Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 23
NDR Pairs

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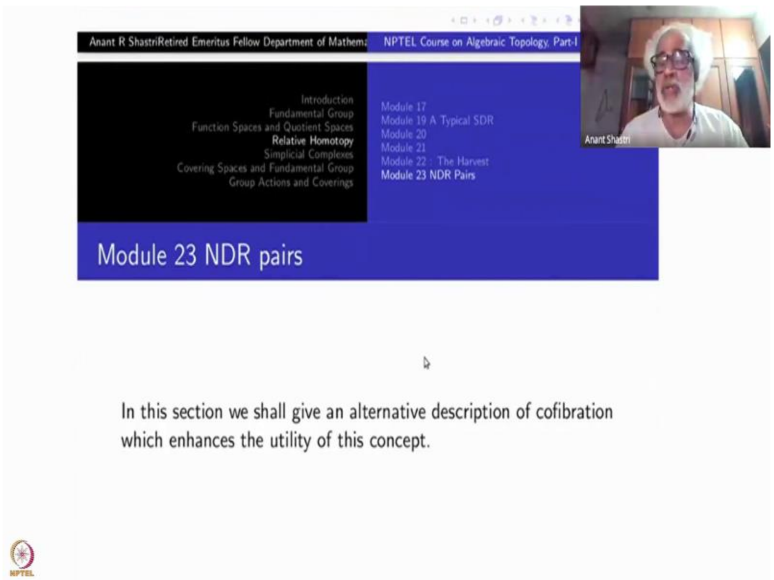

So, in this module which I have named NDR pairs, we shall describe cofibration in a slightly different way which will enhance the utility of this concept. No doubt you must have realized, no doubt now that the cofibrations are very important. Before we actually touch NDR pairs, let us do a little more work on cofibrations.

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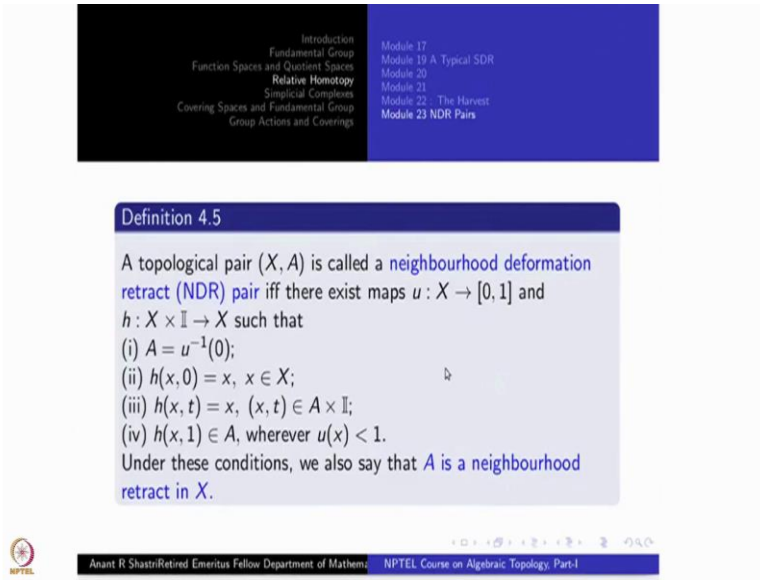

Oh! we have already done that, let me start with the definition of an NDR pair. N corresponds to neighborhood, D corresponds to deformation and R corresponds to retract. An NDR pair is a topological pair (X, A) satisfying a number of conditions,-- this is going to be a long definition, so you have to have a little bit of patience. So, you start with a topological pair (X,A), a continuous function u from X to $\mathbb{I}$ and a homotopy h from X cross $\mathbb{I}$ to X, with a number of properties.

The first property is that A is the exact 0 of the function, A is u inverse of 0. For each point of A has to go to 0 and nothing else. So, that is the meaning of $A = u^{-1}\{0\}$. That is one property of the continuous function here. The homotopy h is the identity to begin with. So, it is a deformation any homotopy which is identity to begin with is called a deformation, remember that.

It is not only a deformation, it is a constant along $\{a\} \times \mathbb{I}$. Each point of a of A, h(a, t) = a for all t. So, points of A never move under h, (a, t) will go to a itself. And the fourth property is that h1 namely when t is equal to 1, h (x, 1) is inside A. So, h is deformation namely h is a homotopy of the identity map to some function from X to A. The entire of X is folded back into the subspace A.

Sorry, not entire of X but only those points where in u(x) is less than 1. Remember, use of a continuous function into 0,1. So, points where in u(x) is less than 1 is an open subset of X and it contains A because on A, u(x)=0. So, this set where u(x) is less than 1 is going to be a neighborhood of this A in X and that neighborhood is pushed into A. The map which pushes is homotopic to identity. Now, you can see why the name deformation retract.

The neighborhood deforms into A and A is a retract of that neighbourhood. So it is a deformation retract of a neighbor and that neighborhood is precisely all points such that u is less than 1. The neighborhood is not given by just some topological condition. But a strong condition, namely it is open set given by a continuous function. It is u inverse of the interval [0,1) , which is open in closed interval [0,1]. So, this looks like too much of a condition. But you will see that this is not a new concept at all.

So, under these conditions, we also say that A is a neighborhood retracting X. The whole X does not retract onto A. But A is a retract of a neighborhood of itself. You can just say that much but that looks to be weaker than giving the neighborhood by a function like this. At least it looks like it. But maybe it is not weaker. So, we have the strongest condition then we will see that this is equivalent to whatever we have been studying all the way under the name cofibration.

So, this notion is due to Steenrod. G. W. Whitehead, developed this notion very thoroughly and he has contributed to homotopy theory a lot. There are two famous Whiteheads in homotopy theory, one is J. H. C. Whitehead, another one is G. W. Whitehead, who has contributed more is difficult to say. So, they are quite comparable in some sense.

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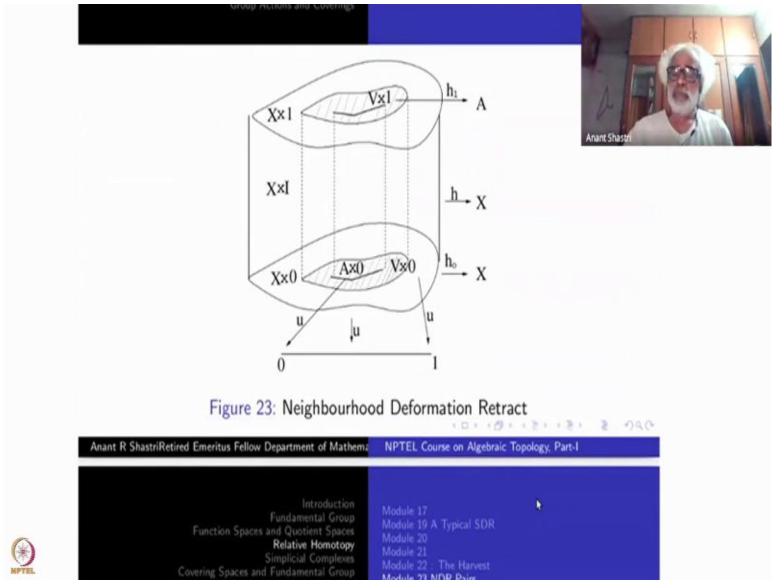

So, here is a picture of those conditions 1, 2, 3, 4. You start with an area in the plane as X. Then this line little bit of a line segment here, is A. It is a closed subset, all right. Then you have a neighborhood of it, open neighborhood here. How is this function u? It is a function from this X to the interval 0, 1, this is the 0, and this is the interval. This u is a continuous function. The entire of this line segment here is going to the single point 0 under u. This portion outside the shaded region is going to 1. That just means that the shaded region here is the inverse image of 0 closed, 1 open, under u.

So, this is the picture of what is happening at X cross 0 inside X. So, we use the inverse image of 0 closed, 1 open under u, so that is V. Now look at X cross I, there is a homotopy $h : X \times \mathbb{I} \to X$ h. It is identity on this $A \times \mathbb{I}$. It is identity on X cross 0 on the bottom, that is h naught. And this entire V cross 1 has gone inside A. So, under this condition, we say that A is A is a neighborhood deformation retract, NDR pair.

So this is just the definition, you can visualize it, in this picture.

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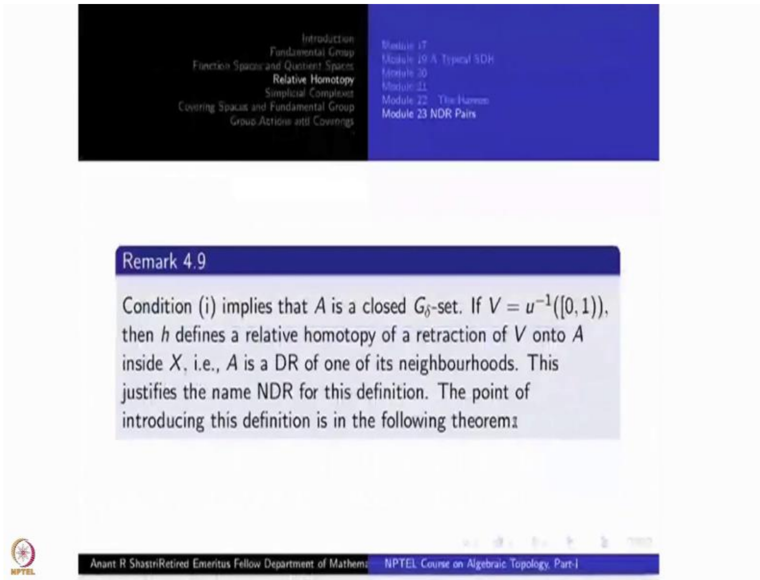

Condition 1 implies that, u inverse of 0 is equal to A implies that A is a closed G_δ-set. Because it is the inverse image of single point, automatically it must be closed. In fact, it is intersection of open sets--- take 0 to 1 by n open, take the inverse image and then take the intersection over all n. That is it. That is why it is G_δ-set . The meaning of G_δ? If you know it, alright, if you do not know G_δ, forget about it. At least you can see that A is a closed subset.

Put $V = u^{-1}[0,1)$. That is what I explained before. Then h defines a relative homotopy. Relative to what? Relative to A of a retraction of V on to A in the side X. You see, that is why you have to be careful here. It is not in V, that the deformation is taking place. A is a deformation retract of V is exactly correct because it is not happening inside V.

So, so, we have to be bit careful about in these kind of wordings here. That is why you have to stick to the definition finally. So, that is what I wanted to draw your attention to. You know h is a relative homotopy of a retraction of V on to A inside X. The homotopy is taking place inside X. A is a retract of one of its neighborhoods and you can say is homotopy equivalent inside X to id of V. This justifies the name neighborhood deformation retract.

The point of introducing this definition is, not because it is a new concept, but the fact that the following 4 conditions are all equivalent:

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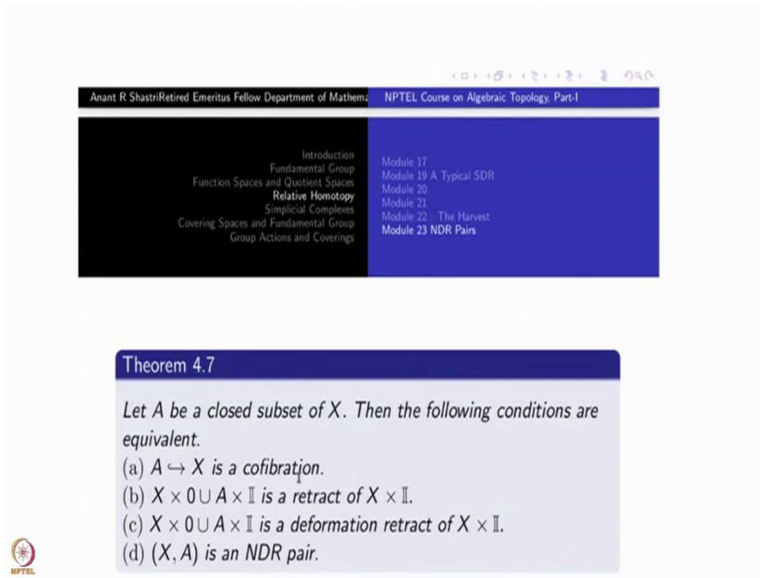

The last condition is the new definition (X, A) is an NDR pair. What is the first one? A is a cofibration. Standing assumption is that A is a closed subset, you start with that. Then (a) A is a cofibration, (b) X cross 0 union A cross I is a retract of X cross I. This is our proposition 4.1 whatever we have been using this one. (c) X cross 0 union A cross I is actually a deformation retract of X cross I. This part, I used in the previous lecture. Maybe I did not prove it so far. So, now, we will prove that one also. So, you do not have worry for this one.

The last condition is (X, A) is actually an NDR pair. So, you see cofibration is giving you all such data, so much of data and it is equivalent.

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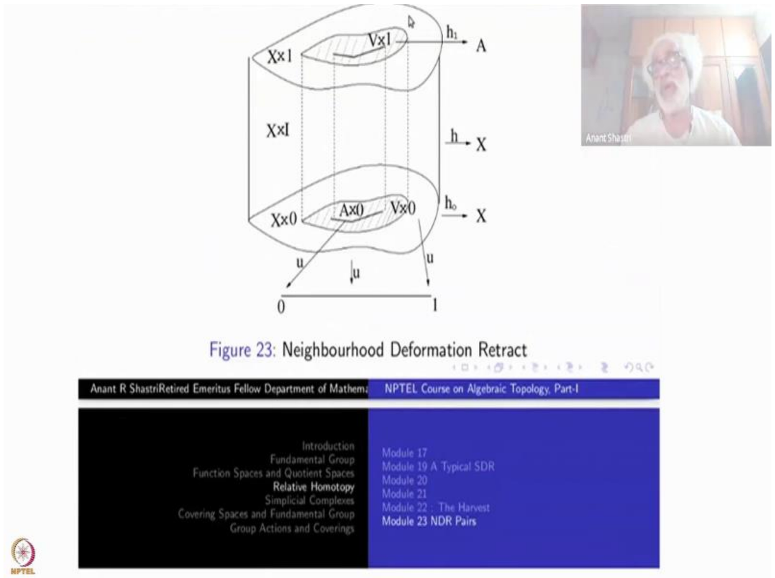

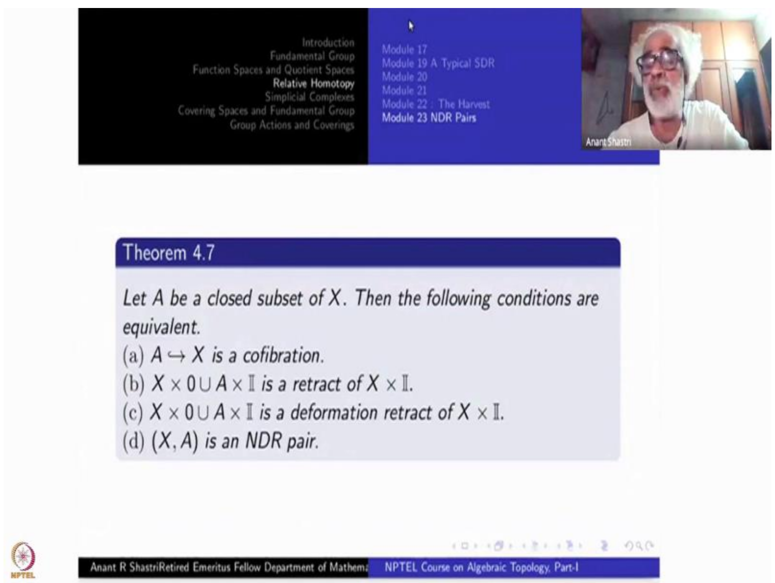

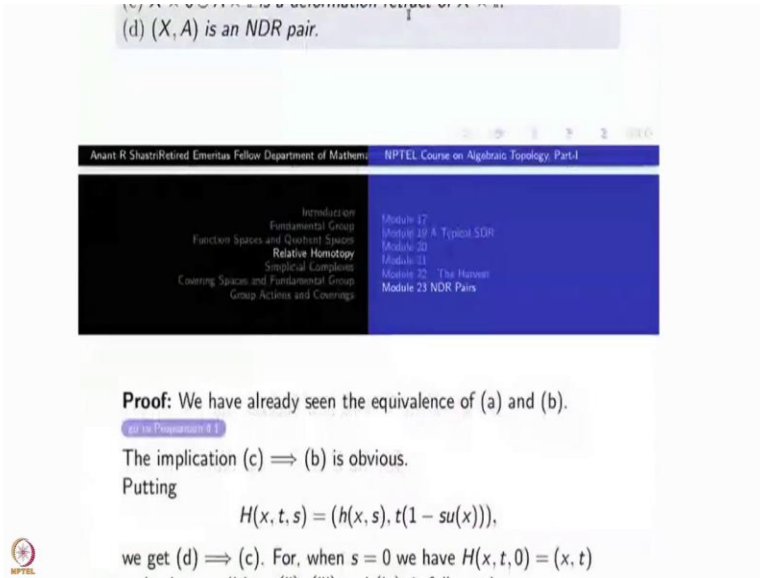

So, just using the cofibration, we have to construct all these things that is the task here. If you have an NDR pair with all these things, proving that is a cofibration is not difficult. So, we shall prove that these 4 conditions are equivalent. Thus, cofibration is a such a powerful notion. That is what we have come to know. Nevertheless, we also know that it is very common, it is occurring in lots of cases. Only very very bad cases like the comb space etc., you will get counter examples, right?

All right. So the proof of this theorem, finally.

We have already seen in proposition 4.1 that (a) and (b) are equivalent. So, this is just a repetition here. So, we don't have to worry about the equivalence of (a) and (b). Now, I want to prove that (c) implies (b). That is also easy, because a deformation retract is a retract.

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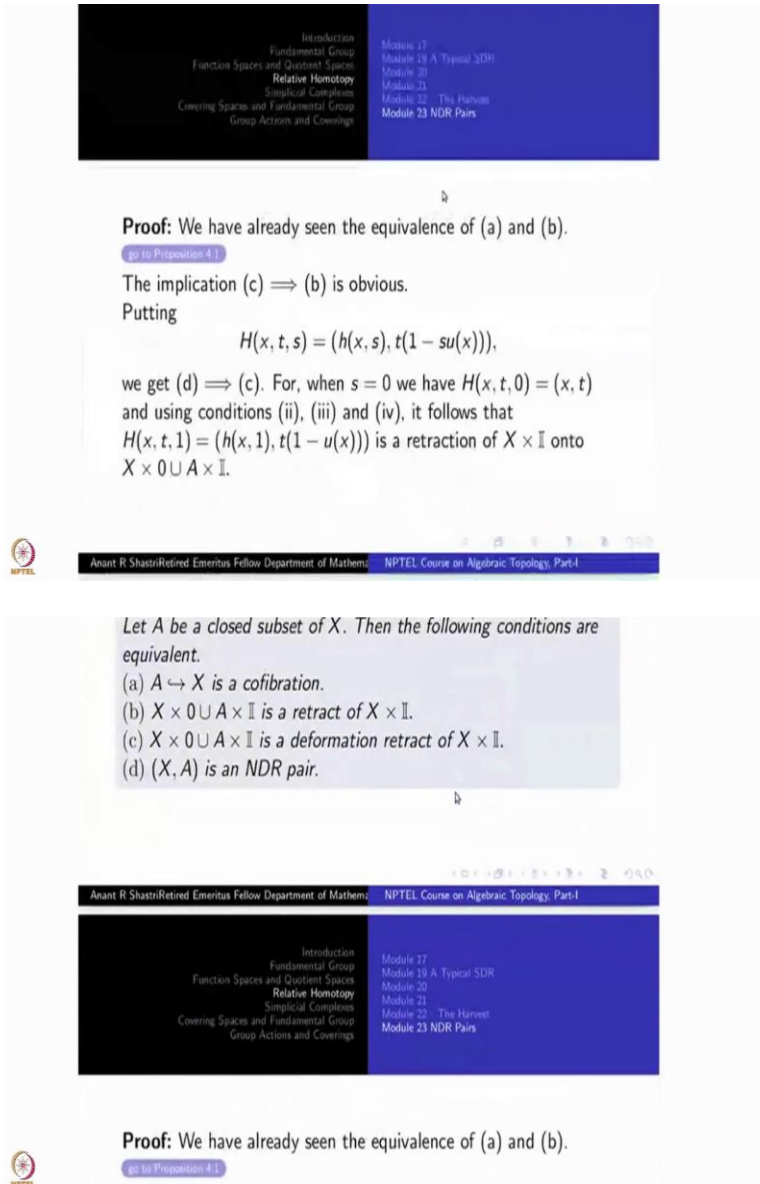


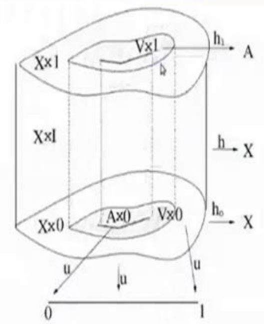
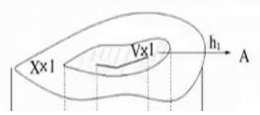
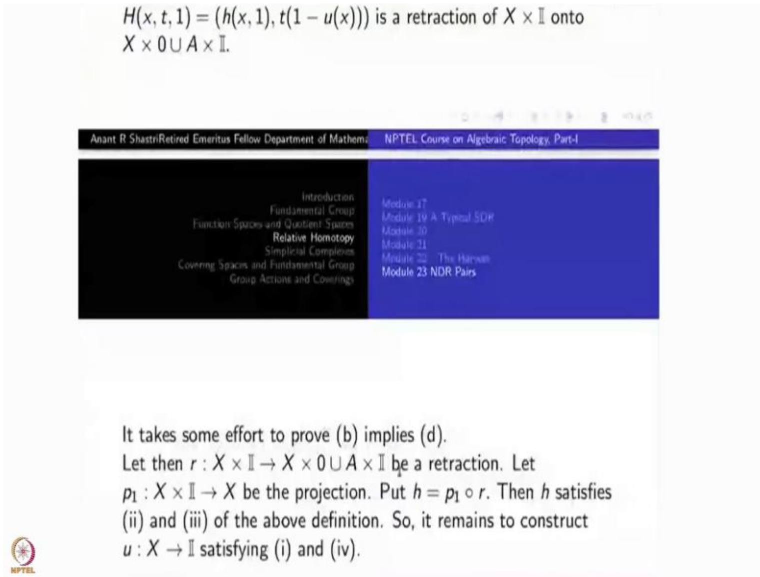


Figure 23: Neighbourhood Deformation Retract

retract (NDR) pair iff there exist maps $u : X \to [0,1]$ and $h : X \times \mathbb{I} \to X$ such that
(i) $A = u^{-1}(0)$;
(ii) $h(x,0) = x,\ x \in X$;
(iii) $h(x,t) = x,\ (x,t) \in A \times \mathbb{I}$;
(iv) $h(x,1) \in A$, wherever $u(x) < 1$.
Under these conditions, we also say that A is a neighbourhood retract in X.





Putting $H(x,t,s) = (h(x,t), t(1-su(x)))$ we get (d) implies (c). For, when s=0, we have H(x,t)=(h(x,0), t)=(x,t). And using conditions (ii), (iii) and (iv), of the definition of NDR, it follows that $H(x,t,1) = (h(x,1), t(1-u(x)))$ is a retraction of $X \times \mathbb{I}$ onto $X \times \{0\} \cup A \times \mathbb{I}$.

[**Ignore the following; There is a mathematical error. The correct proof is already given above.**

Now, we will prove (d) implies (c). For this you have to do a little circus, namely, start with the homotopy h, that is given by (d). We have to define capital H. h is the homotopy which is given by the hypothesis (d). Put the first coordinate of H of x, t, s equal to h of x, ts -- Multiply the t and s here in the second coordinate. Remember h of x, s is inside X. H is from X cross I cross I to X cross I. So the 2nd coordinate is t times 1 minus s of u x; u x is a function from X to I, so s times u of x is also inside 0,1, 1 minus that makes sense. This number is always less than or equal to 1. Therefore, 1 minus that will be also between 0 and 1. You can multiply it by t again you get to point in 0 and 1. Thus second coordinate takes value inside 0 to 1. So, this is a map from X cross I cross I to X cross I. Take capital H to be this map. The two coordinate functions of H have been defined here. H of x,s,t is equal to h of x, st, followed by t times 1minus s into u of x.

I want to say that this will already imply (c). What is c? Just read. What is (c)? X cross 0 union A cross I is a deformation retract of X cross I. I have given this a deformation retract. Let us verify it . If s is 0, what is H of x, t, 0? This is 0, h of x, 0, by definition, h of x, 0 is x. Since s is 0, s into u x is 0, this is just t. With the first coordinate x and second one t, that is the identity map. So, to begin with H is identity map. When you use condition 2, 3 and 4 together, it will imply that H of x t comma 1 at the end, what happens? when you put s equal to 1, this is just t times 1 minus u x. But what is this is h of x, t, right?

So, h of x, 1, t times 1 minus u x. This is a retract of X cross I on to X cross 0 union A cross I That is what I want. Why? This whole thing takes value either in X cross 0 or inside A cross I. When it is X cross 0 means what? This is 0, either t is 0 or 1 minus u x is 0 then this will be 0.

Finally, if the second coordinate is not 0 means what? t should not be 0 and u x should not be 1. That means u x should be strictly less than 1. When u x is strictly less than 1, look at the condition 4 here, when u x is strictly less than 1 implies h of x, 1 is inside of A, this last condition 4. So, coming back here, when this is not 0 it is inside A cross I, this is in A, A cross I. So, this map actually takes values X cross 0 in A cross I. Suppose you take t equal to 0 and t equals 0 this is 0 this will be a H of x, 1 which is x.

So, this is identity X cross 0 on X cross 0 is I, similarly on A cross I as soon as something is in A, remember u is identical is 0 on A. Therefore, it is H of x, 1 comma 1. What is H of x, 1 comma 1? Look at his condition as X is inside A it is identity for all t. Therefore, this is identity on X cross 0 this map is added to an X cross 0 union A cross I therefore it is retraction. This retraction is homotopic to identity therefore it is deformation retraction.

Thinking of this is something new and I myself do not know how to explain this. See there is a circus here, the third coordinate here s second coordinate here H, H of x, s. This is something s into 1 minus s kind of thing but you are not multiplying by 1 minus s you are multiplying by s and then subtracting. These are all quite somewhat weird way of writing. Of course Whitehead does not explain this one even this much whatever I explained.

So I am explaining to the extent I understand this. So it is quite tricky thing. But more or less, you know, forced on you to, if you are not given this one by just looking at this and trying to do what you want to you will have to come to this map. So, what we have done is d implies c implies b implies a, a implies b this much we have done. What we have to show is actually we will directly show that b implies d that the cycle is complete.

We do not need this one. If you use this one it may be easier it is not. **Ignore up till here.]**

So, we will just assume X cross 0 union A cross I is a retract and show directly that (X,A) is an NDR pair.

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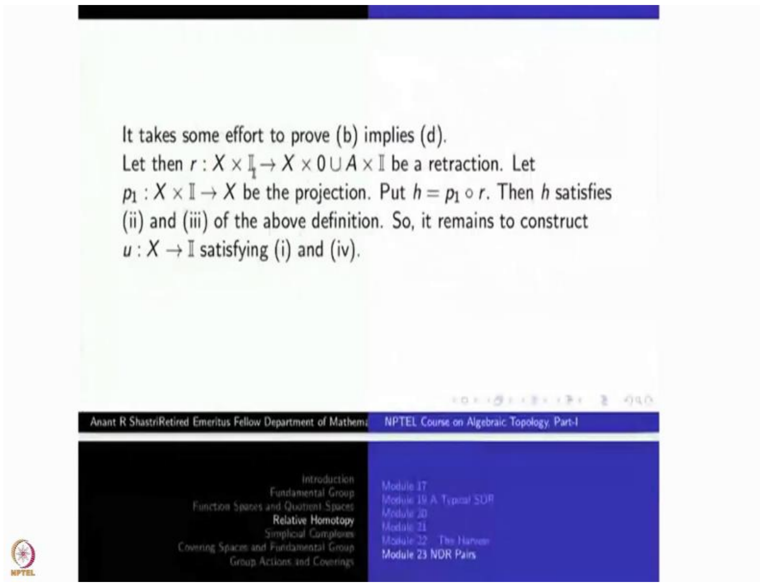

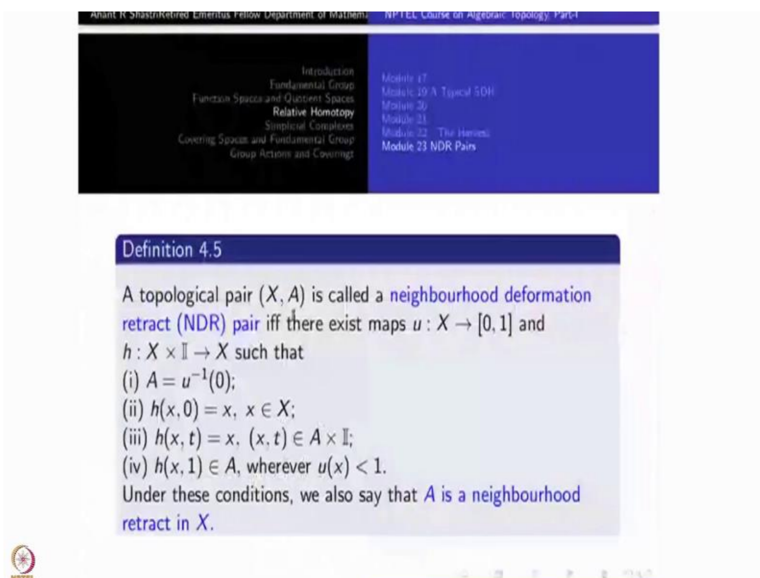

So, this is the hardest part-- to prove (b) implies (d). So, let us denote the retraction from X cross I to X cross 0 union A cross I by r. Take the first projection X cross I to X, say p_1, put $h = p_1 \circ r$. So take r and then follow it by first projection. This whole space is contained inside X cross I. So, you can take p_1 and $p_1 \circ r$. This map h satisfies condition (ii) and (iii) of the above definition. What is condition (ii)?

h of x, 0 is x. On X cross 0, r itself is a identity, so its project into the first coordinate is just x. What is (iii)? h of x, t is x for every x, t in A cross I. For again on $A \times \mathbb{I}$, r is identity and when you project it to the first coordinate you get x. (The second coordinate is t.) So that is what we are doing here. So, it satisfies (ii) and (iii).

So, what you have to do now is to get this continuous function $u : X \to \mathbb{I}$, which is identically 0 on A and satisfies condition (iv), namely the set of points wherein it is not 1 is going inside A by h, by our already defined h. So, we are adjusting u to suit this h, so that condition (iv) is satisfied.

You want to have a continuous function like this you know it looks like Tietze's extension theorem. You are given the 0 function on a closed subset A-- entire A goes to 0. You have to extend it over X. It is more than that of course, namely, I do not want u to have any other zeros. So, I was a bit skeptical about this till I read the proof, this is one thing which I did not prove by myself-- I have read it from Steenrod. It is interesting and not so difficult, finally.

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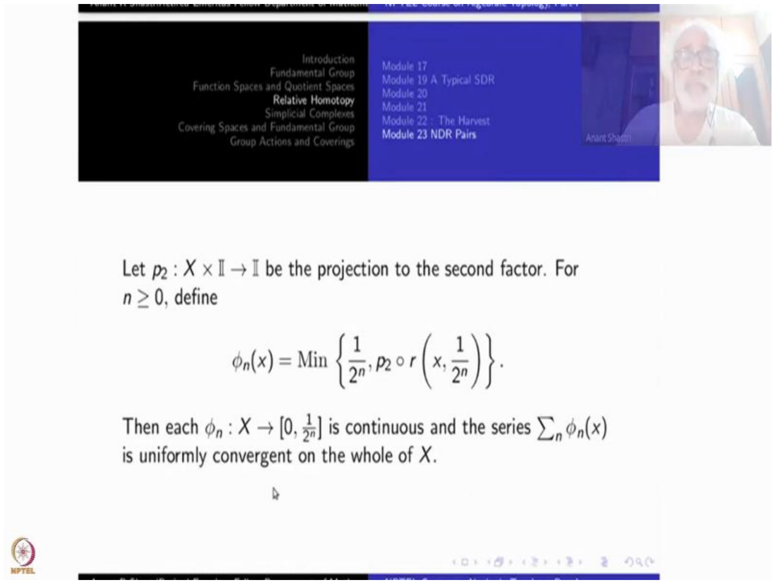

So, let us take the second projection $p_2 : X \times \mathbb{I} \to \mathbb{I}$ the second projection. Now, for each integer n greater than or equal to 0, let us define a function ϕ_n to be minimum of 1 by 2 power n and the value of p2 composite r of x comma 1 by 2 power n. Remember r is a function from $X \times \mathbb{I}$ into $X \times \{0\} \cup A \times \mathbb{I}$. So, I can talk about the second projection of that, that will be an element inside $\mathbb{I}$.

So, this is some real number between 0 and 1 and you see, 1 by 2 power n you take the minimum of the two. This is a continuous function. This is a constant so ϕ_n will be a continuous function, minimum of the two continuous functions is continuous. So, I have got a sequence of continuous functions taking values between 0 and 1 by 2 power n, -- because if something is bigger than 1 by 2 power n, ϕ_n is 1 by 2 power n here.

So, it's maximum is 1 by 2 power n, the minimum of these two. So, it is taking values in $[0, \frac{1}{2^n}]$ 0 to 1 by 2 power n. Therefore, if you take the series summation $\phi_n(x)$, sum it over n, this will be convergent. Not only it is convergent, it is uniformly convergent for all x. Why?-- it is dominated by the summation 1 by 2 power n-- the geometric series. So, Weierstrass's majorant theorem tells that it is uniformly convergent.

Once you have a uniformly convergent series of continuous functions, the limit function, here the summation will be itself a continuous function.

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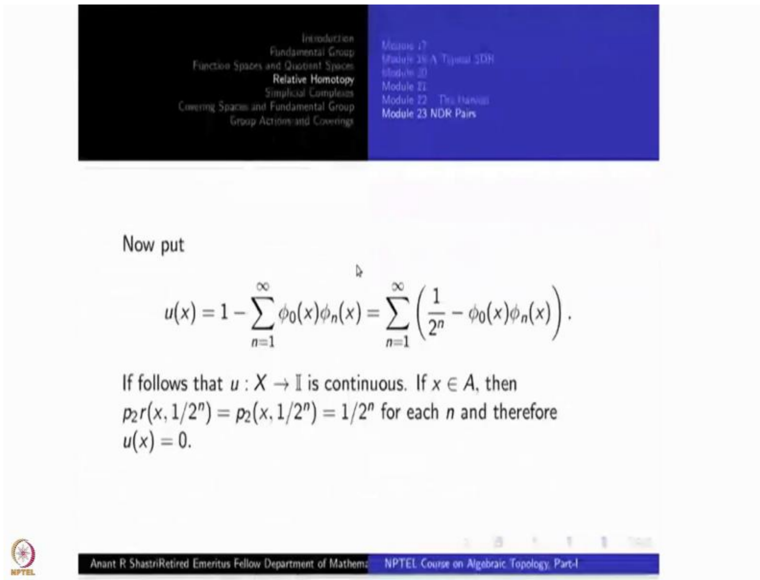

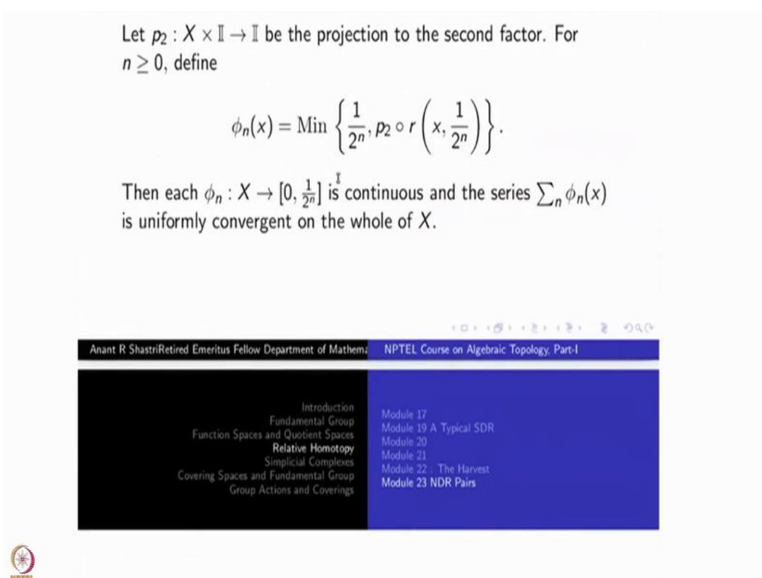

$$p_2 \circ r(x, \frac{1}{2^n})$$

So, most of our work is done now. You put to $u(x) = 1 - \sum_n \phi_n(x)$. To make sure something works fine, I should multiply it by $\phi_0(x)$. So, $\phi_0(x)$ can be multiplied outside the summation also. Because this is the common thing you can take it outside also if you like. This series is series non negative terms which is absolutely convergent. You can try this one on this side here also or you can rewrite the whole thing because one is equal to the summation 1 by 2 power n, 1 to infinity ---1 by 2 plus 1 by 4, etc., equal to 1.

So, you write is 1 by 2 power n minus phi naught of x into phi n of x and then take the summation. So, this u can be defined either this way or defined this way so, it has two formulas. So, this formula defines our function u; why it is continuous? Because summation phi n is continuous multiplying by a phi naught of x is continuous, 1 minus that is continues. So, I have got a continuous function u.

Each of these terms in the summation phi n of x, is between 0 and 1 because they are dominated by 1 by 2 power n summation 1 by 2 power n is equal to 1 so, these are smaller than that. So, this entire thing takes value between 0 and 1. So I have got a continuous function on X namely this u. So, you have got a continuous function. Now if x is inside A, then you get $p_2 \circ r(x, \frac{1}{2^n}) = \frac{1}{2^n}$.

Go back here in the definition of ϕ_n. ϕ_n will be now minimum of these but this is equal to 1 by 2 power n so, ϕ_n(x) will be equal to 1 by 2 power n for all points x inside A. Therefore, u(x) will be 0. If x is inside A then p_2 itself is 1 by 2 power n for each n. It's summation will be equal to 1 and $\phi_0(x)$ is just 1.

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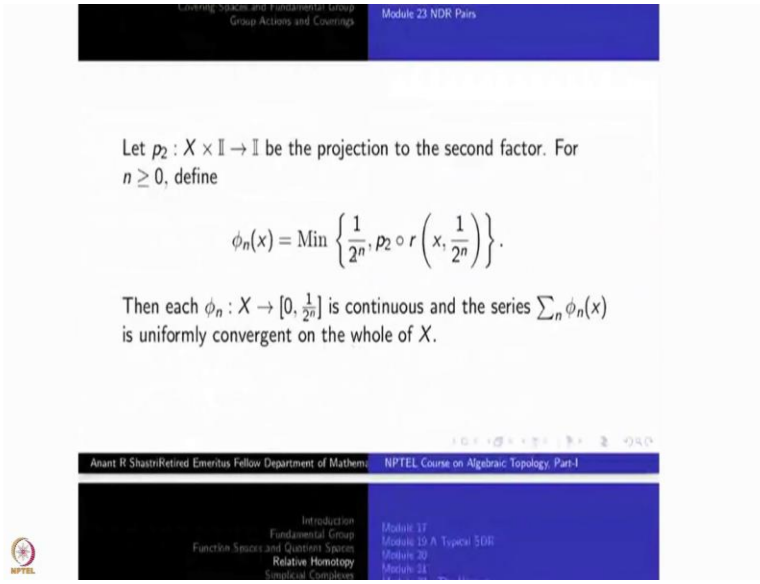

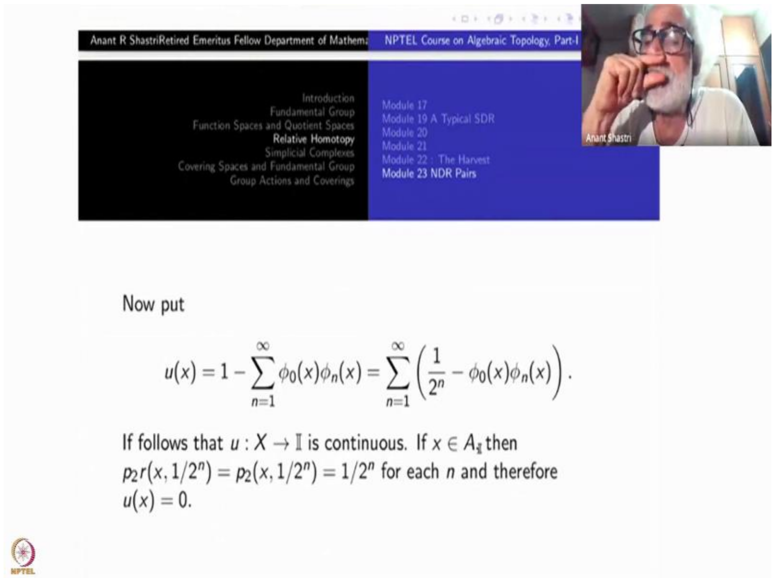

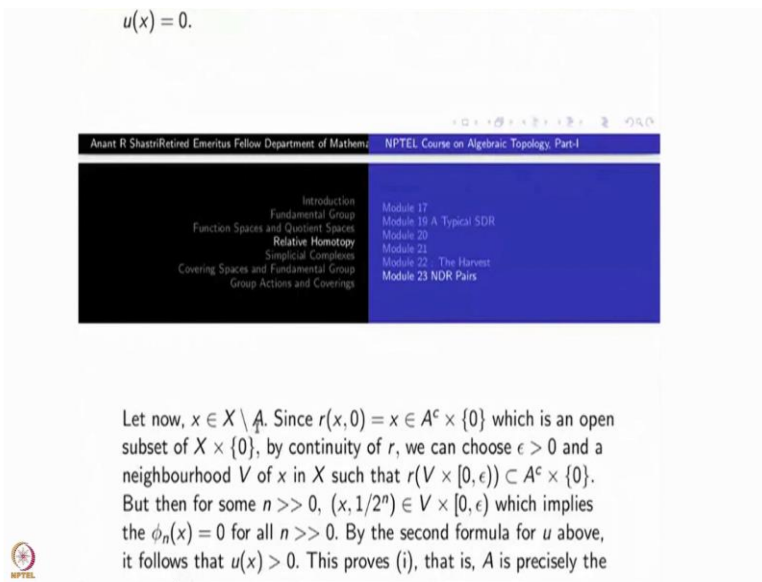

So, therefore, this summation here, this is equal to 1 therefore, u vanishes here. So, A goes to 0. So, u(x) is 0 if x is inside A. Now, I have to prove that if x is outside A then u(x) is not 0. Because I want to prove that A is exactly u inverse 0. So, take x in the compliment of A. Then $r(x,0) = (x,0)$, because r is the identity map on X cross 0. But x is inside compliment of A.

I am taking A cross 0 as a subset of X cross 0, r(x, 0) is x comma 0. Since A is closed in X, A complement cross 0 is an open subset of X cross 0. By continuity of r, we can choose epsilon positive and a neighborhood V have x inside X such that r of V cross 0 to epsilon is inside A complement cross 0. See, (x,0) is in A complement cross 0 and it is an open set, therefore, some neighborhood of this point must go inside this set -- that is all I am telling.

How does the neighborhood of x cross 0 look like inside $X \times \mathbb{I}$? V cross 0 to epsilon, right? Now, this whole thing must be inside $A^c \times \{0\}$. A^c is compliment of A in X. But if this happens, if you take n very large, then x comma 1 by 2 power n will be inside here, in this neighborhood of x comma 0, and under it is going inside $X \times \{0\} \setminus A \times \{0\}$. So, x comma 1 by 2 power n is inside here-- that means r of that takes values in $A^c \times \{0\}$.

What does that mean? p_2 of this point is 0. Therefore, ϕ_n itself will be 0 because ϕ_n is a minimum of 1 by 2 for n and that value phi n is 0 if n is sufficiently large. Look at this, look at this expression here, after a certain stage all these terms will be 0. Even if you take all of them to the maximum possible, the sum will be 1. So, most of them are 0 and only for a few of them are left, some total of them will be less than 1 by 2 power n. Therefore the sum total will never be equal to 1.

That means 1 minus some smaller number so, it is never 0. Therefore, u(x) is positive.

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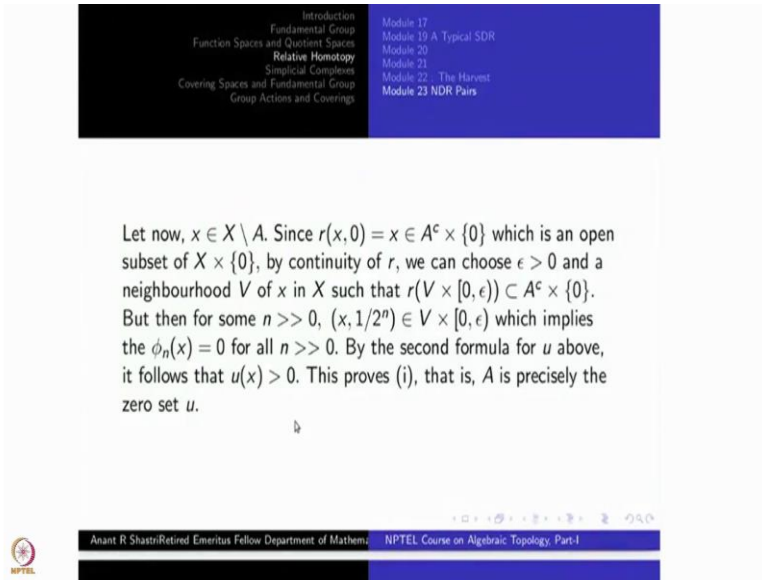

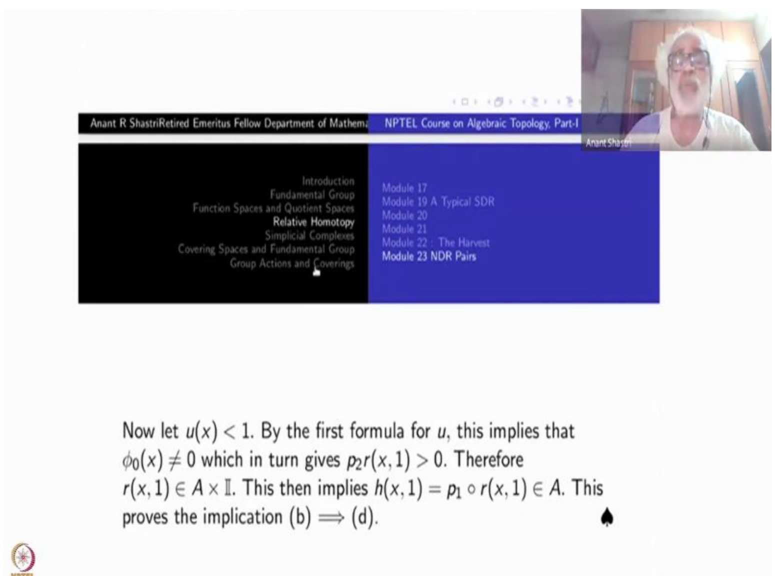

This proves (i) namely, A is precisely the 0 set of u. So, major thing is done. One more thing is there now. When u(x) is less than 1, we have to show that the homotopy, h(x,1) is inside A. Suppose u(x) is less than 1. That is u(x) is not equal to 1 that is all about it.

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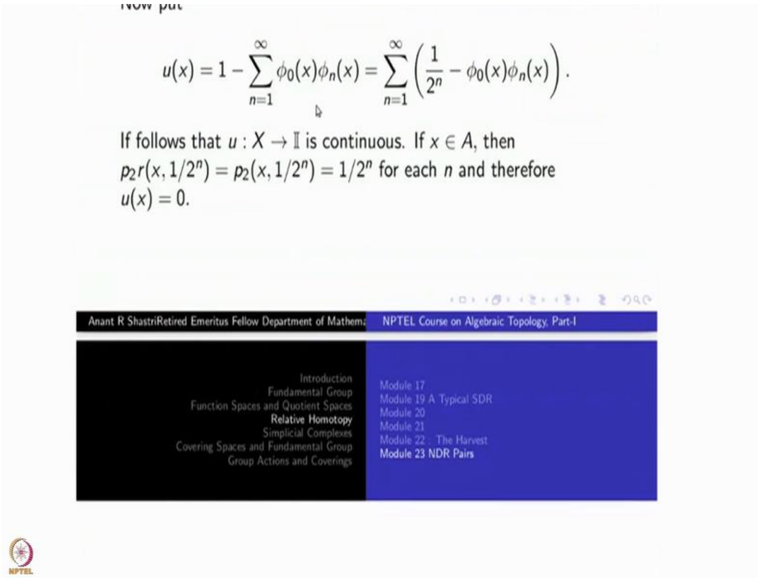

Well, look at the first formula for u. This part it is not equal to one means what? whatever you have subtracted is not 0. u(x) is less than or equal to 1 right? It is not equal to 1 means this is not 0. So, one of this would be not zero. If $\phi_0(x) = 0$ then the whole thing would have been 0. In particular, $\phi_0(x) \neq 0$ right? So, that is what I wanted here.

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So, if u(x) is less than 1, this formula this implies that $\phi_0(x) \neq 0$. Other terms may or may not be 0, phi naught is 0 then whole thing is 0. So phi naught is not equal to 0. phi naught is not equal to 0 would imply $p_2 \circ r(x, 1) > 0$. Look at the definition of ϕ_0, when n equals 0 , is minimum of 1 and p2 of r of x comma 1, the minimum is not 0 then this is not 0. So this implies that the second coordinate, $p_2 \circ r(x, 1) > 0$.

Remember that r takes value in $X \times \{0\} \cup A \times \mathbb{I}$. So, it cannot be in X cross 0, because the second coordinate is positive therefore, So, it must be in A cross I. So, this implies that h (x, 1) is $p_1 \circ r(x, 1)$ which is the first coordinate, must be inside A. p_1 of r x, comma 1, this is the definition of h. Because I have taken h to be p_1 composite r. So, this is inside A.

That is a beautiful proof. So, in summary, simple looking cofibration implies all this stuff.

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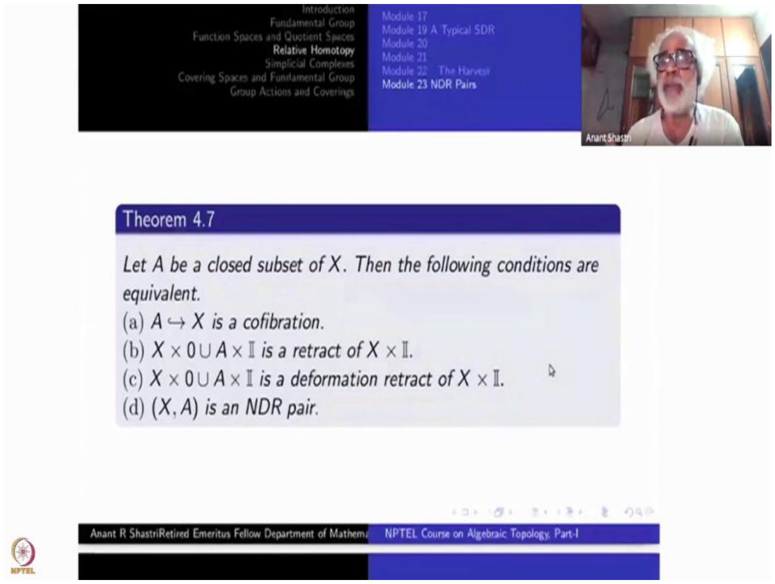

So, somehow, the kind of thing that usually happens inside any Euclidean space has come back to us by just by this assumption cofibration. The fact that A is $G_\delta-$set, for example. You know just you are assuming A is a closed subset you were not even ready to assume that either. Anyway, you assume A is a closed subset then it is G delta set blah blah blah and a small neighborhood of it can be retracted to A, deformation retracted to A.

All these things happen in nice neighborhoods, inside $\mathbb{R}$, $\mathbb{R}^2, \mathbb{R}^3$ and so on for nice subspaces. Inside $\mathbb{R}^2$itself, there are weird spaces like the comb space. Another example called `zebra' has every point a bad point of that space. And this is a sub space of $\mathbb{R}^2$ Every point is a bad, bad point means what? It is not a cofibration, inclusion map of the singleton is not a cofibration. This space is `locally disconnected' at all points.

So, this more or less completes whatever we have planned of homotopy theory. From now onward, we will study special kinds of things namely we will specialize to spaces which are built up of straight lines, segments, triangles, tetrahedrons and so on. So that we have plenty of homotopies, plenty of line segments to join two points and so on, when they are nearby. Thank you.