Introduction to Algebraic Topology
(Part-1)

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The Harvest
Lecture No. 22

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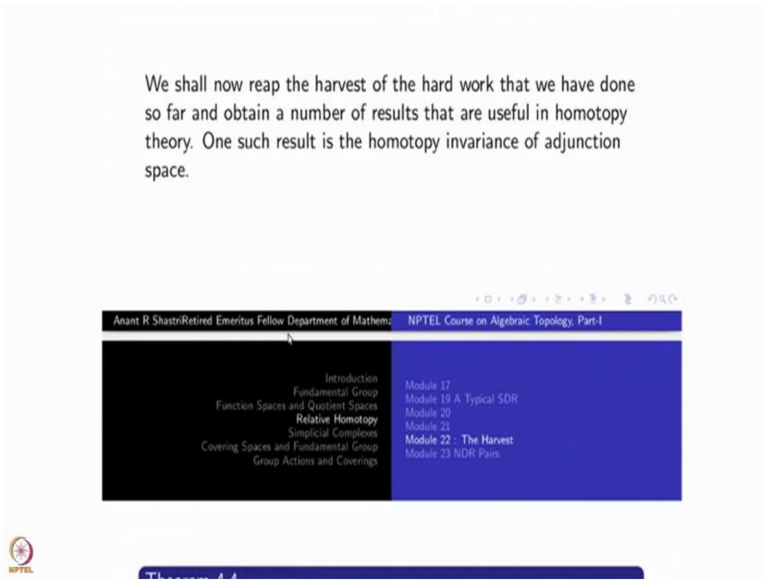

Today's session I have called The Harvest. Because whatever hard work we did for past few days we are going to reap it now. A lot of results will be prove using whatever hard work we have done for past few days. One of the landmark results will be the invariance of the adjunction space. Details we will see in what follows now.

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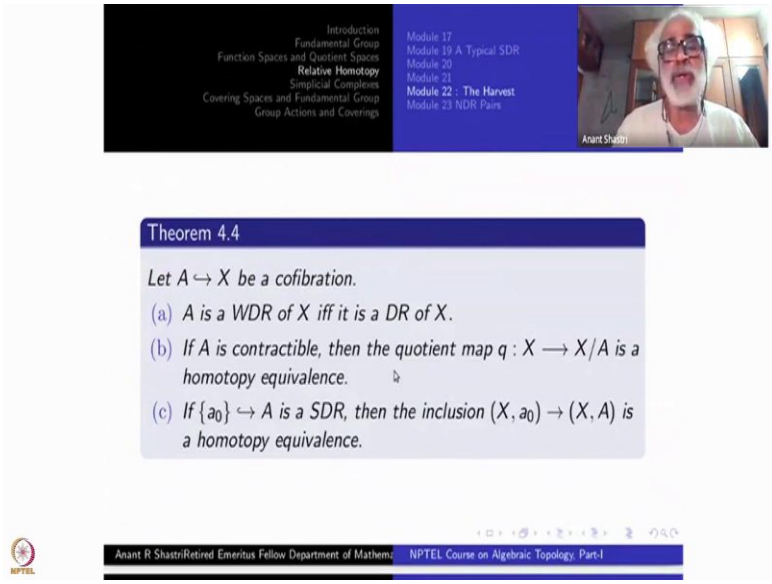

So the first thing is this result which relates the weak deformation retract, the deformation retract and strong deformation retract. We know that these things are one stronger than the other. But when would they be equal to each other? That is the kind of thing we want to see. The first thing says the following: you take a cofibration from A to X. A is a subspace which is closed, A to X is a cofibration. Then the first thing is that if A is a weak deformation retract, then it is a deformation retract.

The `only if' part is what we need to verify. If it is a deformation retract, it is a weak deformation retract-- that is obvious. So if A is a weak deformation retract of X if and only if it is a deformation retract of X. The next thing is namely (b). If A happens to be contractible, then as if the entire subspace A can be thought of as a single point homotopically. So the correct thing is to go modulo A, namely, identify the entire space to A to a single point, take the quotient space, the quotient map itself becomes a homotopy equivalence.

But this you may expect always but it is not always true. It is true under the hypothesis that A to X is a cofibration. So for nice subspaces, this is true. Third one is slightly a different variation of the same phenomena. Instead of quotienting out A, you keep the subspace A as it is but take homotopy relative to A. Then can you say that contractibility of A will imply anything under homotopy relative to A?

That is again not true. You need this time, even stronger hypothesis. Namely, a point a_0 in A, wherever you want to concentrate the whole of A, should be a strong deformation retract. If A is contractible, a_0 is a deformation retract of A, but may not be strong deformation retract. If it is strong deformation retract, then the inclusion map the pair (X a_0) to (X,A) is a homotopy equivalence.

So it is like all homotopy information now inside A. Whenever you want to control A, it can be done by just controlling the point a_0. So these things are themselves very useful in homotopy theory. So we are developing the homotopy theory. Here is a step, --smaller things to largest things. So let us go through the proof of these statements which are now more or less just one-line proofs. Thus the hard work is already done.

So how to say something is a deformation retract when it is only a weak deformation retract?

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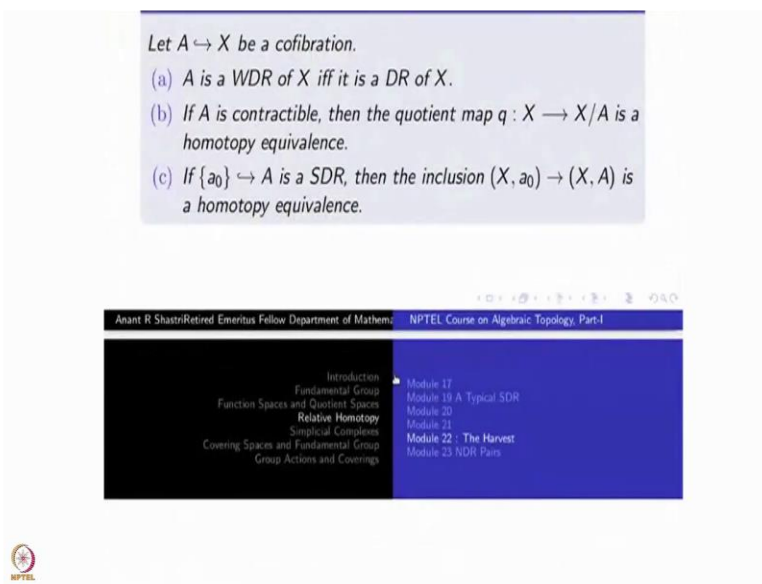

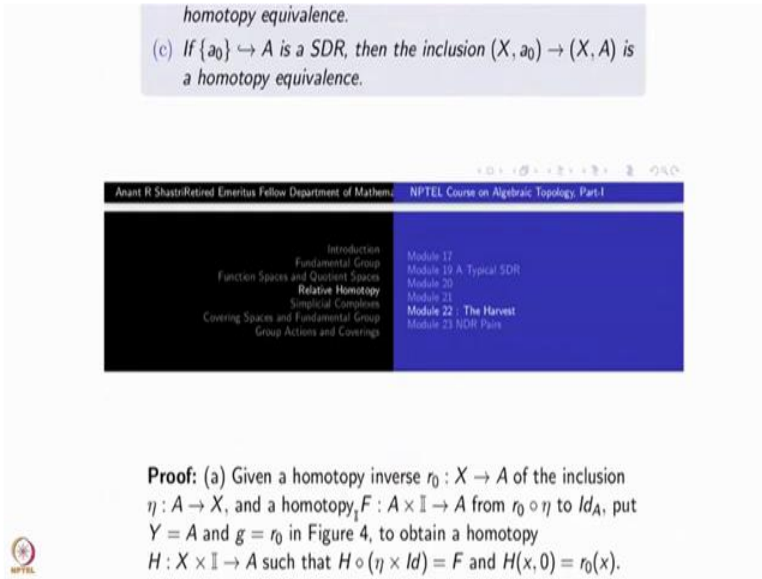

So, given a homotopy inverse r_0 from X to A, remember that the inclusion map is a homotopy equivalence is the same thing as saying that A is a weak deformation retract. So r_0 is not the inverse of η, r_0 is the homotopy inverse for the inclusion map η. $\eta\eta$ is homotopy equivalence is same thing as saying that A is a weak deformation retract, by definition. So take r_0 as its homotopy inverse.

Let F be a homotopy from r_0 composite $\eta\eta$ to identity, because these two are homotopic because this is the homotopy inverse of that. So let F be such a homotopy from $A \times \mathbb{I}$ to A between r_0 composite $\eta\eta$ and identity of A. With this much of data, we have to get a deformation retract. The key is to use the fact that A to X is a cofibration. So go back to your definition of homotopy extension property namely, that is the figure that I will reconstruct it here below.

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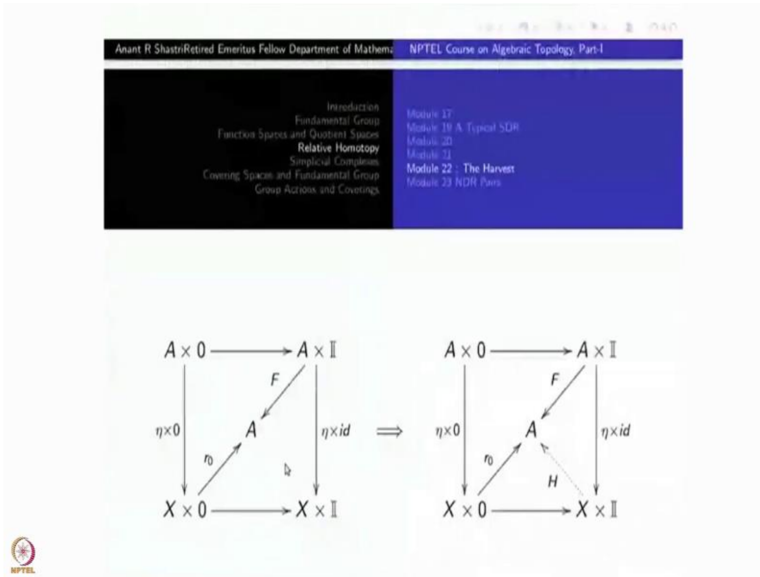
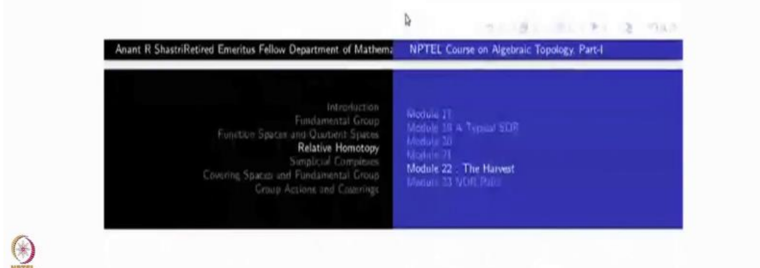
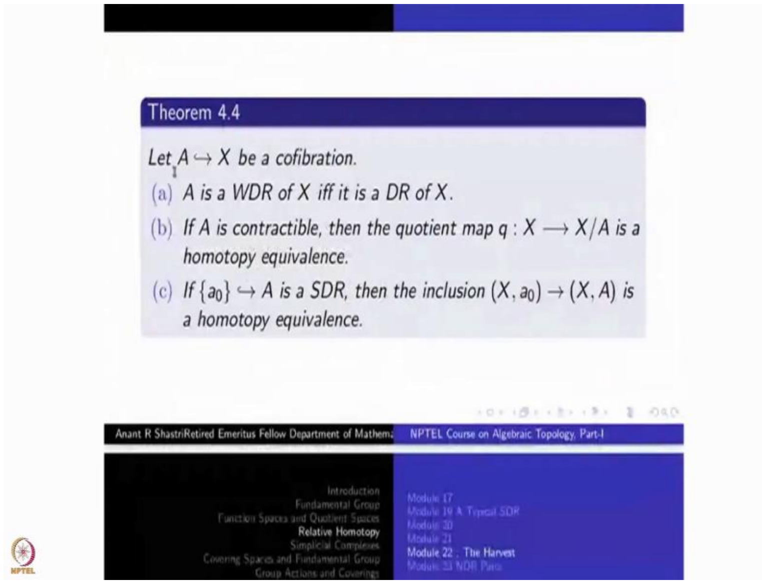


Proof: (a) Given a homotopy inverse $r_0 : X \to A$ of the inclusion $\eta : A \to X$, and a homotopy $F : A \times \mathbb{I} \to A$ from $r_0 \circ \eta$ to Id_A, put $Y = A$ and $g = r_0$ in Figure 4, to obtain a homotopy $H : X \times \mathbb{I} \to A$ such that $H \circ (\eta \times Id) = F$ and $H(x, 0) = r_0(x)$. Then H defines a homotopy $r_0 \sim Id_A$. This also implies $\eta \circ r_1 \sim \eta \circ r_0 \sim Id_X$.





So what I have is A × 0 to X × 0, an inclusion map. Here I have put, instead of g in the original thing, I have put r_0 here now. This is the homotopy inverse of η. And F is the corresponding homotopy from A × $\mathbb{I}$ to $\mathbb{I}$. So we take this commutative diagram which is automatically implied because this is a cofibration-- that is, the homotopy extension property implies that there is a homotopy H from X × $\mathbb{I}$ to A, which fits this diagram.

Namely, restricted to A × $\mathbb{I}$, it is F, restricted to X × 0 it is r_0. So that is what I am going to use now.

Student:

Hello sir, would you explain the diagram one more time, say why it is homotopy extension for what it shows.

Professor:

A to X has the homotopy extension property is the assumption. You see that this is a cofibration on the left, --a blanket assumption is made. Cofibration means what? homotopy extension property, for every space, and every map and so on. This side you can choose. As soon as you choose this so that that is a diagram commutative, there is map here. That is the conclusion. So F is extended to H-- that is the homotopy extension property.

So I am going to use that now. So you have H from X × 𝕀 to A such that H restricted to this A × 𝕀 through $\eta\eta$ × identity is F. Restricted to X × 0, its r_0. But then H defines a homotopy between r_0 and identity of A. Because H restricted to X × 0 is r_0. H(a ,1) will be F (a,1) that should be identity. So this also implies that if you take r_1= to H on X × 1, η composite r_0 is homotopic to $\eta\eta$composite to r_1. But we know that $\eta\eta$ composite r_0 is homotopic to identity of X.

So we have got that r_1 is a deformation retract. So a weak deformation retract became a deformation retract as soon as A to X is a cofibration. That is the key here.

Similar thing we can do on the second part here: if A is contractible, quotient map X to X / A will be a homotopy equivalence. So let us go through this one now.

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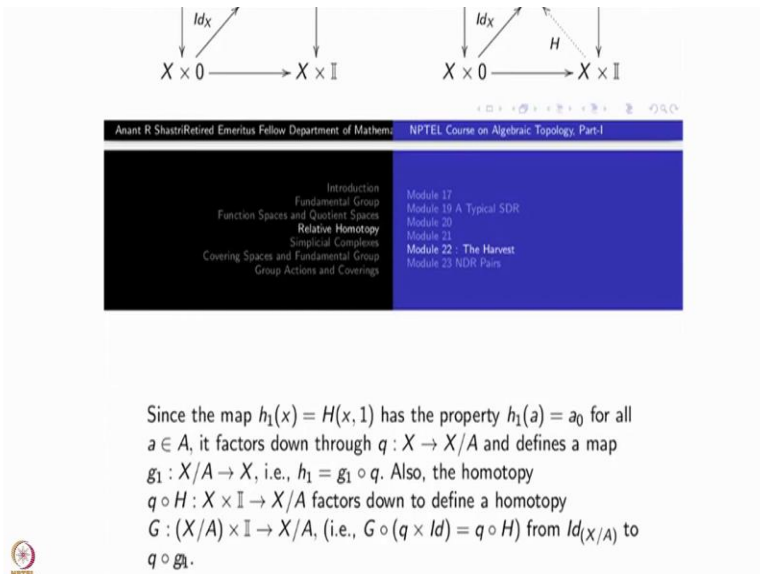

F is a map from A × 𝕀 to A, is a homotopy of the identity map with a constant map, a_0 is a point of A. That is the meaning of A is contractible-- identity η map of A is homotopy to a constant map. This time, instead of taking A here, I take this space to be X here. Actually in the original definition, we have `for any space Y and any map g like this' and so on is there. So instead of that Y, I am putting, this time, here I am putting Y to to be X itself.

And this map will be taken to be the identity map. That is what I am talking in figure 4. Again you get a capital H here to fit the diagram. Here again, I am using the fact that A is a cofibration. So this time it is X and instead of r, we take identity map. Restricted to A it is homotopic to a constant map, F on A,1 will be a constant map here. So that is the homotopy F. This is homotopy data, and A to X is cofibration gives rise to this conclusion, namely there is an H here which fits into this diagram.

So what is the meaning of that now. Restricted to H restricted, sorry, H restricted to X × 0 is identity. On A × $\mathbb{I}$, it is F. So what are the properties this H? $h_1(x)$ which is H(x,1). H on A × 1 here has the property that it is F on A × 1 which is constant map a_0. Which means all of the points of A, belongs to A, they are going to a single point.

Therefore, this map h_1 factors down through the map q from X to X / A and gives you a map g_1 from X / A to X. So that is what I am calling g_1. Because under this h_1, all points of A are going to single point which is the single point identification in X / A, representing one single class. So this g_1 is from X / A to X. This h_1 is X to X, just I am writing what is the property of g_1, $h_1 = g_1$ composite to q.

Also we a homotopy start with q × identity. See, H is from X × $\mathbb{I}$ to X, then follow it by q. That means I have quotiented it out the all the points here. Then q composite H factors through q × Id to define a homotopy G from X / A × $\mathbb{I}$ to X / A. The points of A which have been identified here, they are identified here also. That is what you have is q × identity followed by G is the same thing as q composite H.

So what is this map G? G is a homotopy on (X / A) × $\mathbb{I}$ to X / A. On (X / A) × 0, it becomes identity map from X / A to X / A. That is the starting point. And the end is q × g_1, q composite g_1. So it is a homotopy between identity and q composite g_1. Which means q has g_1 as its right inverse, a homotopy inverse.

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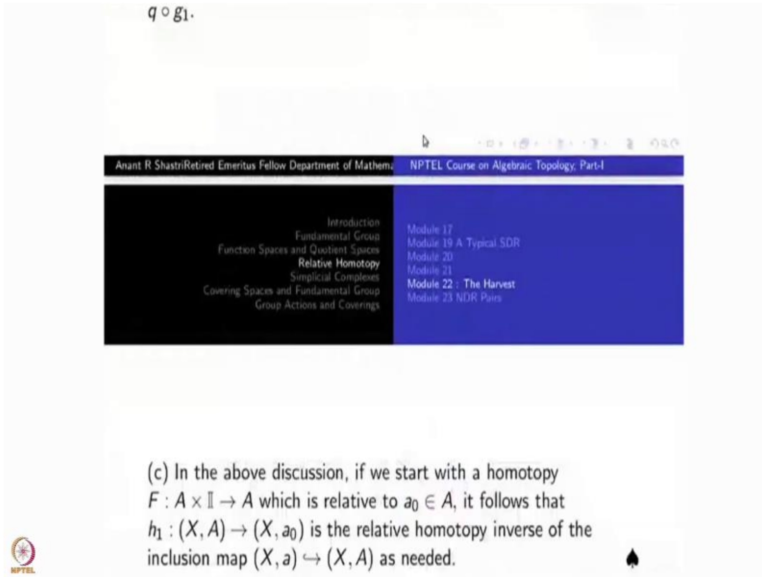
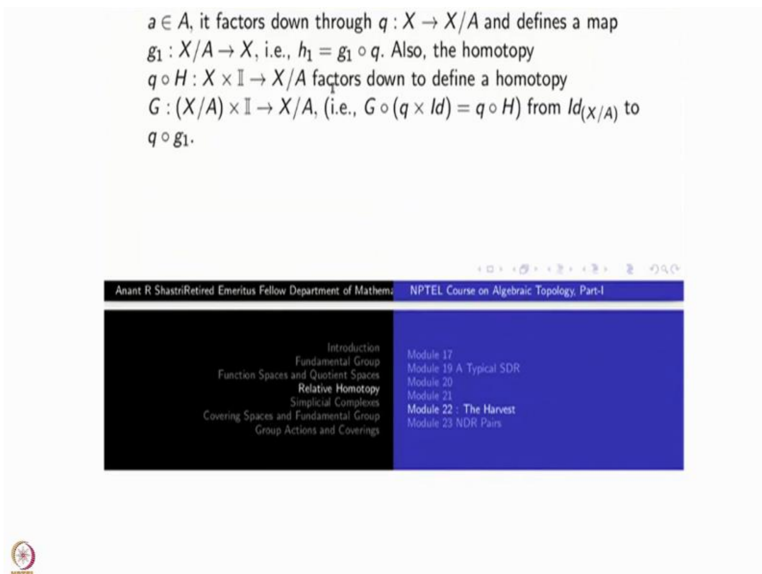




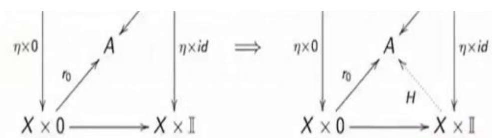
Theorem 4.4

Let $A \hookrightarrow X$ be a cofibration.

(a) *A is a WDR of X iff it is a DR of X.*

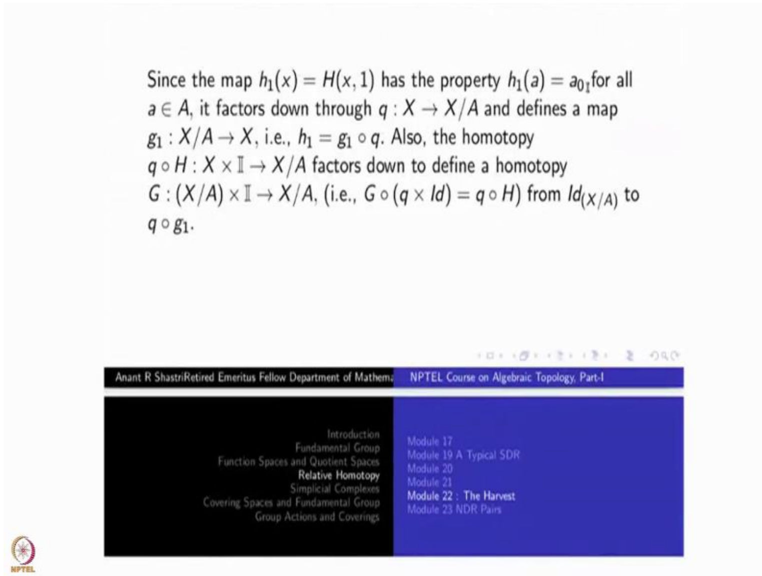
(b) *If A is contractible, then the quotient map $q : X \longrightarrow X/A$ is a homotopy equivalence.*

(c) *If $\{a_0\} \hookrightarrow A$ is a SDR, then the inclusion $(X, a_0) \to (X, A)$ is a homotopy equivalence.*



Anant R ShastriRetired Emeritus Fellow Department of Mathema NPTEL Course on Algebraic Topology, Part-I
Introduction
Fundamental Group
Function Spaces and Quotient Spaces
Relative Homotopy
Simplicial Complexes
Covering Spaces and Fundamental Group
Group Actions and Coverings
Module 17
Module 19 A Typical SDR
Module 20
Module 21
Module 22 : The Harvest
Module 23 NDR Pairs

(b) Let $F : A \times \mathbb{I} \to A$ be a homotopy of the identity map with the constant map at a_0. Put $Y = X, g = Id_X$ in Figure 4 to obtain a homotopy $H : X \times \mathbb{I} \to X$ such that $H(a,t) = F(a,t), a \in A$ and $H(x,0) = x, x, \in X$.



And that is enough to conclude that q a homotopy equivalence. So g_1 composite q is this h_1 which is homotopic to identity.

The last one (c) is special case. What is this definition of SDR? You have, instead of arbitrary contractibility, for contractibility you are have just a homotopy from identity to constant map, strong a deformation retract means this homotopy is relative to a_0.

The same argument as in (b) will give you, same g_1 and so on, all these things are same as in the previous case, but now I have the homotopy being constant on a_0. So that is the extra thing here. So that will tell you that h_1 is a relative homotopy, it means that the homotopy from h_1 is constant on a_0 that is all. Whatever we have got there is then a relative homotopy on a_0 ,with respect to a_0.

So this will conclude that we have maps (X ,A) to (X, A). In this case you do not have to go down to X / A at all. Half way here in this proof, before going down, as soon as you have got this h_1 which is a homotopy inverse relative to A. So stronger hypothesis gives you an easier conclusion, a stronger conclusion.

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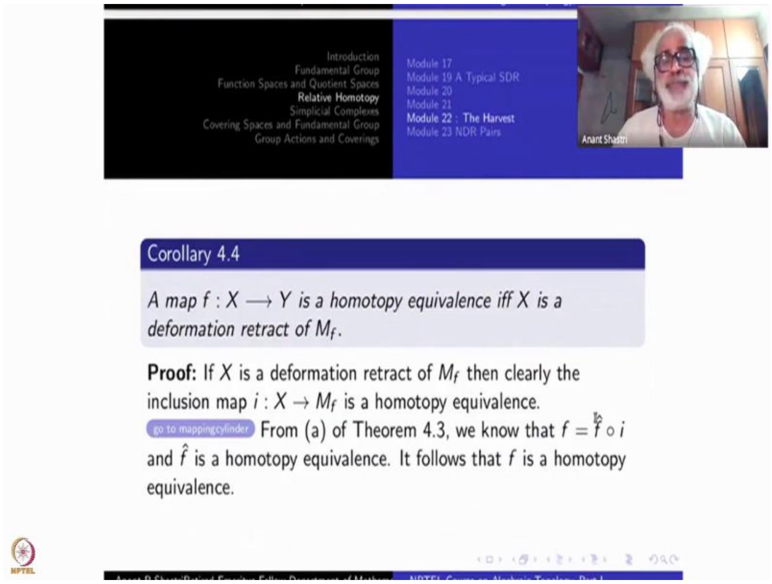

Next, many books just say `contractible therefore you can collapse' without giving any proper reason. I have given you a proof and the proof is correct only if A to X is a cofibration. In fact we will have easy counter examples when this is not a cofibration. You cannot just take X / A and say that it is a homotopic to the original space X.

So next result: Take a map from X to Y. It is a homotopy equivalence if and only if X is a deformation retract of M_f.

Remember M_f is a quotient of $X \times \mathbb{I}$ and disjoint union Y by the identification x ,1 is identified with its image f x in Y. We also know that Y itself is a strong deformation retract of M_f. Now results says that f is a homotopy equivalence iff X is a deformation retract of M_f.

Suppose you assume that X is deformation retract of M_f. Then X to M_f the inclusion is a homotopy equivalence, f hat from M_f to Y is a homotopy equivalence. The composite will be a homotopy equivalence then. That is precisely f. Inclusion map followed by whatever you have got, viz, f hat is identically is equal to f. You have seen that right? So one way is clear that if it is a deformation retract then f must be homotopy equivalence. For the converse, hope you can prove the converse. I have already given you here the prove of one way: namely the inclusion map is a homotopy equivalence (is a same thing as saying that X is a deformation retract of M_f and this) implies f is a homotopy equivalence. Inclusion map followed by f hat is f. What is f hat? f hat is strong deformation retraction from M_f to Y. So that is why the composite will be a homotopy equivalence. So let us prove the converse.

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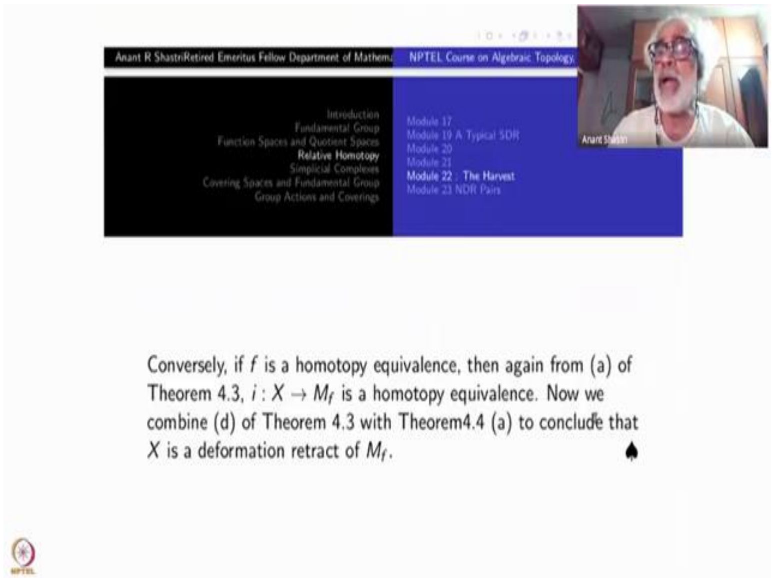

If f is a homotopy equivalence, then again from theorem 4.3(a), the inclusion map X to M_f will be homotopy equivalence. Because f is i composite f hat. The result says that if two of them are homotopy equivalences then the third one is homotopy equivalence. I think this is one of the exercises (but not a theorem) in the very beginning. Hopefully you have done that exercise by now. Now you combine part of d in 4.3 with theorem 4.4, one which we have just seen. To conclude that X is a deformation retract of M_f.

What are these results? First thing is that inclusion map of X in M_f is a cofibration. This is the first one of the results, part (d) of theorem 4.3. And the previous theorem says that if a cofibration is a weak deformation retract then it is a deformation retract. Weak deformation retract is same thing as inclusion map is homotopy equivalence. I repeat: once the inclusion map is a homotopy equivalence and it is cofibration then it is a deformation retract.

So this was the theorem that we just proved here a, b, c three parts of the theorem. So use that, so we get the converse here. So the corollary is proved namely, a map is a homotopy equivalence between two spaces is the same thing as saying that the domain is a deformation retract of the mapping cylinder.

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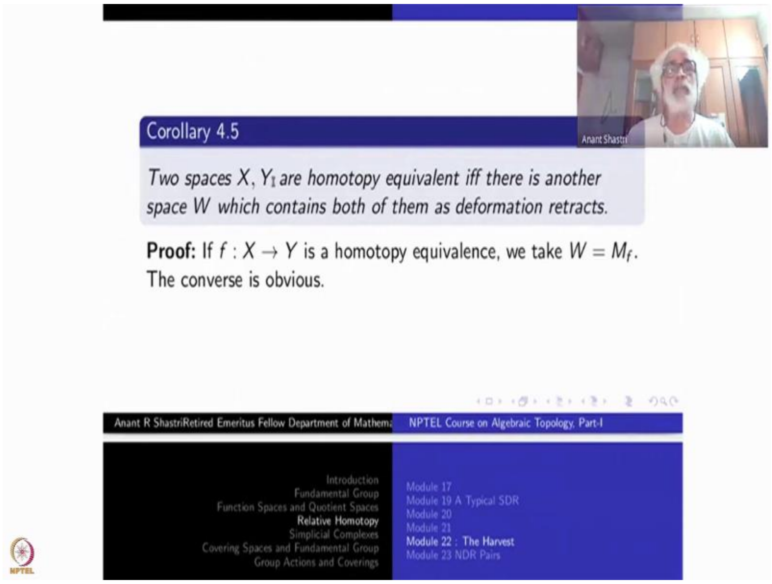

Now we come to another result--- all landmark results in homotopy theory. Two spaces are homotopy equivalent, if and only if there is another space W which contains both of them as deformation retracts. This is not just for fun. Sometimes there is no other way to show that two given spaces are homotopy equivalent. This theorem is used heavily in differential topology namely, in Cobordism theory. A lot of hypothesis is there with which you want to construct a homotopy between X and Y, they are manifolds.

What you do then? That inclusion maps you construct are into a `cobordism' W. Then you show that these inclusion maps are deformation retractions, and hence homotopy equivalence. Plus in the case of manifolds, they are cofibrations also. Therefore, because this theorem X and Y become homotopy equivalent. So one way is obvious. But here we are insisting that you are not l anything more, you are not doing anything great. It is `if and only if'. This will always happen.

So that if and only if part what we have seen. Suppose f from X to Y is a homotopy is equivalence. You take W to be the map in cylinder and we have already seen that the inclusion map followed by f hat is f. Now f is a homotopy equivalence, $\hat{f}$ is homotopy equivalence, so i will be a homotopy equivalence, that is all. And prove the converse here: now what is the statement here, what is the converse? The converse is obvious I have not written down here. The converse is obvious.

Why because X to W is homotopy equivalence and Y to W is homotopy a equivalence means there is a homotopy inverse map from W to Y, you compose the two-- that is all. So this W could be anything. But to get one, you have to take M_f. If this f is homotopy equivalence you take W to be M_f. It may not be M_f always. It could be something bigger also, something smaller also does not matter. If it has the property that inclusion maps are homotopy equivalences from both X and Y then X, Y will be a homotopy equivalent. That is the whole idea. X and Y will be homotopy equivalent. There is no f given in this. You have to construct one. But if f is given then you take the M_f as the space W that is all.

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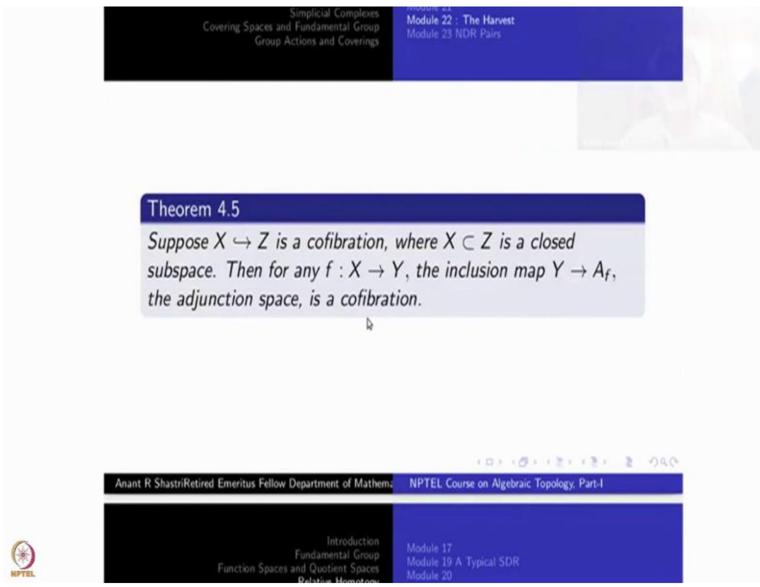

Now we will get some more results: Take a cofibration X to Z. So here I have changed the notation-- usually we take A to X . I have taken now X to Z is a cofibration. Every time I am assuming it is a closed subset--- X is a closed subspace of Z. Then for every continuous map from X to Y the inclusion map of Y into the adjunction space is a cofibration. What is the adjunction space of f, f is from X to Y. So it is the quotient of Z disjoint union Y; points of X are identified with points of Y through f; x in X and f (x) are identified. That is the adjunction space A_f

So, adjuction space of f always contains Y as a subspace. Z may not be a subspace, X may not be subspace, because f may not be injective. Y to A_f is an inclusion map, that becomes a cofibration under the hypothesis that X to Z is a cofibration. Let us suppose that f from X to Y is an inclusion map. Then we are getting an extended cofibration Y to Y union Z from X to Z .

So X to Z is a cofibration implies Y to A_f (is a larger one) becomes a cofibration. So, more generally than the case when X is a subspace (Y), for any map, f from X to Y. So that is the importance of homotopy theory.

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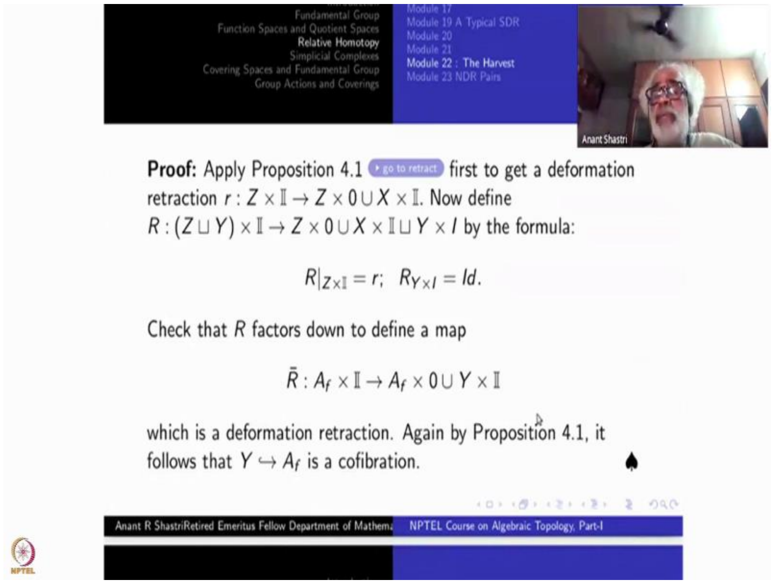

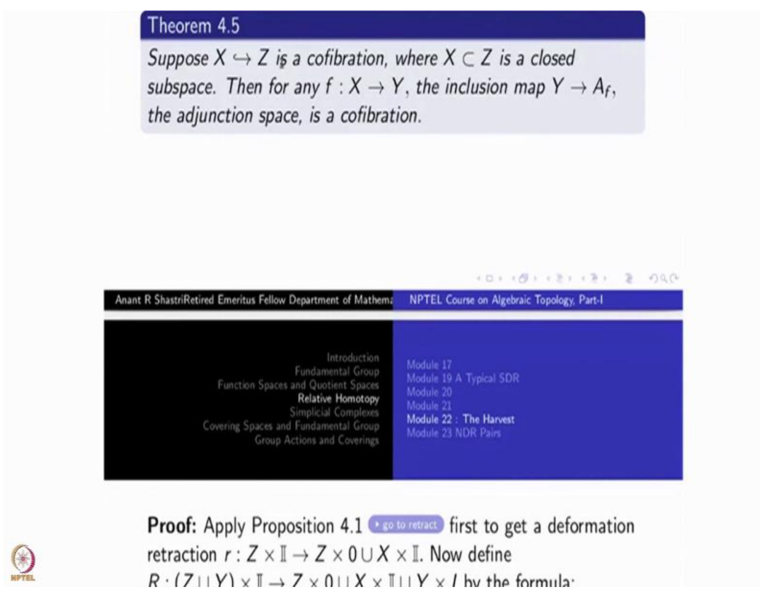

So again to check cofibration what we have to do? you cannot go on verifying it for homotopy extension property for all functions and all spaces. You have a ready-made proposition there 4.1 namely, you show that $A \times \mathbb{I}$ union $X \times 0$ is a retract of $X \times \mathbb{I}$. In this case what you have to show $A_f \times \mathbb{I}$ sorry, $Y \times \mathbb{I}$ union $A_f \times 0$ is retract of $A_f \times \mathbb{I}$ that is what you have to show? So it is very easy from corresponding statement: The corresponding hypothesis here is that X to Z is a cofibration.

What I get? Z× $\mathbb{I}$ sorry, Z × 0 union X × $\mathbb{I}$ is a retract of Z × $\mathbb{I}$. So this proposition says first we get just a retract. That's is good enough. Then we will become a deformation retract. This is what we have seen already. So just retract is enough, a retraction from Z × $\mathbb{I}$ to Z × 0 union X × $\mathbb{I}$. Retraction means what? On this subspace, it is identity. Now you just consider Z disjoint union Y × $\mathbb{I}$ -- remember the adjunction space is defined as quotient of Z disjoint union Y.

So if you want to construct some map on a quotient space, you go back to the mother space here. with certain additional property, you have to define a map so that it will goes down to the quotient space. So here I am using the fact that the product I am taking with is a compact space, the unit interval and therefore whether you take first the quotient here then take product or first take the product and then quotient, they are same. This is one of the fundamental result we have proved and we have used several times by now.

So I am defining a map R on Z disjoint union Y × $\mathbb{I}$ to Z cross 0 disjoint union X × $\mathbb{I}$ disjoint union Y × $\mathbb{I}$. Here later on, points of X will be identify with f(X), - same thing will be added then here. The second factor $\mathbb{I}$ there is not problem there. Take R on Z × $\mathbb{I}$, (Z disjoint union Y × $\mathbb{I}$ is Z × $\mathbb{I}$ disjoint union Y × $\mathbb{I}$), take R to be your earlier ready-made function r here. On Y × $\mathbb{I}$, take R to be identity. Over. After that you have to check that it goes down to define a map from A_f × $\mathbb{I}$ to A_f × 0 union Y × $\mathbb{I}$. This part will be A_f × $\mathbb{I}$ 0 union Y × $\mathbb{I}$. On Y × $\mathbb{I}$, you have already taken identity.

So if you try to prove this cofibration directly you will see how difficult it is-- proving the homotopy extension property. I am not saying that it is not possible. But here, this proposition 4.1 makes the life very easy.

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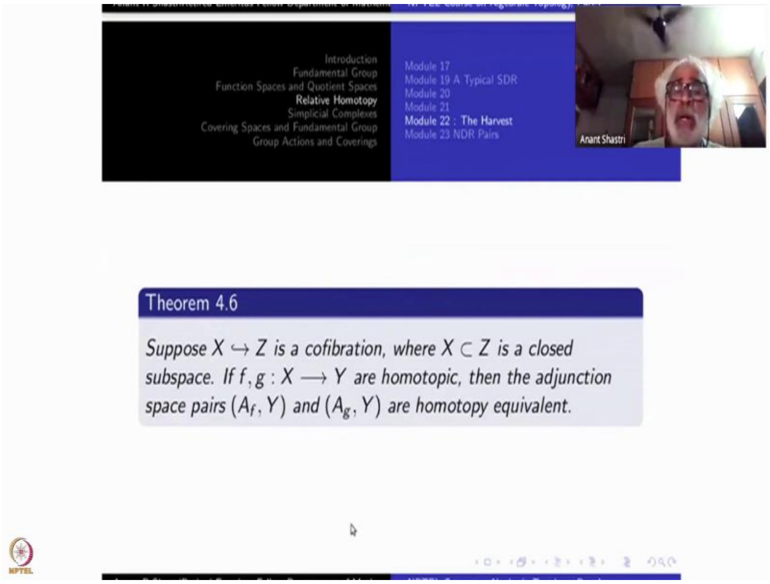

Finally, the homotopy invariance of, of what? Of the adjunction spaces. Take X to Z a cofibration, where X is a closed subset of Z. Given two functions f and g from X to Y, I am going to take adjunction spaces. But if these two maps are homotopic, then the adjunction spaces are homotopy equivalent. Even the stronger statement is true namely, as relative pairs (A_f,Y), (A_g,Y), Y is a common subspace here, (A_f,Y), (A_g,Y) as pairs are homotopy equivalent.

This was done right in the beginning remember. What is the meaning of this? There is a homotopy, there is a map from A_f to A_g which is the identity on Y. Similarly, a map from A_g to A_f which is the identity on Y , these two are homotopy inverse of each other relative to Y. So that is the strong statement namely adjunction spaces do not depend on exactly what f is but strongly depend on the homotopy class [f].

So this helps a lot while doing homotopy theory. Often you start picking up a map here but you do not pick up the map, you are actually interested in the homotopy class of that map. And then you want to do something but then you want to say that whatever you have picked up may induce something whereas if you pick another one it may do something else. No, this won't happen for adjunction space-- as far this is concerned. The homotopy type does change. So this is very useful result. So let us go through the proof of this one.

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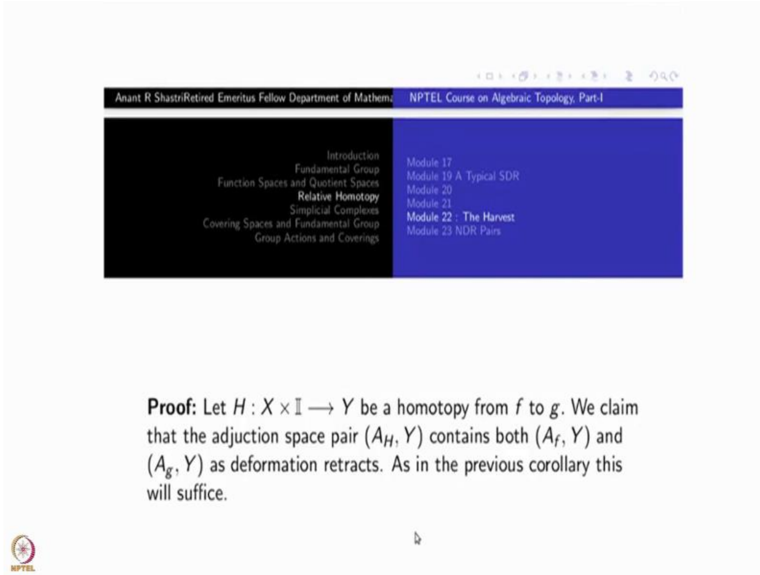

Start with the homotopy X × 𝕀 to Y from f to g. Now you can define the adjunction space of H, of this H itself. You see it is Z × 𝕀 disjoint union Y, X, points of X × 𝕀 being identified with the image under H. So that is A_H. That also contains Y. So what we are saying is, A_f, Y as a pair, a topological pair and A_g ,Y another topological pair both are subspace of A_H, Y and the inclusion maps are homotopy equivalences. This is enough to say that they are themselves w homotopy equivalent. But what we are saying that deformation retracts.

Student question:

Hello sir, Can you say one more time how you made A_H I mean yes how, what is A_H?

Professor: A_H is the adjunction space of H. Adjunction space is defined for every map.

Student question:

Yes, you used Z × 𝕀 right?

Professor: Yeah. Because X × 𝕀 is not contain inside Z, but Z × 𝕀. So that will come. I have just claimed something here-- explanation will come now. So what is A_H?

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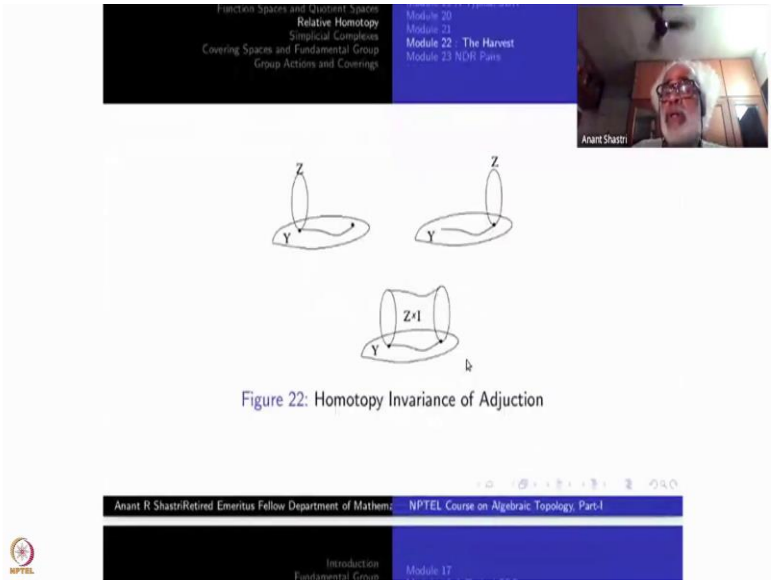

Figure 22: Homotopy Invariance of Adjuction

This is the picture. See this is Y, this Y here, X is a single point as shown in the diagram, this circle is your Z. You get it? X first goes to this point y. And then this y, we have chosen a path, goes this point. A homotopy of a point map means it is a path here. A homotopy of point is nothing but a path as in the picture. I have to take the simplest picture. So the first point is f, the end point is g. When singleton, X is a singleton, map is nothing but a point. Now homotopy is nothing but a path here. Here I have taken A_g, and A_f here. This is Y disjoint union Z but X is identified with its image. Is this clear? In this picture I have taken A_g. I want to say both of them are subspace of A_H here. What is A_H? It is from $Z \times \mathbb{I}$, $X \times \mathbb{I}$ identified with its image here. You see it is not a straight line now-- you have take the path, whatever path the capital A_H is, the adjunction space for H.

Now is this picture clear? Now what I have to show is this part is subspace here that is subspace there, they are deformation retracts. The argument is exactly symmetrical if I show one of them is deformation retract another one will of deformation retract for the same reason. In fact there is a t factor, you have to reverse t to 1- t. Then the role $Z \times 0$ and $Z \times 1$ will be interchanged, that is all.

So this part is $Z \times 1$ here, this is $Z \times 0$, when you take the homotopy here. (You see we are actually doing much more than Cobordism theory. This looks like a Cobordism when things are manifold here they are not even manifolds but we have `Cobordism' like things here. So results will be used in very deep differential topology.)

So here is the proof. Finally you have to write down proof. This is only an idea.

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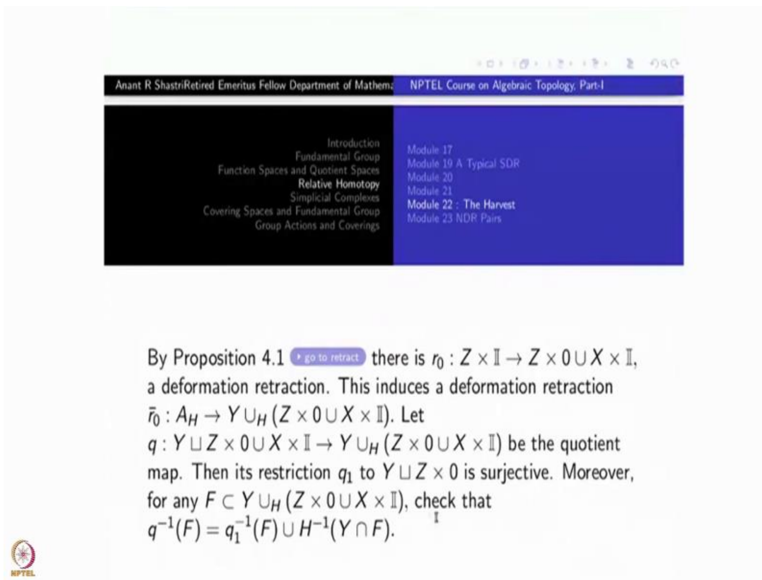

Again by this 4.1 what is the assumption? Assumption is that X to Z is a cofibration. X to Z is a cofibration that is the starting assumption. We have a retraction r_0 from $Z \times \mathbb{I}$ to Z cross 0 union $X \times \mathbb{I}$. This induces a deformation retraction r_0 bar from A_H which is the adjunction space of H to Y union over H, $Z \times 0$ union $X \times \mathbb{I}$.

Let q be the quotient map from Y disjoint union $Z \times 0$ union $X \times \mathbb{I}$ to Y union over H (Z) $\times$ 0 union $X \times \mathbb{I}$-- this is the quotient space. This is disjoint union, this is where this identification is taking place. Its restriction q1 to Y disjoint union Z cross 0, (forget about this part.) that itself is a surjective map. Moreover, for any closed subspace F contained Y union this part, check that q inverse of F is $q_1^{-1}(F)$ union $H^{-1}(Y)$ intersection F.

This part I have written down only for people who have forgotten that result on product of quotients-- first take the quotient and then the product with the identity map when $\mathbb{I}$ is compact space, is the same as first take the product and then the quotient. This part is not needed if you know that result, because I have actually proved that one. But here, independently, I am actually proving that this a quotient map.

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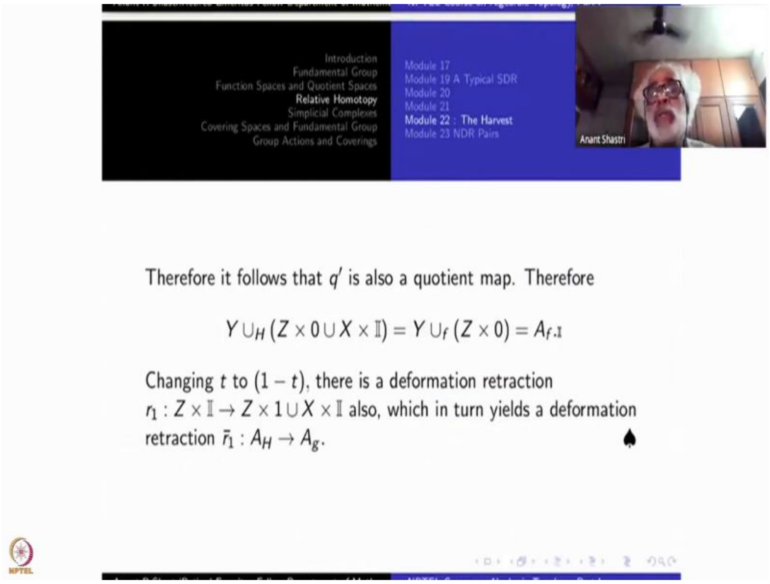

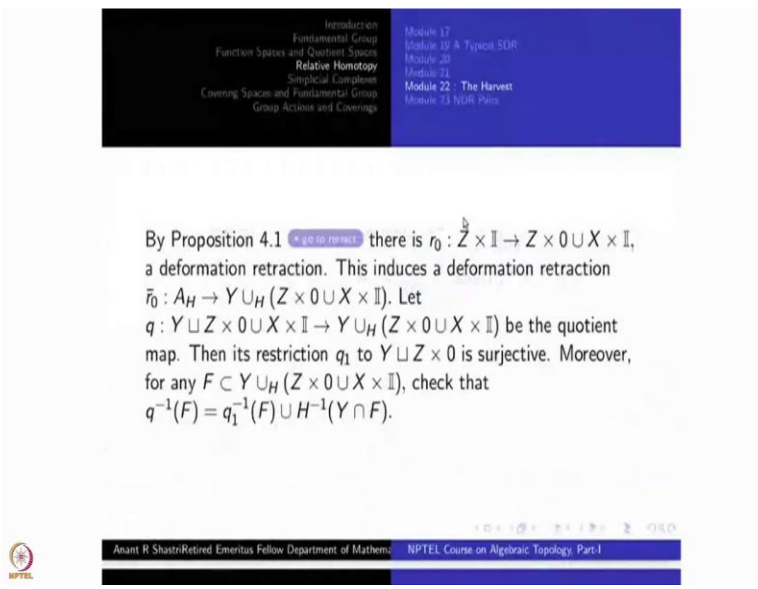

So therefore what you get is that q_1 is a quotient map after taking the product. Therefore, this part is nothing but Y union over f, $Z \times 0$ because $X \times \mathbb{I}$ part is already covered by Y. So that is A_f. So this will show you that A_f is deformation retract. This part, what is this part? This was deformation retract so that is A_f what it is Y union H this was a deformation retract.

This I showed you is equal to A_f. So that care of it. It may take some time to understand why such things work. And you may get lost in the notations here. Proof is clear from this picture: you can see what is happening. Once you have proved that Af is a deformation retract, Ag is also a deformation retract; just change t to 1- t; --- symmetric argument. So we can stop here.