Introduction to Algebraic Topology (Part-I)
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Lecture 24
General Remarks

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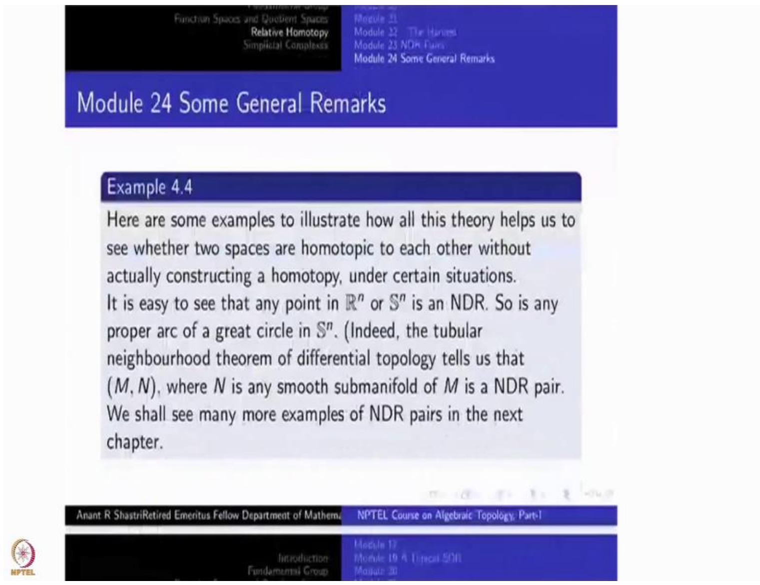

So, in this module we will try to sum up a number of things that we have done about relative homotopy. In particular, I will try to illustrate how this theory helps you to see that two given spaces are homotopic whenever they are of course, in somewhat easy situations without writing actual homotopy equivalence. To begin with you take any point in $\mathbb{R}^n$. It is an NDR, a deformation, neighbor deformation retract.

The same thing applies to Sn also, similar to that for $\mathbb{R}^n$. What is the meaning of neighborhood retract? Neighborhood deformation retract? you know all those conditions: a function u, function h and so on... Essentially it says that there is a neighborhood of the given subset, that neighborhood actually deformation retracts to the subset inside a slightly larger neighborhood. In fact, this is the case with many other examples of subsets of $\mathbb{R}^n$. For example, any smooth arc will be a deformation NDR as a subset of $\mathbb{R}^n$.

So, the tubular neighborhood theorem in differential topology if you have learned, will tell you that a sub manifold of a manifold, compact or whatever, in general being a submanifold is good enough--- forms an NDR pair. Later on, when you study simplicial complexes, you will have many examples NDR pairs.

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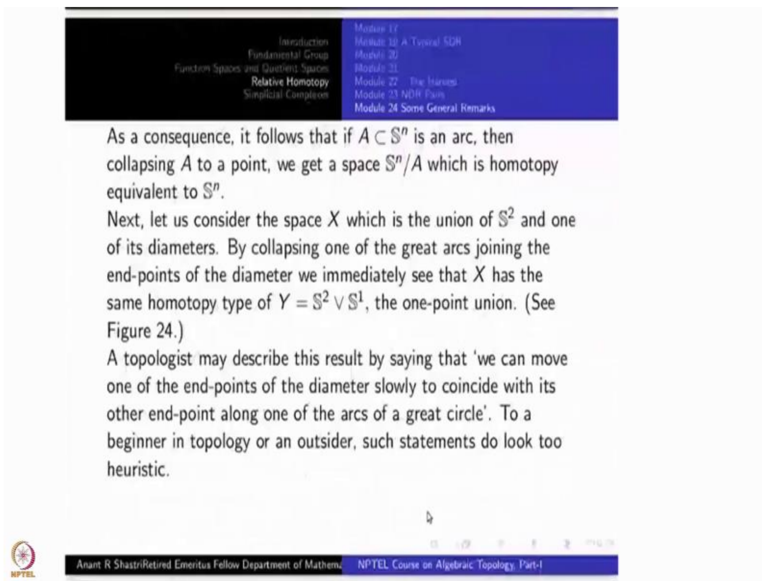

In particular, when you have an NDR pair, the inclusion map is a cofibration. That is an easier way to express the same thing right? If A is a subset of $\mathbb{S}^n$ homeomorphic to an arc, ---arc is contractable –so, I can use the theorem that the arc which is contractible can be collapsed to a single point to get a space namely $\mathbb{S}^n$ by A which is homotopy equivalent $\mathbb{S}^n$. You take any closed convex subset of Sn, collapse it into a single point, again what you get is $\mathbb{S}^n$ up to homotopy.

If we are asked to write down the homotopy equivalence every time, you will see how horrendous the task will be. But the theory is easy to remember and helps us in this. This is the point I want to tell you. Let us consider a little more complicated example. Let us take the sphere along with one of its diameters, so that is my space X. By collapsing one of the great arcs which joins the end points of the diameter, --- end points can be joined inside the sphere through a great arc right? So, take such an arc-- you collapse that arc to a single point. Earlier we have seen that $\mathbb{S}^n$ ($\mathbb{S}^2$ here in the picture) $\mathbb{S}^2$ modulo that arc is again homotopy type of $\mathbb{S}^2$. But now you have an arc there, the two end points of which come to a single point. Therefore, there will be a copy of $\mathbb{S}^1$, now a circle.

And this circle and the sphere will have only one point in common. such a thing is called one point union, it is denoted by $\mathbb{S}^2 \bigvee \mathbb{S}^1$, one point here and one point there are identified.

So, this quotient space Y is a one point union of two space, $\mathbb{S}^2$ and $\mathbb{S}^1$. So, our original space $\mathbb{S}^2$ along with this diameter is homotopy type of $\mathbb{S}^2 \bigvee \mathbb{S}^1$--which is the one-point union not disjoint union-- only one common point. Just like a bunch of balloons. If you are asked to write down homotopy equivalence here, you would have to go through the whole lot of troublesome steps of writing formulas and so on. There is no need, here is the theory.

Many topologists have this habit of not explaining this to you at all. They will just say-- Oh! this obvious! That is what they say if at all you ask. They will say --oh these 2 points can be moved on $\mathbb{S}^2$ so that they come together. That is all. We can move one endpoint of the diameter slowly to coincide with other endpoint along the arc we have chosen. But this is not a proof --if you write down all the intermediate positions, that will give you a homotopy.

In fact, all in between stages are all homeomorphic to each other except the end result, which is not homeomorphic to the original, because the two points of the diameter, the ends have come together. So the original space and this final result $\mathbb{S}^2 \bigvee \mathbb{S}^1$ are not homeomorphic to each other, but they have the same homotopy type. So, to a beginner or an outsider, all these things are matter of getting used to `hand waving and so on, but an expert topologist knows exactly what is the proof also, though he will give you heuristic arguments like this.

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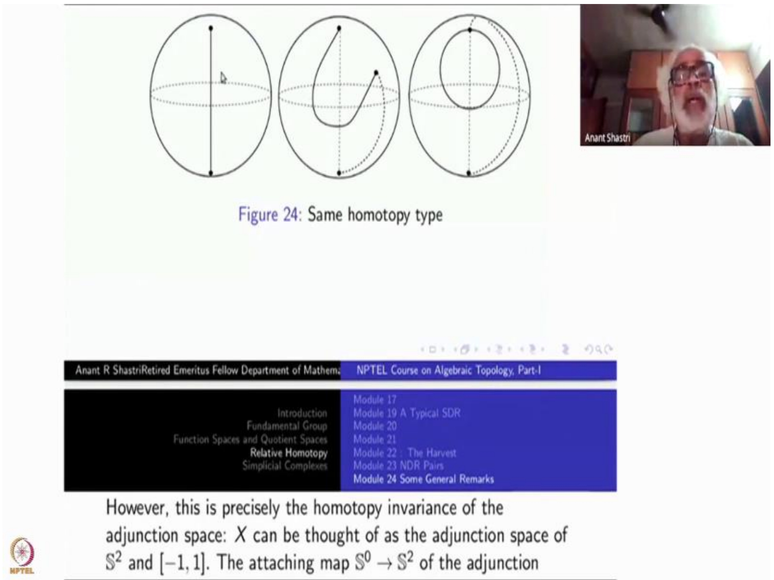

So, here is a picture. So, I have taken the sphere and this diameter one of the diameter for this space X. So, I start moving it along this arc all the way here, on the great arc. So, this is a middle stage here, the diameter has become like this, so finally it will go and all the way to coincide with this one. So, this will also become a space homeomorphic to a circle. And along with the sphere, this point is on the sphere, which if you like, may take it at the North Pole.

So, at this point, there is a circle attached to the sphere. So, this is $\mathbb{S}^2 \bigvee \mathbb{S}^1$. So, these two picture have the same homotopy type. This and this are actually homeomorphic pictures. So, you can have a homeomorphism mapping any point of the circle, any point of the sphere to any other point, keeping this point fixed, such homeomorphisms are there, of the sphere. So, you can describe this one, one way I told you-- namely you will collapse this arc but then you have to know what happened to the sphere, but that space is X by A, we are collapsing an arc-- it is same homotopy type. That is ensured by the theorem.

Alternatively, what you can do Is: You can think of this as an adjunction space. $\mathbb{S}^2$ is Y, your Z is the interval [-1,1], the diameter. What is X? X is the endpoints, what is F? F takes one endpoint to the south pole, other endpoint to the north pole. When you take the adjunction space what you get is this one. So, your X is two point space, endpoints of an interval. From that you can have different maps, mainly in this picture one point always goes to this point.

But the other point goes to different points here. Think of this as a map from $\mathbb{S}^0$ to $\mathbb{S}^2$, the two maps are homotopic to each other. The final picture is when both the points are mapped to the same point. That is also homotopic to the original map, inside $\mathbb{S}^2$. The theorem on the homotopy invariance of adjunction space says that all these spaces --whatever the adjunction space you have got, – they are all of the same homotopy type.

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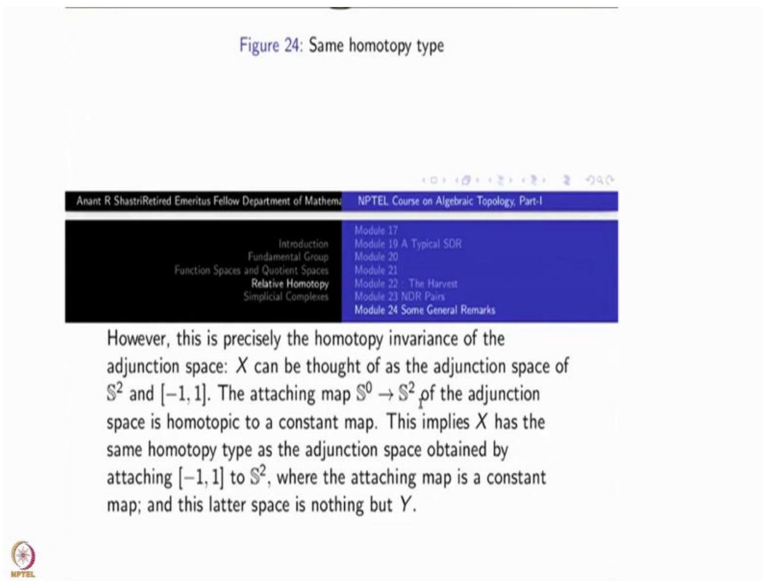

So, that is what I have explained. So, you can think of this as boundary of [-1,1] that is $\mathbb{S}^0$. The maps from $\mathbb{S}^0$ to $\mathbb{S}^2$, which keep one of the points fixed-- they are all homotopic to each other because $\mathbb{S}^2$ is path connected. Now one can give many such examples similar to whatever we have done just now.

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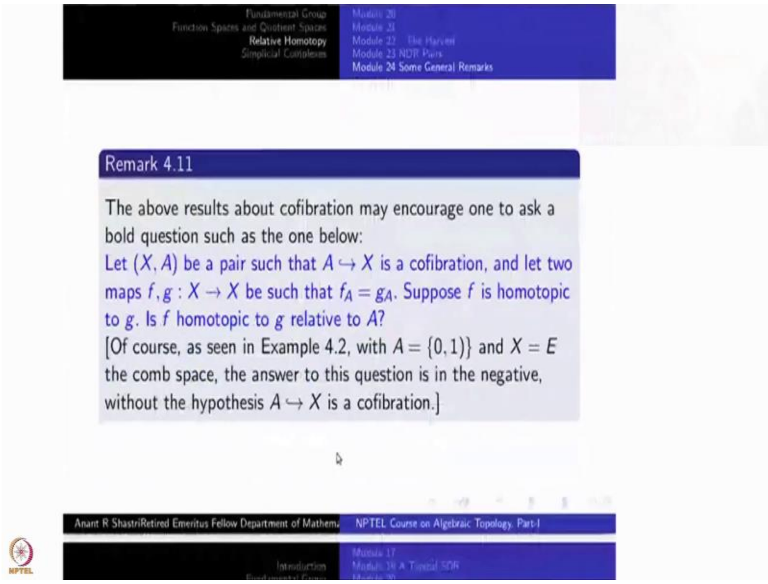

So, I come to a slightly deeper question here now, the above result about cofibration. Cofibration was essential-- necessary hypothesis in the homotopy invariance. Adjunction space performed on a subspace from which the inclusion map is a cofibration allows you to do all this.-- remember that. So, the above result of cofibration may encourage one to ask a bolder question. Suppose you have X, a topological space and a subspace A and inclusion map a cofibration.

Now, let two maps f and g from X to X be such that on A they agree,-- f restricted to A equals g restricted A. $f(a) = g(a), a \in A$; that is the meaning of this f restricted to A, and g retricted to A. Suppose f is homotopic to g. Then is f homotopic to g relative to A?

This question was motivated by the result that a weak deformation retract is a deformation retract and that if inclusion of a singleton point is a cofibration, then any deformation is strong deformation retract and so on. Remember that theorem? So, when you have just an arbitrary homotopy will it give a relative homotopy? Of course, without A being a cofibration, we know that this is not possible. So, under the assumption that A to X is a cofibration will this be true? The answer is in the negative again. That is why I said it is a bold question, but the answer is in the negative. So, this is where a real expert and a hand-waver will be distinguished.

If a person has learned only hand-waving, he will go and do this kind of mistakes, this is only an example. There are lots of people who have fallen into this kind of traps while doing algebraic topology. So, it is important to learn where your theorems come from-- how they have originated. The fundamental concepts should be very clear.

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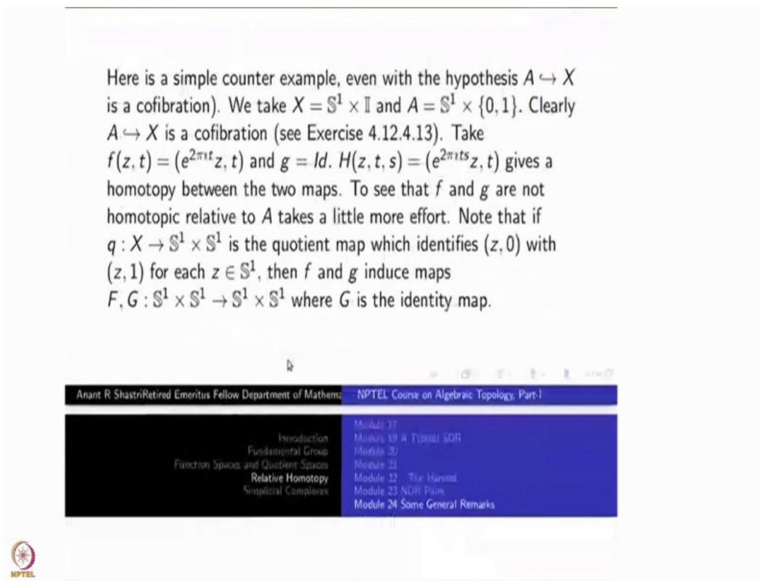

So, here is a counter example, the proof of which cannot be completed at this stage because you have to do some computations. I have given those exercises. If you have not done those exercise, then you will not be able to understand it completely. But modulo that I will explain this to you. So, what is this counter example? It is also a simple one. You start taking the cylinder $\mathbb{S}^1 \times \mathbb{I}$. X is S1 cross I, $X = \mathbb{S}^1 \times \mathbb{I}, A = \mathbb{S}^1 \times \{0, 1\}$, namely the two brims --the boundaries $\mathbb{S}^1 \times \mathbb{I}$, the 2 circles.

A to X is a cofibration so, this you have seen before. Double points in minus 1 to plus 1 or 0, 1 contained inside $\mathbb{I}$, that is a cofibration. Then you take the product of this one with $\mathbb{S}^1$. So, this is one way. There are several ways of seeing this one. The first thing is that the boundary included inside S1 cross I that inclusion map is a cofibration this is first thing to note. Now, you take the function f defined by $f(z, t) = (e^{2\pi\imath t} z, t)$.

So, I am multiplying the first coordinates that I am thinking this as a complex number of unit length, first coordinate is complex number of unit length, second coordinates real number; e power 2 pi i t times z will be another complex number of unit length-- multiplication will be again a complex number of unit length. So, f of z, t is equal to e power 2 pi i t times z comma t. Geometrically this just means that at time t, I have rotated z by an angle 2 pi t, that is the meaning of this.

At t equals 0, e power 2 i t is 1 so, there is no rotation. At t equal to 1 what will happen to this one? It would have rotated through 2 pi. So, it comes back to z. So, this map at t equal to 0 and t equal to 1 is the identity map. $f(z,0) = (z,0); f(z,1) = (z,1)$, both are identity maps. So, take this f and let g to be the identity map of $\mathbb{S}^1 \times \mathbb{I}$ to $\mathbb{S}^1 \times \mathbb{I}$.

So, this f, g are from X to X, I have two maps here. Let us look at H (z, t, s)--- namely, we want a homotopy $\mathbb{S}^1 \times \mathbb{I} \times I \to \mathbb{S}^1 \times \mathbb{I}$--- $H(z,t,s) = (e^{2\pi\imath ts}z, t)$.

If s, or t is 0, t plus s. write it as t plus s, yeah. This is not ts, this is t plus s, there is a typo here. If s is 0, I have this as identity. If s is 1, t plus s times 2 pi i, the bracket should be there, is also 1 so it will be identity. H gives a homotopy between the two maps.

I want that when s is equal to 1, I want it to be 2 pi i t times z, t. Okay okay, it is multiplication only. Sorry, no typos there. It is s multiplied t. If t is equal to 0, it is e raised to 0 which is 1 and hence yields identity map g, t equal to 1 it is f. Sorry for the confusion. This is correct. So the first map is for t equal to 0, it is identity map. Then this f when t equal to 1. This gives you a homotopy between the two maps, g to f. Finally I want to say that f and g are not homotopic relative to A. The points of A here in this homotopy are moved , they are rotated.

So relative to A they are not homotopic. If you want to keep them fixed, they are not homotopic to each other. When you want to say they are `not' homotopic, it is not just that you cannot construct one, or I cannot construct one and so on. We should show that there cannot be any homotopy relative to the endpoints, $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{1\}$, all the points should be kept intact while homotopy takes place.

How to prove that? So, that takes a little more effort. So, let us go to the quotient space namely identify $\mathbb{S}^1 \times \{0\}$ with $\mathbb{S}^1 \times \{1\}$, z comma 0 with z comma 1. Then what do we get? We will get $\mathbb{S}^1 \times \mathbb{S}^1$. So, you get a quotient space identifying z cross 0 with z cross 1 for each point z in $\mathbb{S}^1$. Then what happens to this f and g? Since they are identity maps on $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{0\}$ with $\mathbb{S}^1 \times \{1\}$ because the points corresponding to z, 0 and z, 1 are identified under this map there images will also get identified correctly.

So, let capital G, capital F be the induced maps on $\mathbb{S}^1 \times \{0\}$ to $\mathbb{S}^1 \times \{1\}$. G will be identity; F will be some other map given by f, that there is a map like this is what you have to see. Because z, 0 and z, 1 are not moved by them, it is e power 2 pi i t. That is why that makes sense here. But what is the difference between F and G? F fixes $\mathbb{S}^1 \times \{1\}$, the circle here, the image of the boundaries of this one, two boundary components.
On that both F and G are identities. But along the circle here, on the other circles here F is twisting. F is twisting the other circle exactly once whereas G is identity. So, somehow you have to distinguish these 2 phenomena and this can be distinguished by looking at the fundamental group-- application of fundamental group to show that there cannot be a homotopy between F and G if there is a relative homotopy between little f and little g namely identity, that too boils down to give you a homotopy of capital F and capital G.

Therefore, by showing that this capital F and capital G are not homotopic here, you would have proved that f and g cannot be homotopic relative to the boundaries. So finally how to do this one? For this, you have to compute the corresponding induced homomorphisms on the π_1. You have computed $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$. It is $\mathbb{Z} \times \mathbb{Z}$, that is, the free abelian group of rank 2, because $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

Using that you can compute the homomorphisms induced by F and by G. For G, it is an identity map because G itself is an identity. So what happens to F? That is what you have to figure out.

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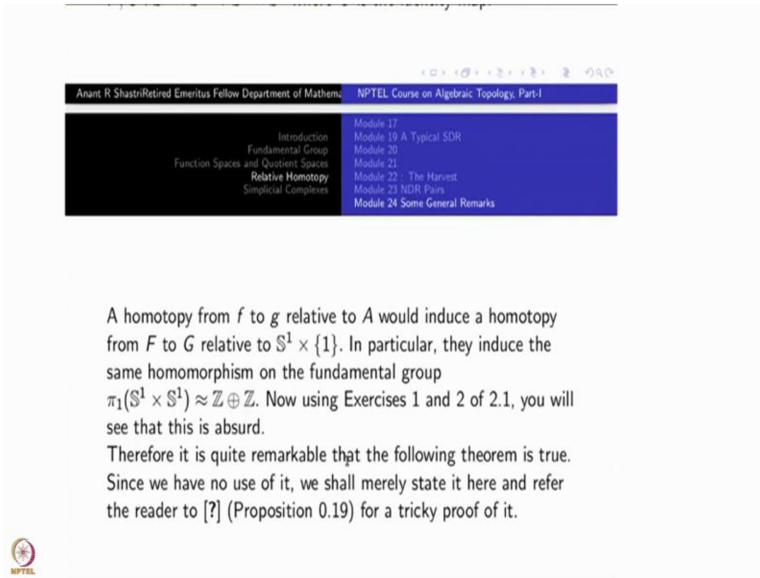

So, I have left it at this stage, say use those two exercises. You can see what are the steps. Because at the fundamental group level, the generator of one, first generator goes to identity, second generator goes to this, this generator plus that generator not just identity. First generator goes to, second generator goes identities is for G, for F, it will be it is a basically a comma b are the generators then F check of a will be a and F check of b will be a plus b. Whereas G check a is A, but G check of b is just b, because G is identity. So, this is what you have to verify.

So, finally I conclude with one of the theorems that I have read from Hatcher's book, which says that though arbitrary homotopies are not possible, homotopy equivalence is possible. Which is a very remarkable result because it is just the borderline thing and the proof is quite a tricky one. Therefore, I want to say that since I have nothing to add, I have given you a reference here. If you want to know, if you want to learn you can read it from Hatch's book.

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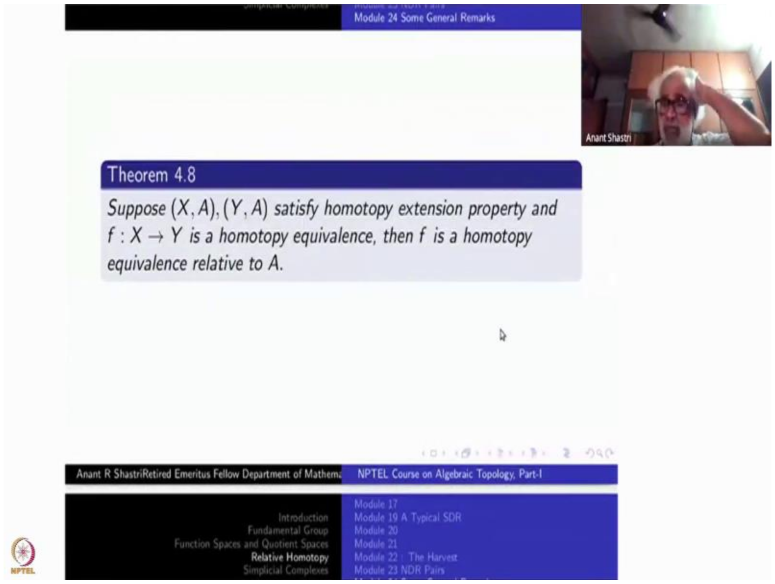

What is that? I will state the theorem. The theorem is: The pairs (X, A) and (Y, A) satisfy homotopy extension property, they are NDR pairs. Suppose F from (X, A) to (Y, A) is a homotopy equivalence. Then F is a homotopic equivalence relative to A as a pair. So, this is a wonderful result. But, the proof does not follow from anything that you have done so far. You have to cook up, cook it up by using some tricks. So, I am not going to give this proof because I am not going to use this result either.

But this was one of the questions: how to determine relative homotopy? If they are just homotopy equivalences between X and Y. Then they will be homotopy equivalent to each other relative to A. What is this A? A could be any common subspace, inclusion maps are a cofibrations, that is all. Alright, so this is the end of this session. Thank you.