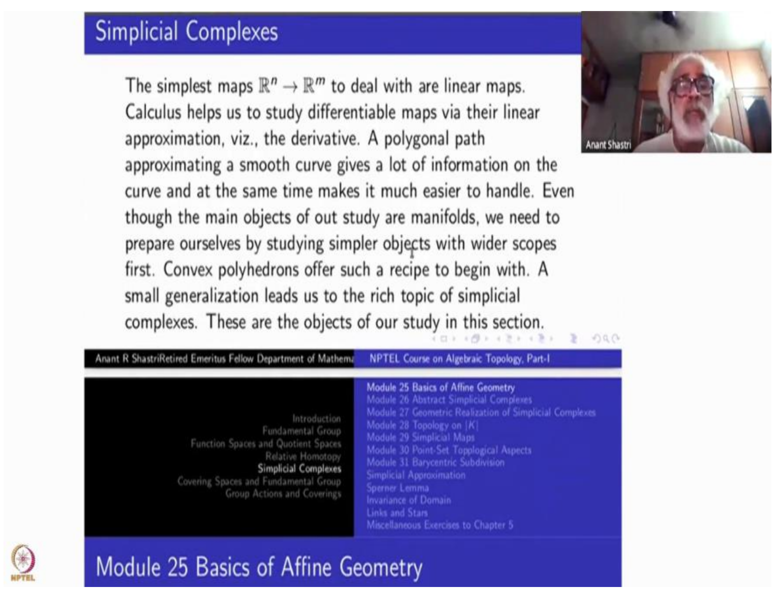

Today we are starting a new chapter, a new topic namely, Simplicial Complexes. Motivation for the Simplicial Complexes is manifold, several, you know, several motivations. First of all, simplest maps to understand are linear maps other than constant maps. Calculus was introduced basically to convert so called differentiable maps into linear maps. The derivative is a linear approximation to a differentiable function at a given point. A polygonal path approximating a curve gives a lot of information on the curve itself.

In fact, all your fonts in computers and everywhere you see they are made up of line segments. Similarly, even though the main objects of our study in topology are manifolds, we need to prepare ourselves by studying similar objects with wider scopes first. Manifolds are themselves built up of chunks of Euclidean Spaces, but the building the process there and what we are going to do now is quite different.

We are also going to build up our objects of study namely what are called a Simplicial Complexes, out of closed simplexes or more generally, what are called convex polyhedrons. So, let us understand these objects first in their own way when they are isolated before bringing Simplicial Complexes. The key word here is Basics of Affine Geometry.

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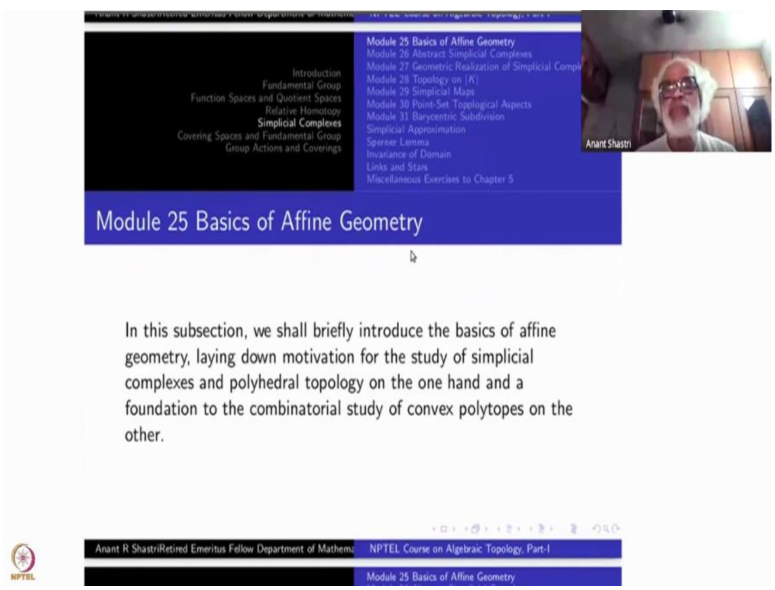

Affine geometry is nothing the study of simple ideas from linear algebra only thing is, the vector space structure has no origin. In other words, we just pretend as if we do not know which point is the origin. Conceptually, this is so, fundamental, but technically everything can be transferred to linear algebra by picking up any point as your origin; that is the advantage. Instead of one fixed origin you can pretend any other point of origin and so, statements which are independent of choice of the origin, you know which point you choose as origin does not matter, such properties are more fundamental.

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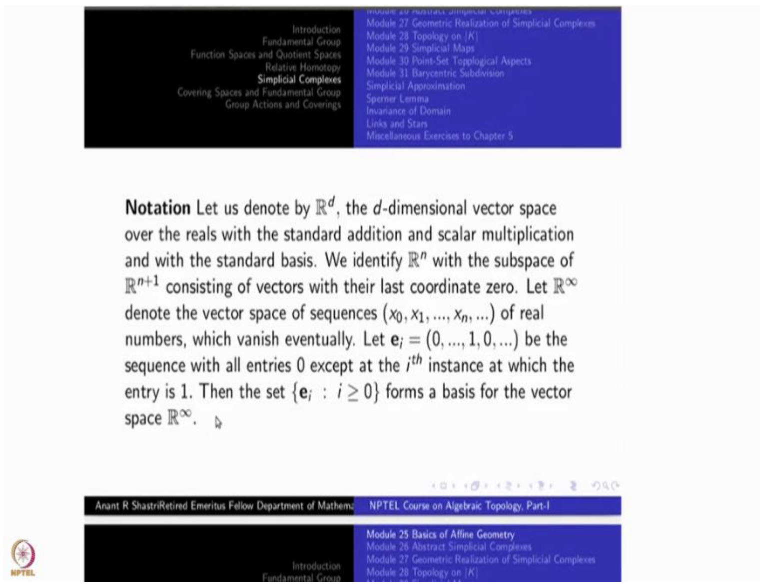

Let us fix up some notation so that again and again we do not have a confusion. $\mathbb{R}^d$ denotes d-dimensional vector space over the real numbers. The standard basis is $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\cdots,(0,0,\ldots,1)\}$; they are denoted by e_i. Every element of $\mathbb{R}^d$, we will start denoting them by $(x_0, x_1, \ldots, x_{d-1})$. In fact, we will keep including

$\mathbb{R} \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^d$

by adding one extra zeros.

So, real numbers are included inside $\mathbb{R}^2$ by $x \mapsto (x, 0)$x; inside $\mathbb{R}^3$ by $x \mapsto (x, 0, 0)$ and so on. So, this will allow us to study all the R power d simultaneously. the notation is $\mathbb{R}^\infty$, which consists of all finite sequences of representing, infinite sequences, but which are eventually 0 after certain stage only zeros, which is actually, in terms of vector space, is a direct sum of infinitely many copies of $\mathbb{R}$. So, that is $\mathbb{R}^\infty$.

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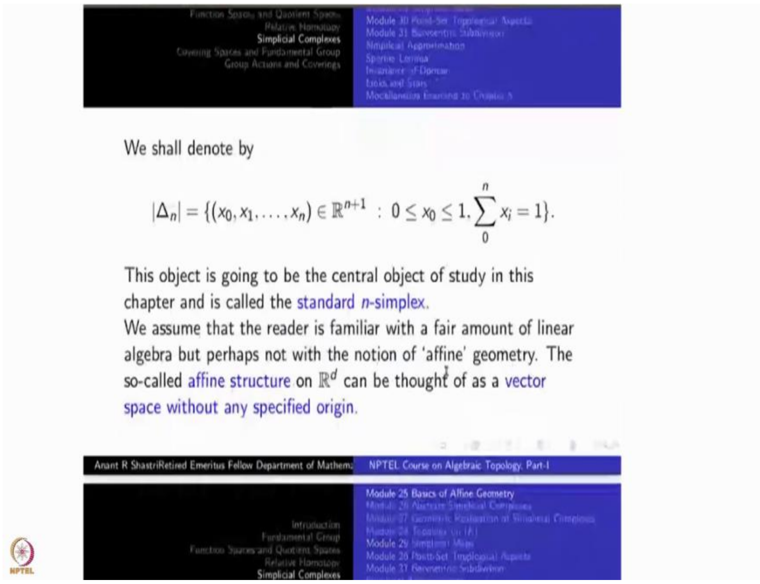

We shall denote by mod delta-n, $|\Delta_n|$, the set of all points $(x_0, \ldots, x_n)$ belonging to $\mathbb{R}^{n+1}$, which satisfy the following property, namely, (I) all x_i are non negative and lie between 0 and 1, and (ii) the sum total $\sum_i x_i$ is equal to 1. The sum total is equal to 1 condition defines a co-dimension 1 affine subspace inside $\mathbb{R}^{n+1}$. If you just take only this one, this is not a vector subspace, but it is what is called affine subspace. So, it is of dimension n inside $\mathbb{R}^{n+1}$. Further, I am taking only those which have all coordinates non negative, i.e., lie all the coordinates lie to 0 and 1.

So, this will be denoted by $|\Delta_n|$. mod delta-n. You may wonder why this mod and so on? Why not just delta n? Soon we will see all this. Later on, when we get familiar with this, maybe you can just write it delta n for this one. And this space is called the standard n-simplex. Why is n instead of R n plus 1 it is very immediate because the dimension of this affine subspace is n. So, this is a subspace of that of full dimension.

Let us just look at the one case namely, here n equal to 1 we are working $\mathbb{R}^2$. So what will be the subspace? x naught x1 plus x1 is equal to 1. This will be the line segment joining $(1, 0)$ and $(0, 1)$

inside $\mathbb{R}^2$. it is 1-dimensionL and that will be called the standard 1-simplex. The 1-simplex is embedded in $\mathbb{R}^2$ in a particular way.

Likewise, the standard 2-simplex will be embedded in $\mathbb{R}^3$, the standard 2-simplex will be a triangle spanned by $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. So those 3 will be vertices. We assume that the reader is familiar with a fair amount of linear algebra, but perhaps not with this affine geometry. That is why we are recalling some of these basic things.

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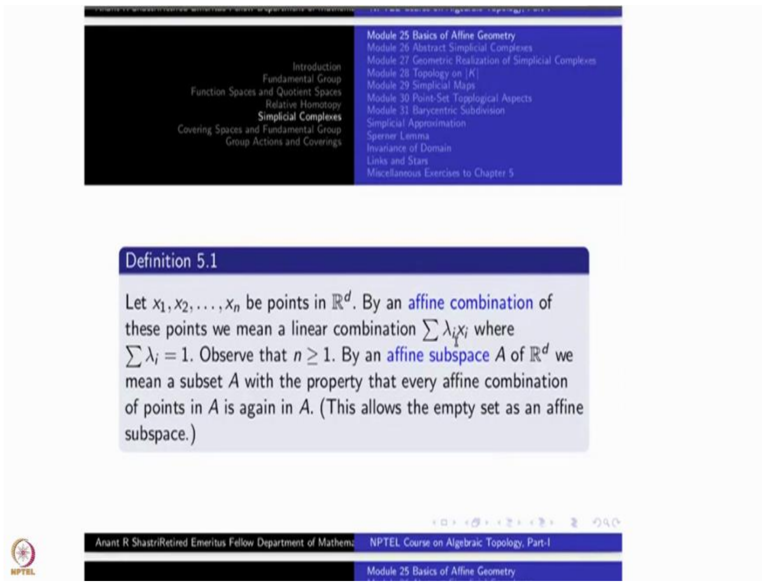

So starting with some n points inside $\mathbb{R}^d$, number n has nothing to do with dimension d here, any finite set of points inside $\mathbb{R}^d$. Affine combination means what? It is like a linear combination $\sum_i \lambda_i x_i$, where λ_i is a real numbers, but with the sum total equal to 1. So this is like the parameterization of a line segment t times v plus 1 minus t times u. Why t and 1 minus t?

Sum total is 1. That is the whole idea. So, that will give you the entire line but it may not be passing through the origin. There is no other conditions total must be equal to 1. So there are only 2 vectors. x_1 and x_2 here are not coordinates. They are points, t times x1 plus 1 minus t times x2, will be the line passing through x1 and x2. So, there is affine combination of x_1 and x_2. Similarly, if you take 3 points, it may give you a plane, may or may not be, if the 3 points are already collinear then affine combination will not give you anything more than a line.

The affine combination always makes sense. By an affine subspace A of $\mathbb{R}^d$ we mean a subset of $\mathbb{R}^d$with the property that every affine combination of points is inside A again. Given some finitely many points of A, take any affine combination of them, it must be inside A. Closed under taking affine combination. This is exactly similar to how you define linear subspaces? Only difference is, in the linear combinations, there is no condition on lambda's, they are arbitrary scalars. Here sum total must be 1-- that is the difference.

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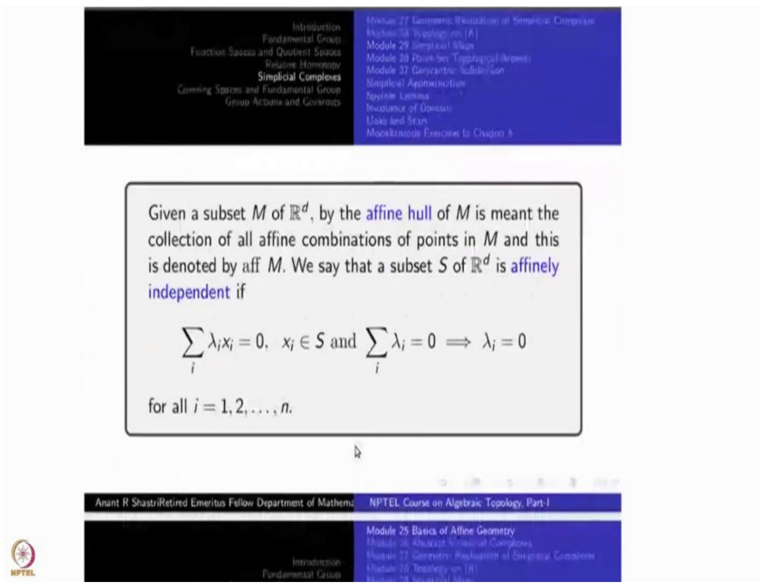

For a subset M of $\mathbb{R}^d$, by the affine hull of M is meant the collection of all affine combinations of points in M and this is denoted the aff M. This is similar to taking L(M), which is the linear space spanned by M; similar construction. Now, I am taking only affine space. We say that a subset S of $\mathbb{R}^d$ is affinely independent, -- once again the corresponding to the notion of linearly independence in linear algebra.

So, what is it? Summation lambda_i xi equal to 0, along with sum total must be equal to 0 here. (watch out, for affine combination is different while here summation lambda equal to 0) Then each lambda i must be 0. If you do not put this condition what you get is linear independence. So, this is the definition now, affine independent. This is not cooked up, this is forced on upon us. We will see in a moment.

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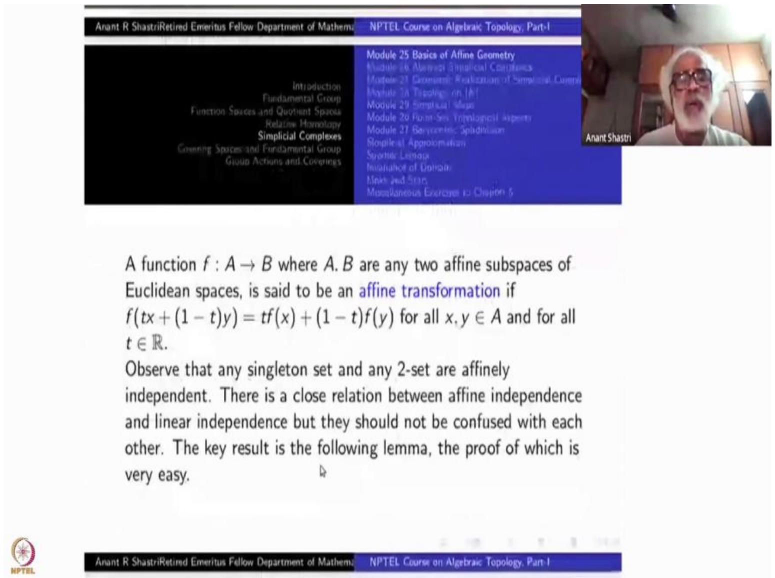

A function $f : A \to B$ between affine subspaces is said to be an affine transformation if it satisfies the following affine linearity. Namely $f(tx+(1-t)y) = tf(x)+(1-t)f(y)$. We are not saying that f of x plus y is equal to fx plus fy neither we are saying that f of tx equal to t times fx. That will be linearity. This is affine linearity, that is the difference, t is any scalar and x and y some points of A. A and B are affine subspaces so, that whenever x and y are inside A t times x plus 1 minus t times y is also inside A for all t.

Now, suppose you have a singleton set or a doubleton set, any set. They are affinely independent. In linear algebra $\{0\}$ is not independent and any non-zero vector is independent. In affine geometry, such a difference is not there. Every point singleton set is independent.

See, this is the beauty of affine algebra. It has better, richer structure than a vector space. Any 2 set, distinct points, is independent. The set with 0 and some vector is not linearly independent. In fact as soon as 0 is there in a set, it is not independent in the case of linear independence. So, that is the difference. So, this is a close relation between, there is a close relation between affine independence and linear independence.

But they should not be confused with each other. The key result is the following lemma, the proof of which is very easy we will see it afterwards. What is the correspondence between affine independent and linear independent? So, this is a lemma.

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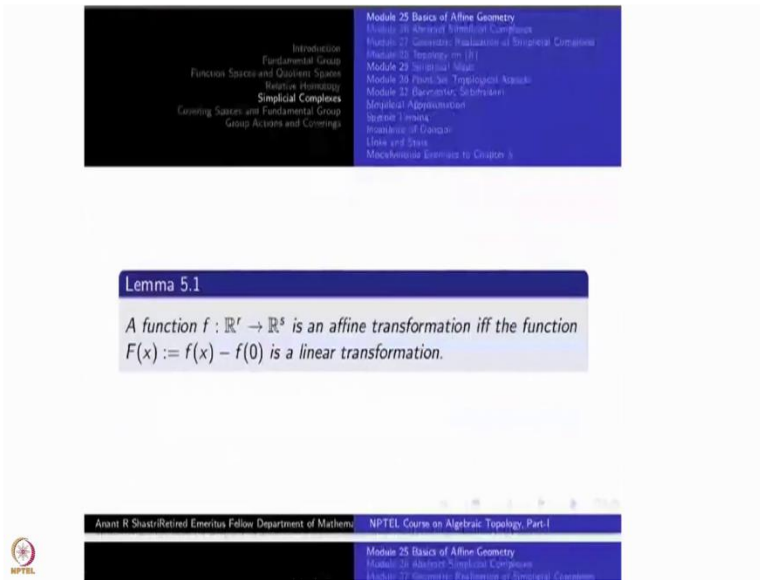

Take a function f from R raised to some power to R raised to some other power. $\mathbb{R}^m \to \mathbb{R}^n$, which is an affine transformation. Now, I am thinking of $\mathbb{R}^d$ affine space. Remember, what is an affine space? Affine combination of any 2 elements should be inside it, that is it. So, $\mathbb{R}^r$ is affine space also is a vector space does not matter. If you do not know the origin, if you have forgotten it then it becomes an affine space is the point.

So, take now an affine transformation, not a linear transmission. F is an affine transformation if and only if you look at the function fx minus f 0 that must be a linear transformation. In a linear transformation 0 is mapped to 0. f might not obey this, because it does not treat any point as distinct that is the whole, whole business. So, now you know what is 0, look at the image of 0 under f subtract that, the corresponding function will be linear and conversely.

So, in other words, $x \mapsto ax$ is a linear map whereas $x \mapsto ax + b$ is affine linear map. So, any degree 1 polynomial defines an affine linear transformation; if the constant is 0 then it is a linear map.

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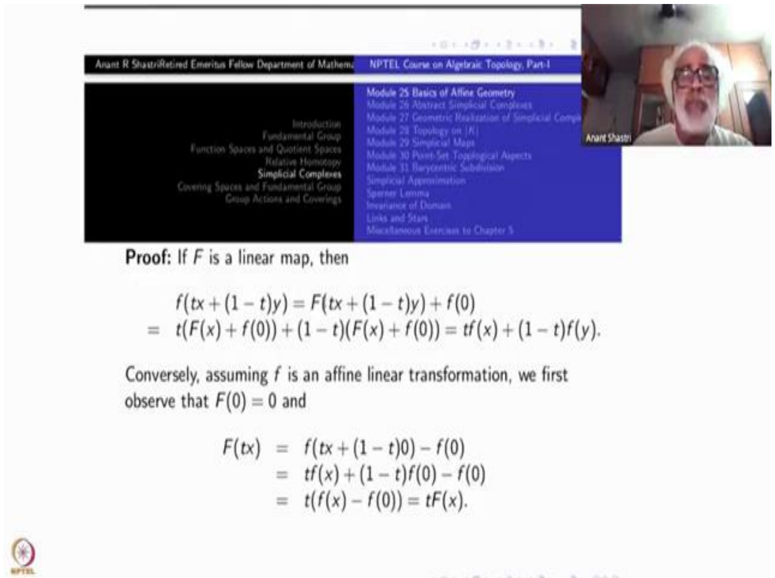

I have given a proof also here. It is capital F, after subtracting f0 is a linear map, then go back to the origin function f, f of tx plus 1 minus ty 1 minus t times y. You see by definition f of tx plus 1 minus t times y plus f0 because capital F of whatever it is, this is f of that minus f0 so, add f0. Now this is linear, that is given. So, it will be t times fx plus f0, I can add, 1 minus t times fx plus f0, I can add to the 1 minus t times fx, 1 minus t times fx are subtracted.

But this is nothing but little fx, little fy. The converse is got just by reversing the steps here. But you can also do it like this. Assume f is an affine linear transformation. Then F of 0 by definition is f0 minus f0 is 0. After that F of tx is by definition f of tx plus 1 minus t0 you can write like this, minus f0. But when you write like this, it is t times fx plus 1 minus t times f0 minus f0.

But that is nothing but t times fx minus f0 which is by definition t times F(x). So t comes out of that. Finally, you have to see the addition F of x plus y equal to Fx and Fy.

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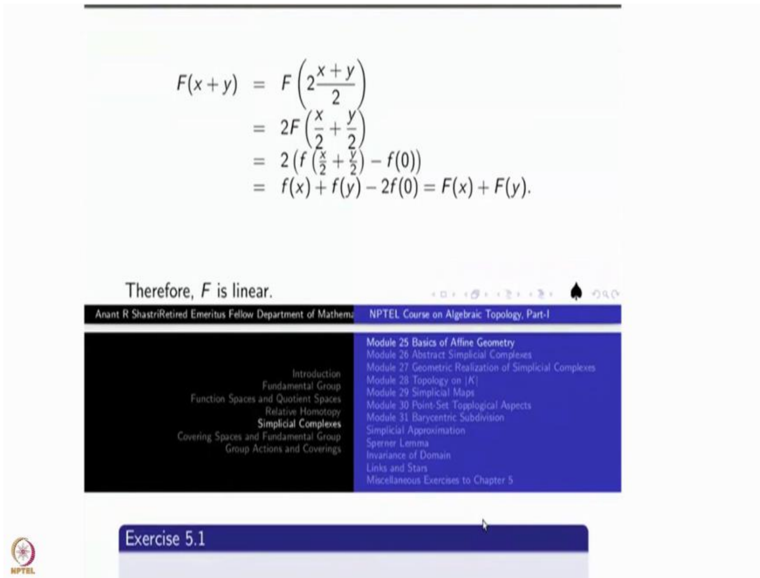

F of x plus y, you can write it as F of twice x plus y by 2. When you write like that this 2 comes out F of x by 2 plus y by 2. But now this is an affine combination, 1 by 2 plus 1 by 2 is equal to 1. So, that is equal to half of f of x plus y minus f0, which you can simplify this, there is 2 outside-- fx plus fy minus twice f0 equal to fx minus f0 plus fy minus f0 which is capital Fx plus capital Fy. Therefore F is linear, capital F is linear, little f is affine transformation.

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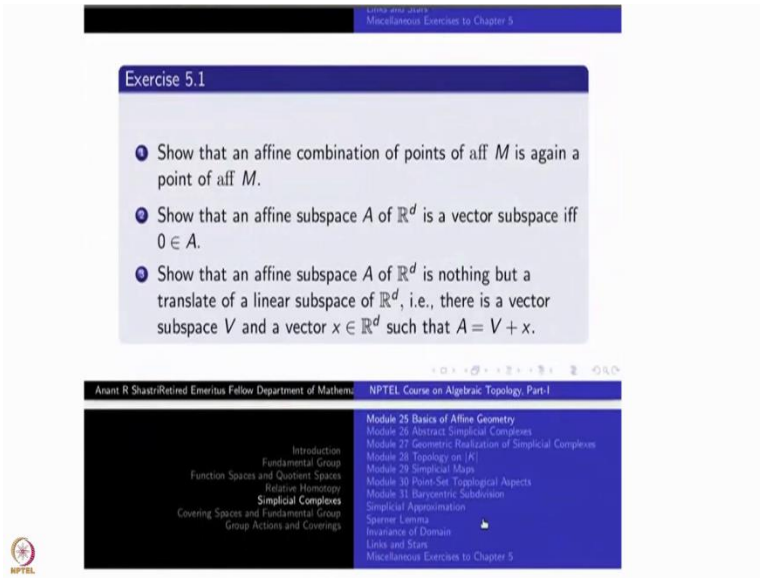

So, here are a few exercises, affine combination of points of aff(M) is again a point of aff(M). Show that affine subspace A of $\mathbb{R}^d$ is a vector space if and only if it is passing through the origin, that is all. All these exercises are very straight forward. Show that affine subspace A of $\mathbb{R}^d$ is nothing but a translate of a linear subspace by a vector. So, this exercise 3 will follow from 2. every A can be written as some vector space V plus some vector. If you can choose that vector as 0 then A itself will be a linear subspace, that is the previous exercise.

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So we will define the dimension of an affine subspace to be the dimension of the vector space A minus x, where x is a point in that A. Take a point x inside A and throw it away sorry, not throwing but subtract it from all elements, in particular A minus x will have 0 in it. So, it will become a vector space. Get a dimension of that. So, this is an easy way of defining dimension, by using the linear algebra, you can do it without going to linear algebra also, independently of all this.

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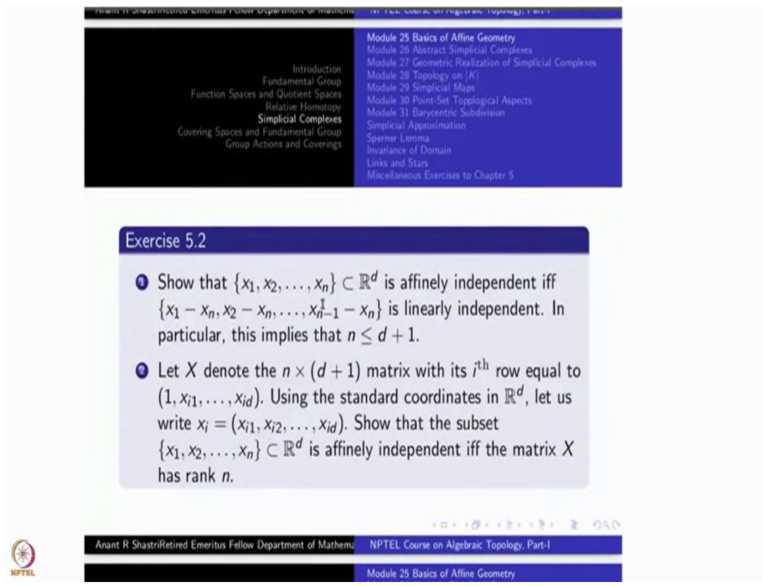

So, here are some other set of exercises. All these exercises should be separately sent to you. You do not have to copy it from here or anything.

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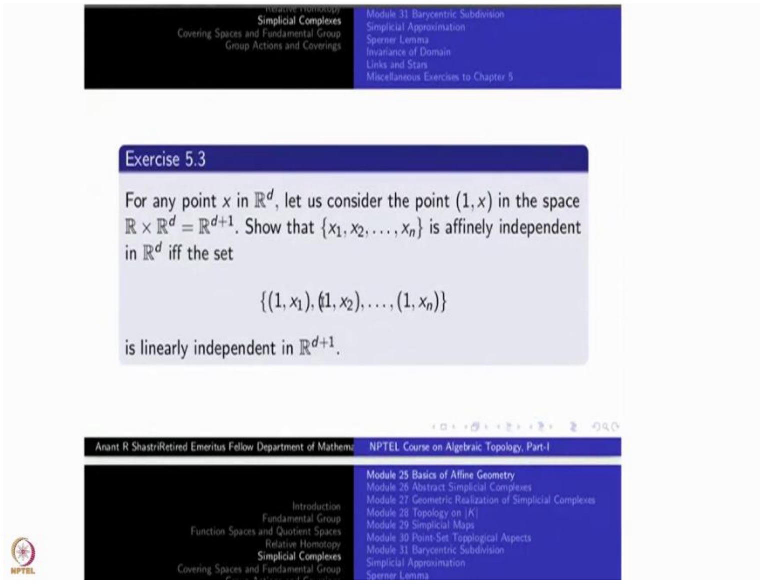

I do not want to discuss each one of them now. But before we get to the next session, we should be familiar with these things. So do not waste your time be firm that you know all these things.

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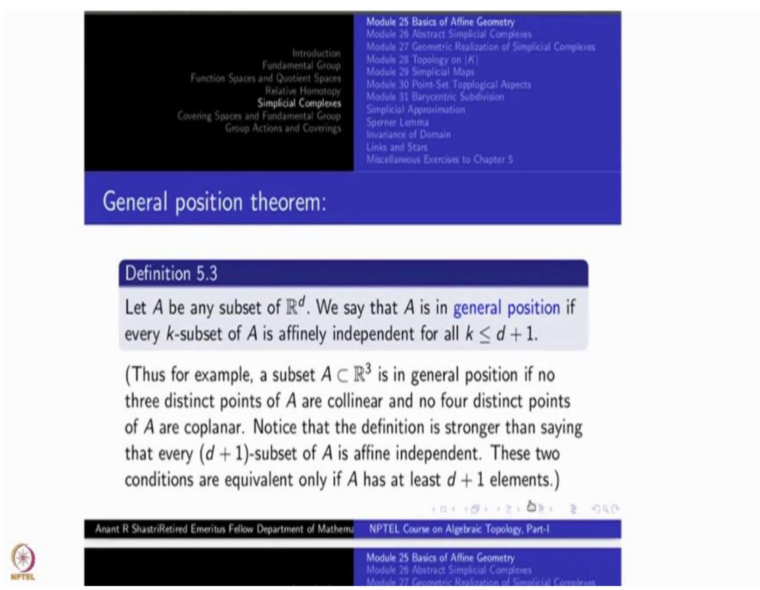

Here is one more concept I want to introduce namely which you might not have learned in linear algebra. So namely take a subset A of $\mathbb{R}^d$, we say this subset is in general position if every k-subset of A is affine independent, where k itself is less than or equal to d plus 1. Every k subset is affinely independent. A itself may be an infinite set. What do you mean by k-subset? A k-subset means a subset which has exactly k elements.

For example, take A as a subset of $\mathbb{R}^3$. This will be in general position if no three distinct points are collinear and no four distinct points are coplanar. A single point is always independent, even every 2-set is always independent. So that would not give you any trouble. Starting with 3 or more, there is need to put some condition. 3 points may l be collinear. That should not happen because our k less than 3 is less than d plus 1 here. Therefore, every k-subset affinely independent means they should not be collinear.

Similarly 4 points. Take any 4 points. They should not lie in the same plane. So this is what the meaning of general position in $\mathbb{R}^3$. So if it is $\mathbb{R}^4$, you have to have one more condition. In $\mathbb{R}^n$ you have to go all the way up to n plus 1, n plus 1 points should be every n, every n plus 1 point should be affinely independent.

Notice that the definition is stronger than saying every d plus 1 subset of A is affinely independent, d plus 1 subset you take maybe affinely independent but another subset with smaller number of points may not be. So, you may put the condition ever every d subset is affinely independent, every d minus 1 subset is affinely independent and so on. Every two subsets is independent, every one set is independent easy easy. That is all those statements are built up here in one row that is why this statement is stronger. Whether you just make this one and then arrive at the other one that is different.

These 2 conditions are equivalent only if A has at least d plus 1 elements. So, you can check that.

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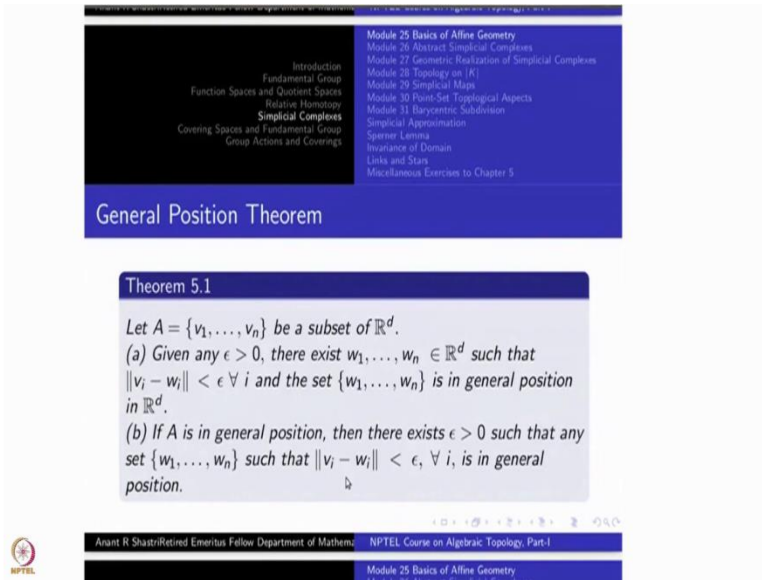

So, here is a general position theorem, which uses a little bit of linear algebra. Namely independence of sets is an open condition. In other words, dependency, linear dependency is a closed condition. So, it will be an exercise. This is what is ignored in this theorem. Take any n set $v_1, v_2 \cdots, v_n$ inside $\mathbb{R}^d$, given an epsilon positive, there will be points $w_1, w_2 \cdots, w_n$ such that they are epsilon close to corresponding v_i; $\|v_i - w_i\| < \epsilon$. Of course, if that is the only condition I could have taken wi equal to vi. But these $w_1, w_2 \cdots, w_n$ are in general position now. Say, uou take 3 points in $\mathbb{R}^3$, if they are collinear what you will do? You will move one of the points slightly away and then they will be in general position; this slightly away can be as small as you please. Epsilon could be any positive number that is the meaning of this.

Next thing is part B. If A is already in general position then there exists epsilon positive ---there every epsilon, here there exists epsilon positive such that any set $w_1, w_2 \cdots, w_n$ which satisfies this property $\|v_i - w_i\| < \epsilon$ is already in general position. So, general position is very dense that is a part A and the second part says general position is stable property if you perturb the whole thing slightly then it is still really general position. So, that is the interpretation of these two A and B here.

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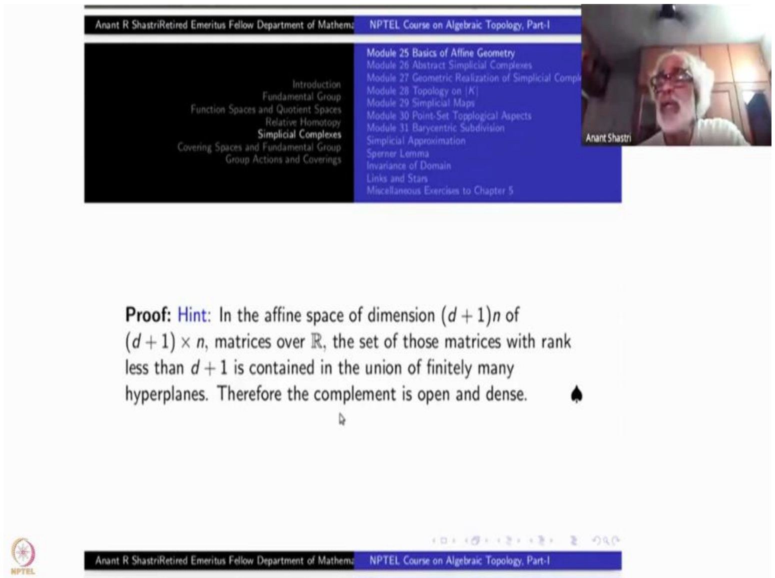

The proof is as I told you, it just follows by the dependency of vectors it is a close condition. An affine space of dimension $(d+1)n$. In this affine space $(d+1) \times n$ matrices over $\mathbb{R}$. I am taking d plus 1 rows and n columns. In other words, I am taking $\mathbb{R}^n \times \mathbb{R}^n \cdots$ d plus 1 rows. Look at that. That is of dimension $(d+1)n$. This is actually linear isomorphic to $\mathbb{R}^{(d+1)\times n}$. All those matrices with the rank less than n are contained in union of finite many hyperplanes given by vanishing of all n times n minors. That is the union of finitely many closed sets. So, for arbitrary small epsilon , you can find matrices with non vanishing n times n minor.

And once they are in the open set, you can find an epsilon such that all small neighborhoods of that will be inside epsilon neighborhoods will be inside that open set. So, that is the second part. So, I think we will stop here and next time we will start studying simplicial complexes. Thank you.