Introduction to Algebraic Topology (Part-1)
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Lecture Number 21
A Theoretical Application

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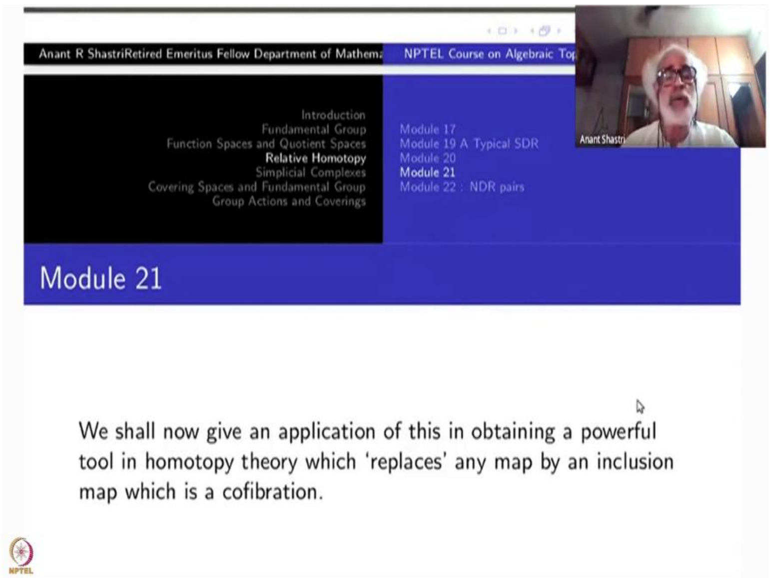

As a student, I was wondering why the several books give definition of cofibration for inclusion maps only, whereas fibration is for arbitrary map not necessarily for quotient maps. The answer was in this result that we are going to do. But there is an even more satisfactory answer. Much much later, only some 10 years back, I read it from Hatcher's book that every cofibration is actually an inclusion map.

So what are we going to do? Because we are studying things up to homotopy, in homotopy theory, this result says that any map can be replaced by an inclusion map. And that inclusion map is a cofibration. So this was so powerful that now you can think up to homotopy, every map is a cofibration.

So this completely justifies the somewhat artificially introduced the notion of cofibration. We are not claiming that every map is a cofibration but up to homotopy it is. So let us go through this one. So, later on, we will see a lot of applications of this and so, this itself is somewhat an hard work you have to do now.

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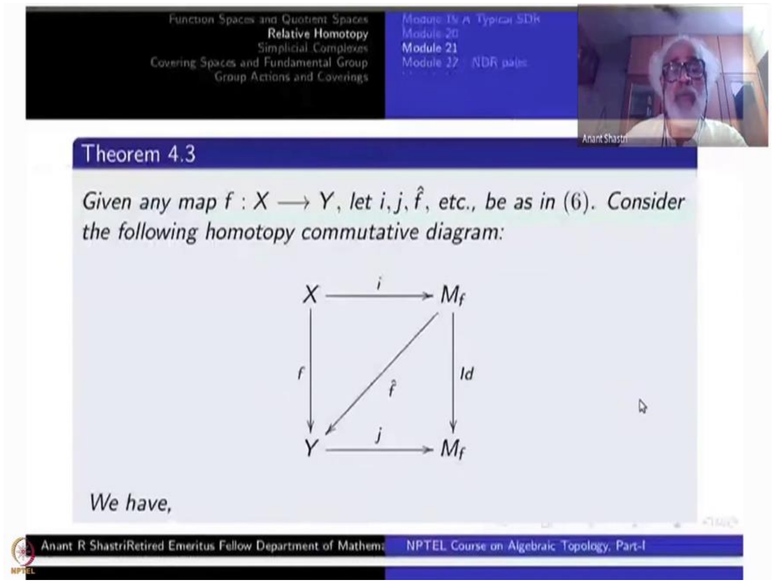

So here is the statement. Given any map f from X to Y let us construct the mapping cylinder M_f. Recall what is M_f. M_f was obtained as the quotient of $X \times \mathbb{I}$ disjoint union Y by the identification namely, each (x,1) is identified with this image f (x). So I am making this diagram of various functions here, X is included in M_f.

This map X to Y and M_f to Y. There is an $\hat{f}$ and then Y is also included in M_f by an inclusion map j. So this $\hat{f}$ is actually an extension of f, because X is thought of as a subspace M_f. So all these things I am writing down carefully namely $\hat{f}$ composite i is f. In the first triangle it is commutative,-- top triangle.

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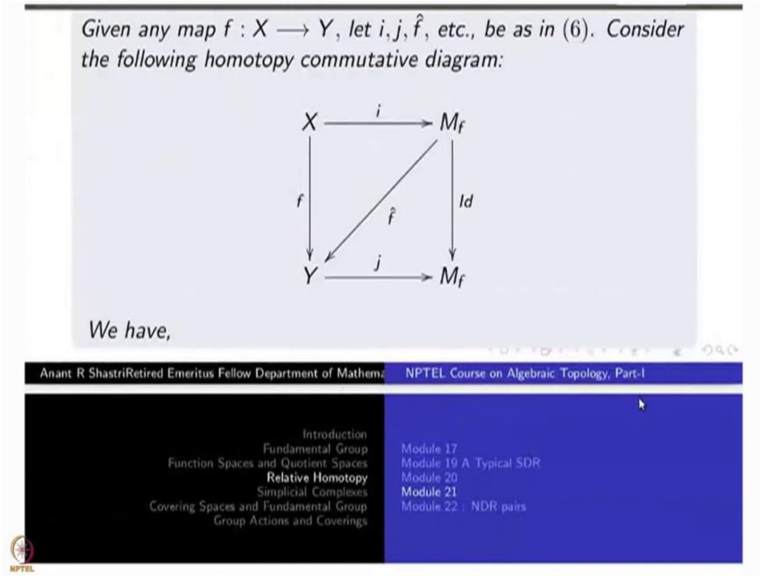

j composite $\hat{f}$, in here from M_f to M_f.

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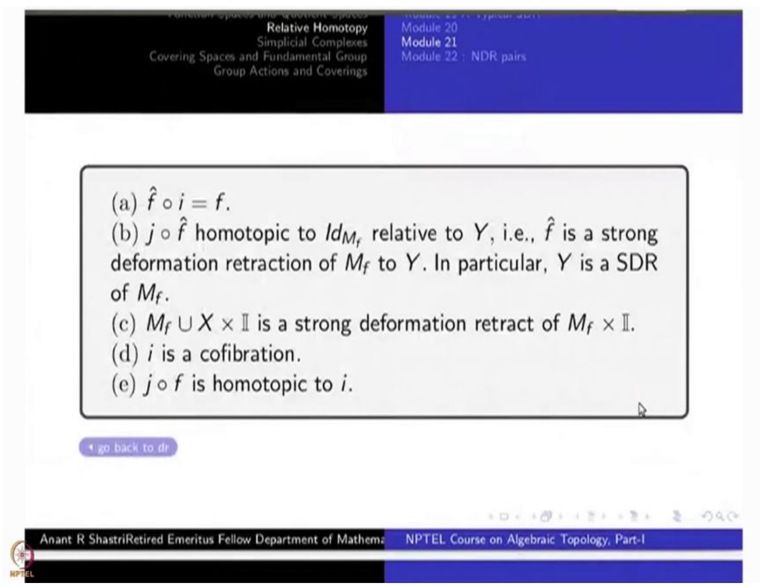

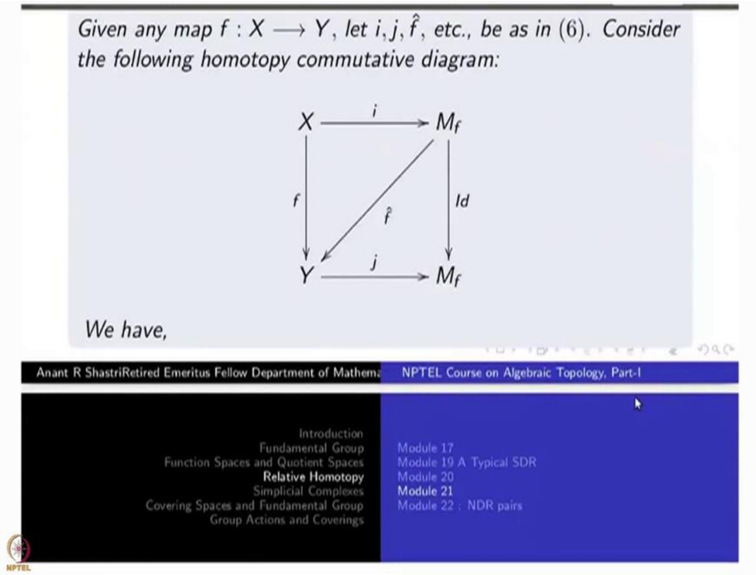

This map is homotopic to Identity of M_f. So the second triangle here the bottom triangle---- it is not a commutative triangle, it is homotopy commutative. j composite $\hat{f}$ is homotopic to identity. And this homotopy is relative to Y, that is $\hat{f}$ is a strong deformation retraction of M_f to Y.

In particular Y is a SDR of M_f. M_f union $X \times \mathbb{I}$ is a strong deformation retract of $M_f \times \mathbb{I}$. So i is a cofibration, the inclusion map, from (c) to (d)--- you know how to go-- because we have seen this one. j composite f finally this j composite f coming from X to M_f, j composite f, it is homotopic to the inclusion map itself.

So when you put this, this arrow identity, if you do not put this one that diagram is commutative, if you put this one it will be homotopy commutative. If you come from here to here and come from here to here, it is homotopy. Because this is homotopic to the identity, precisely the last thing it says--j composite f is homotopic to the inclusion map i.

So let us prove them one by one. Some of them are obvious. Namely, the first thing that f hat composite i is f, we have already seen. What is f hat? Each point of M_f is a class-- (x,t or bracket y. Remember that means equivalence class of elements from $X \times \mathbb{I}$ union Y. On Y part, it takes bracket y to itself. If it is (x , t), it is taken to fx, so it is well defined because whenever there is an identification, namely, at $X \times \mathbb{I}$, $x \times 1$ is identified with fx only. So it makes sense. So it is an extension of f which is defined on $X \times 0$. So first part we have already seen.

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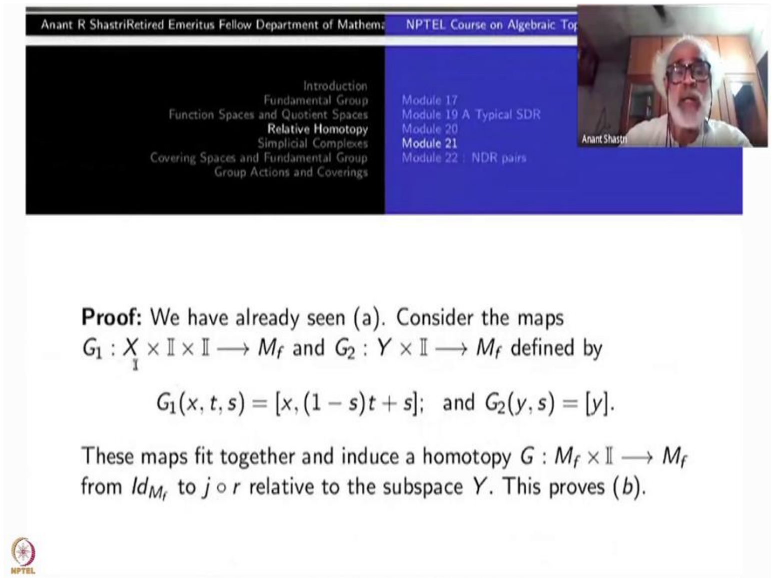

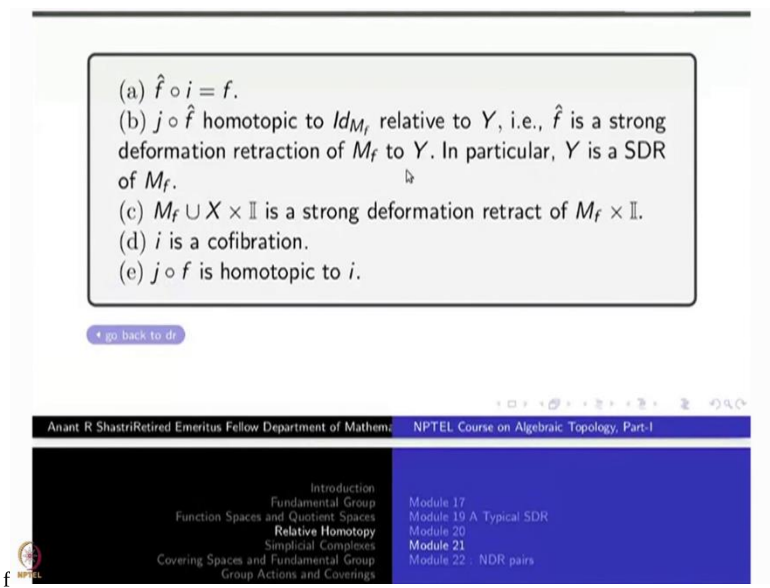

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Secondly, we have to define a homotopy here. So take G_1 from $X \times \mathbb{I} \times \mathbb{I}$ to M_f, and G_2 from $Y \times \mathbb{I}$ to M defined as follows: $G_1(x, t, s)$ goes to x , something. x is as it is you see , everything is happening in $\mathbb{I} \times \mathbb{I}$. So that `something' is 1 - s times t + s--- it joins t and 1--- 1 - s time t + s times the constant 1--- that is the identity function and the constant function are being joined. That is first one. The second function G_2 (y) , s is just y, --- ignores the s coordinate.

These maps together induce a homotopy from $M_f \times \mathbb{I}$ to M_f. On $X \times \mathbb{I}$, when t = 1, it is identified with fx. If t is 1 this is just x , 1. And this is just y. So these maps fit together to define maps from $M_f \times \mathbb{I}$ to M_f. Before we go down to the quotient spaces, G_1 and G_2 are easy to understand. On M_f

$\times \mathbb{I}$, it is from identity of M_f to j composite r relative to Y. On Y always it is the identity function. What is j composite r?

Let just see that, what is this r I have got? A typo, this is not correct. j composite f hat, it should instead of j composite r. That is not correct. Identity of M_f to j composite f hat relative to Y.

Now the proof of (c). M_f union $X \times \mathbb{I}$ is a strong a deformation retract of an $M_f \times \mathbb{I}$. So what do you have to do? You have to do just same thing -- what you have done.

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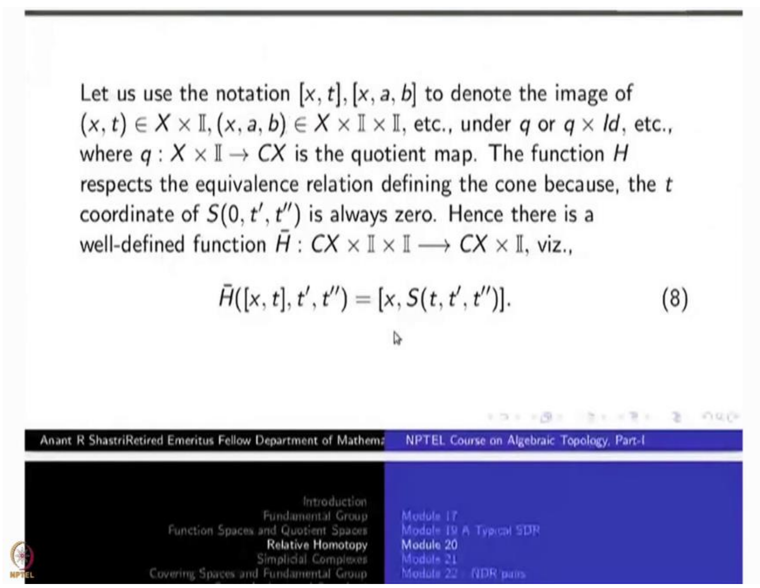

I have to recall that the construction that we have done in the previous module where is it? Module 20 or 19 whatever. So let me go back to the formula (8) here all the way this one. This map will be used again.

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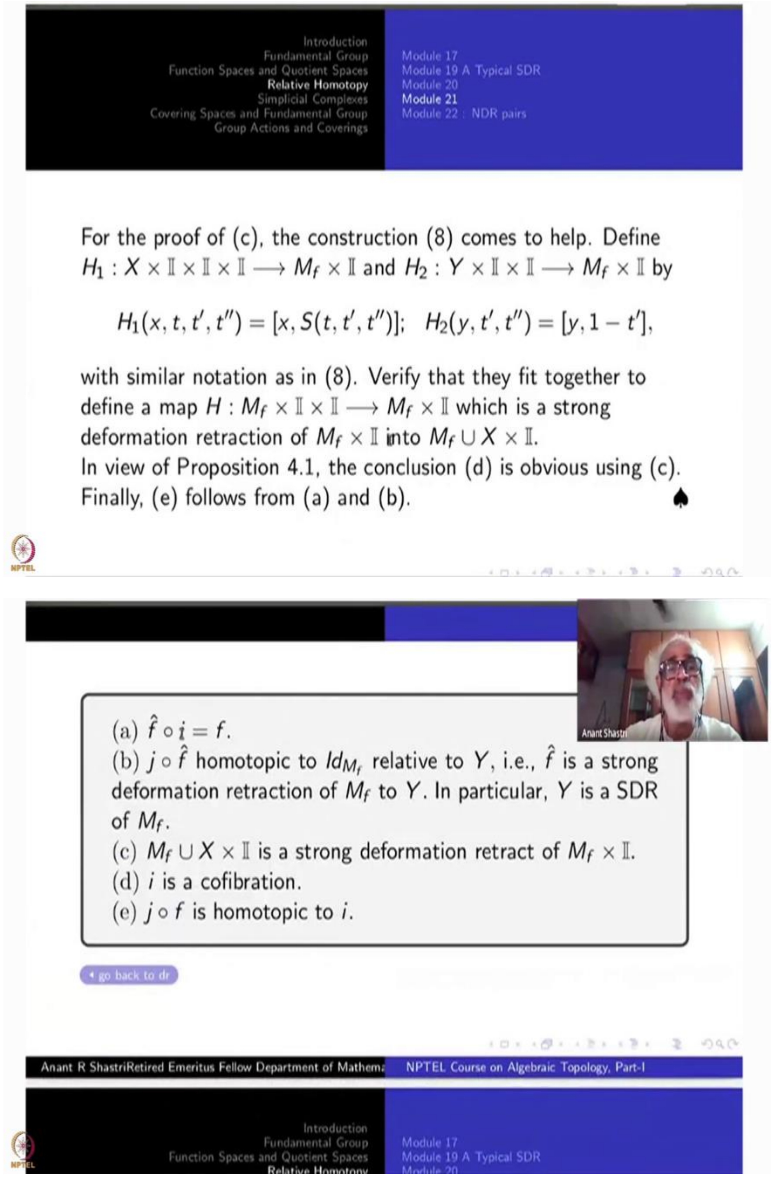

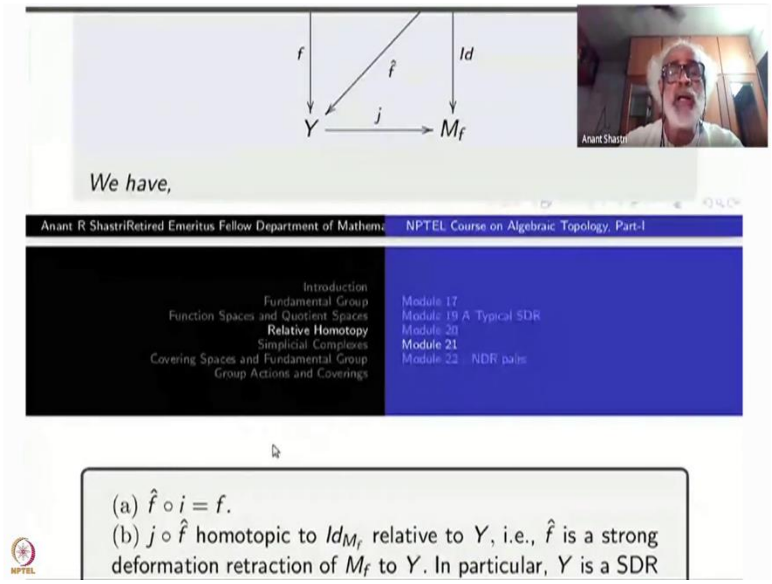

Exactly similar to that. So, H_1 is from $X \times \mathbb{I} \times \mathbb{I} \times \mathbb{I}$ to $M_f \times \mathbb{I}$. Actually it is from $Y \times \mathbb{I} \times \mathbb{I}$ to M_f . $H_1(x, t, t''$ x , S similar to what we have done in the case of arbitrary X. That is precisely what we are doing here exactly same way. S part is taken here, x part is undisturbed $\times \mathbb{I}$. H_2 y, t′, t″is-- y remains as it is and we are ignoring t″ here 1 - t′.

Taking the reverse of that, if you do this, just like the same, similar notation as in that one. All that you have to do is verify that they fit together, fitting together means what? Y is a subspace of M_f. So on that part, you have to verify that wherever this, in this Y is instead of M_f here $X \times \mathbb{I}$ f X when $X \times 1$ is identified in M_f where it is like Y. So there also you have to verify what happens there.

So then they agree as a function. So then continuity follows because all these, wherever they agree all those spaces are closed subspaces. So they fit together to define a map H on $M_f \times \mathbb{I} \times \mathbb{I}$, M_f is the quotient of this part $X \times \mathbb{I}$ disjoint union $Y \times \mathbb{I}$ one more factor $\mathbb{I}$ are also taken--- then the identification. You have to verify. So it gives you a map from $M_f \times \mathbb{I} \times \mathbb{I}$ to $M_f \times \mathbb{I}$ which is a deformation retract of $M_f \times \mathbb{I}$ into M_f union $X \times \mathbb{I}$

Once you have these the proposition will tell you that the inclusion map of M_f into $M_f \times \mathbb{I}$ is a what, sorry, inclusion (X) into M_f is a cofibration, so that is (d). (d) tells you that i is a cofibration. What is i? i is the inclusion map (X) into M_f. Finally, what is a (e)? (e) is j composite f is homotopic to i. j composite f =j composite f hat composite i.

Because f hat composite i is f. Now you have already shown that j composite f hat is homotopic to identity of M_f. Therefore, the whole thing will be homotopic to identity composite to $\mathbb{I}$, this is same thing as the inclusion. So (e) follows easily if we put these two things together, (a) and (b) together.

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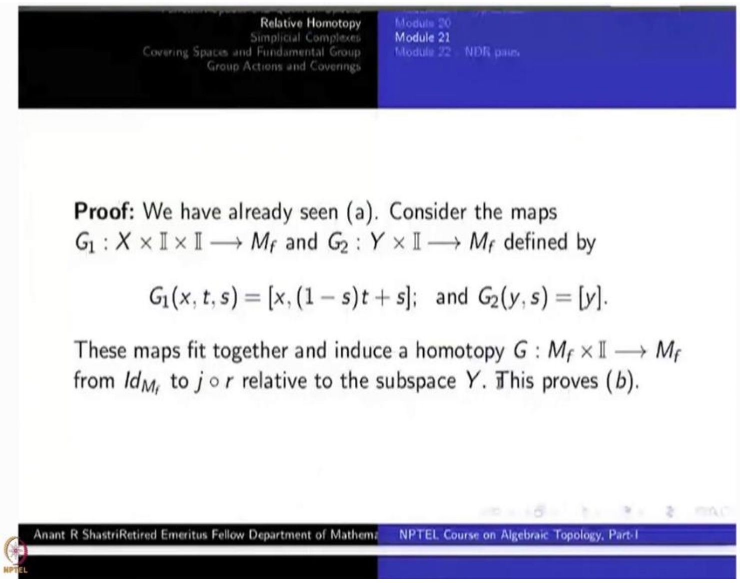

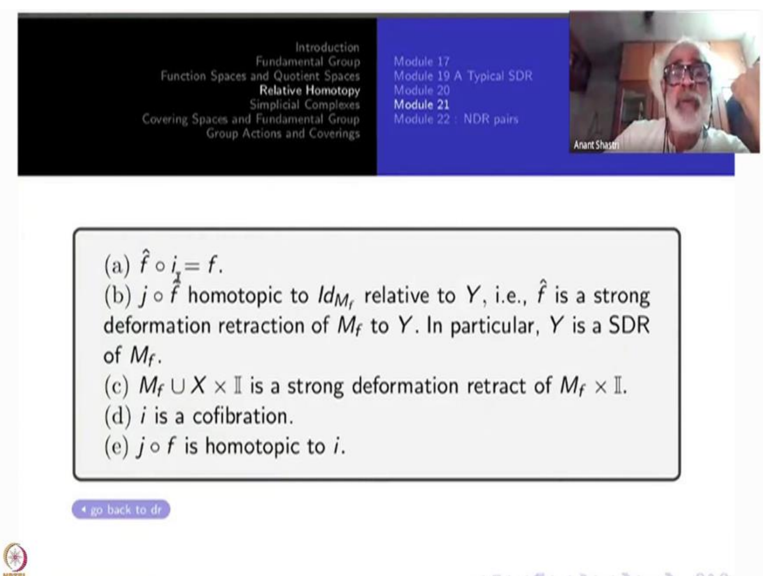

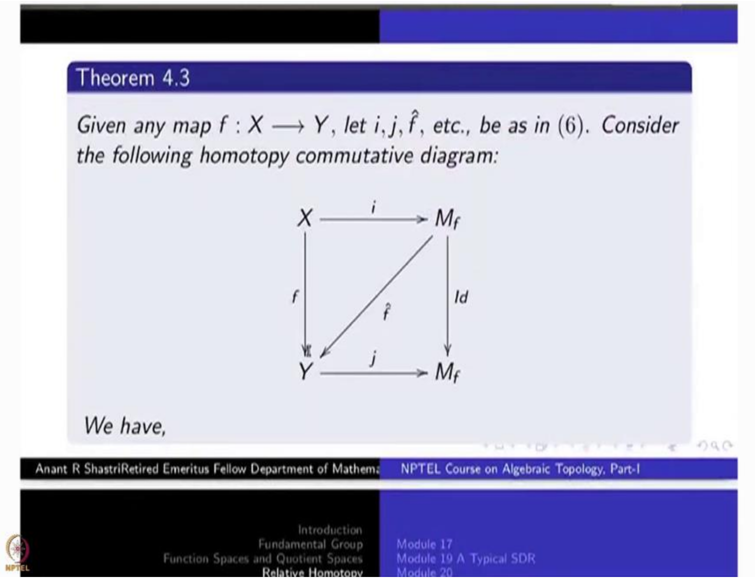

So this is the essence of all this. It may be difficult to remember. So, let us go through it once again. What you have done? start with f any map, it is getting replaced by the inclusion map. This f is getting replaced by the inclusion map. The inclusion map is homotopic to this one ---as if it is homotopic to f. It is not exactly homotopic to f because things are not taking place here, but inside here.

So, j composite to f is homotopic to i that is the meaning of saying that f is replaced by i , inclusion map up to homopty. Instead of Y we have M_f, but what does it represent? M_f and Y they are of the same homotopy type--they are homotopy equivalent to each other. In fact, this is not an ordinary homotopy equivalence. This is a strong deformation retraction. Y is a strong deformation retract. Up to a strong deformation retract, this f becomes i.

We cannot define an inverse of f here. But the map j plays the role of homotopy inverse, that is also seen. So f composite j, mapped inside M_f, that is homotopic i. So, in that sense an arbitrary continuous function has been replaced by an inclusion map into some other space. And what is this space? this is of homotopy type of the codomain Y. Y is strong deformation retract, codomain is a strong deformation retract of new one.

The new space contains the codomain also. So it is an enlargement of the codomain of this map. f has some codomain Y. Now that Y we have enlarged and the enlargement does not lose the homotopy information, I mean, homotopy information is not lost. So this is the gist of this theorem and essentially we use the construction of this number (8), namely, going back all the way, of taking $\mathbb{I} \times \mathbb{I}$ and constructing a strong deformation retract there. That was the fundamental idea in all these things.

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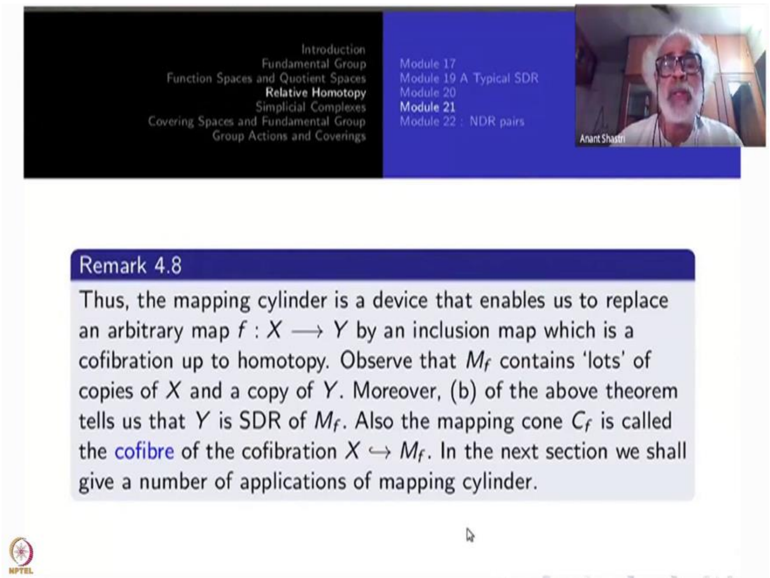

Thus the mapping cylinder is a device that enables us to replace an arbitrary map f from X to Y by an inclusion map, which is a cofibration, up to homotopy. Observe that M_f contains lots of copies of X and one copy of Y. X $\mathbb{I} \times \mathbb{I}$, so for each t, $X \times t$ is there. Except at $X \times 1$ there are no identifications. From (b) of the above theorem, we have that Y is a strong deformation retract of M_f. The mapping cone, there is a mapping cone construction, C_f is called the cofiber of this fibration.

In the next section, we shall give a number of applications of the mapping cylinder. So, to sum up, the hard work is somewhat over now. So, in next module, we will reap the harvest, lot of interesting applications will be there now for this. So that is all. We will not continue now, we will stop here. So let us take the next thing later. So there are some exercises here.

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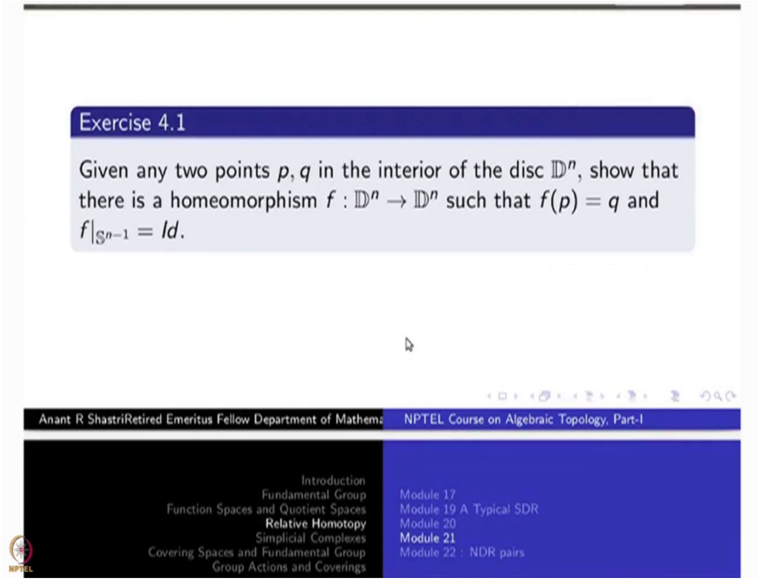

As usually I will just go through a few of them, but I am not giving you any solutions here. The first thing is, you should know all these things, these are all more or less point-set-topological stuffs.

You begin with two points in the interior of the disc $\mathbb{D}^n$. There is a homeomorphism f from $\mathbb{D}^n$ to $\mathbb{D}^n$ such that f(p)= q. p and q are both interior points. On the boundary f is identity.

The boundary is undisturbed, any point p goes to any other point q. First prove this one for a closed interval a b, a and b are fixed. Give me a homeomorphism which takes any point inside a b to another point. And it must be homeomorphism. I think you will be able to do this. Then, for all discs you must able to do this. For the general case,, once you do it for the unit disc, it will be done for all the discs.

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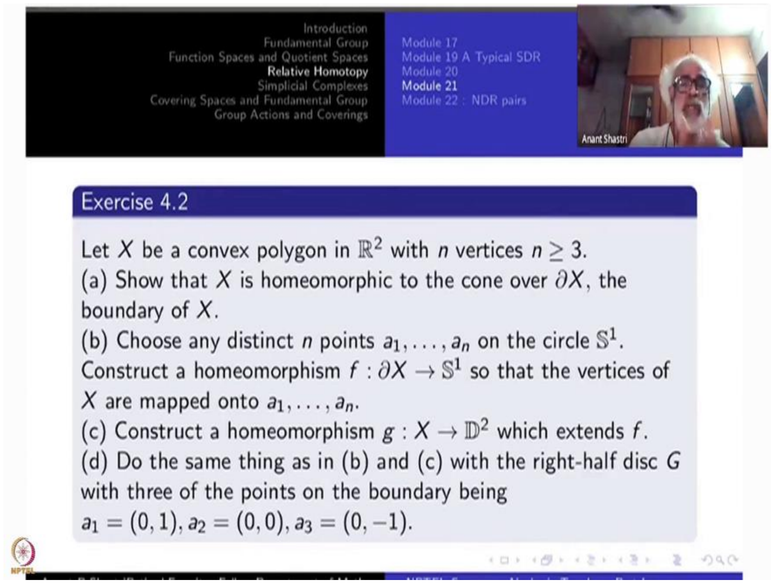

Next thing is about how far are these convex subsets related to the standard convex set namely a disc. Let us look at that one with X be a convex polygon inside $\mathbb{R}^2$ with n sides. This is n ≥3. A convex polygon is like a triangle or a quadrilateral and so on. I am not assuming any regularity fro them,-- any convex polygon. So that X is homeomorphic to the cone over the boundary of X, What is boundary of X? When you say a triangle, there can be some confusion-- is it the `full' triangle or the boundary triangle?

So, here I mean by the convex polygon I mean the entire convex set, the whole thing. And the boundary consists of only sides of the polygon (not a convex set!) So, on the boundary, which is a topological space, you take the cone over that. Then you have to show that the cone is homeomorphic to the convex polygon itself. The full triangle is homeomorphic the cone over the triangle. So that is a first exercise here.

Now choose any n distinct points, a_1, a_2 dot dot dot a_n, on the standard circle $\mathbb{S}^1$. Now construct a homeomorphism f from the boundary of this polygon to $\mathbb{S}^1$, so that the vertices of X, all the n vertices are mapped into a_1, a_2, and here you will have to, you are forced to take the a_1, a_2, a_n, in a particular order whichever way, suppose v_1, v_2 to v_n are the vertices of the polygon written in a particular order but consecutively.

Then a_1 to a_n must be also consecutive. You cannot shuffle them, you cannot separate them. So that is understood here. So v_1 should go to a_1 v_2 should go to a_2 etc v_n should go to a_n. And the entire thing must be a homeomorphism of the boundary of X to $\mathbb{S}^1$. Next construct a homeomorphism g from X to the entire disc inside $\mathbb{S}^1$, together with $\mathbb{S}^1$, which extends the given f in (b). That must be an extension-- on the boundary, it must be your f and the entire thing must be a homeomorphism g.

Do the same thing as in (b) and (c) with the right half disc G, three of the points on the boundary being 0 1,0 0, 0 - 1. So I am taking the half disc here inside $\mathbb{D}^2$, inside $\mathbb{R}^2$. Instead of taking a convex polygon I am taking another kind of convex set, the half disc is also convex set.

Its boundary consists of a line segment on the y axis namely - 1 to 1 and a half circle on the right, right half disc you have taken. So take 0 1, 0 0, 0 - 1 these three points. Map them to three distinct points on the circle, construct the homeomorphism on the boundary. And then construct the homeomorphism of the half disc to the full disc. That is what you have to do in the (d)

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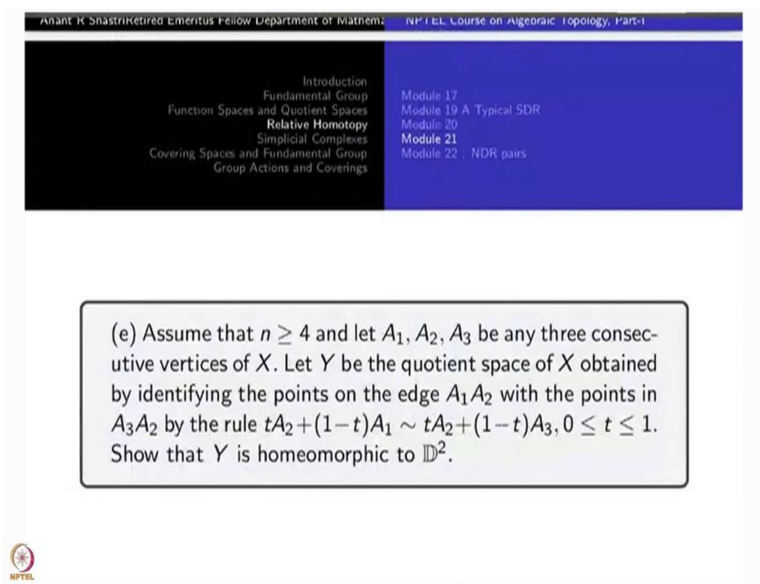

Similarly, now this exercise (e). Assume that n is greater than or equal to 4 now, at least 4. Let A_1, A_2, A_3 be three consecutive vertices of X. Let Y be the quotient space of X obtained by identifying the points of the edge A_1A_2 , with those of A_3A_2, in that reverse order, - identify them by the rule: t times A_2 + 1 times t times A_1, that will lie on the edge A_1A_2, should be identified with to t times A_2 + 1 - t times A_3.

In other words, A_2 will remain as it is, a_1 and a_3 are getting identified. So this is the only identification the two edges are identified. So for all t between 0 and 1, we make this identification. Whatever Y you get out of the entire convex polygon X, after this identification again is homeomorphic to $\mathbb{D}^2$. For this, you are assuming (but don't have to assume) that there are more than three vertices.

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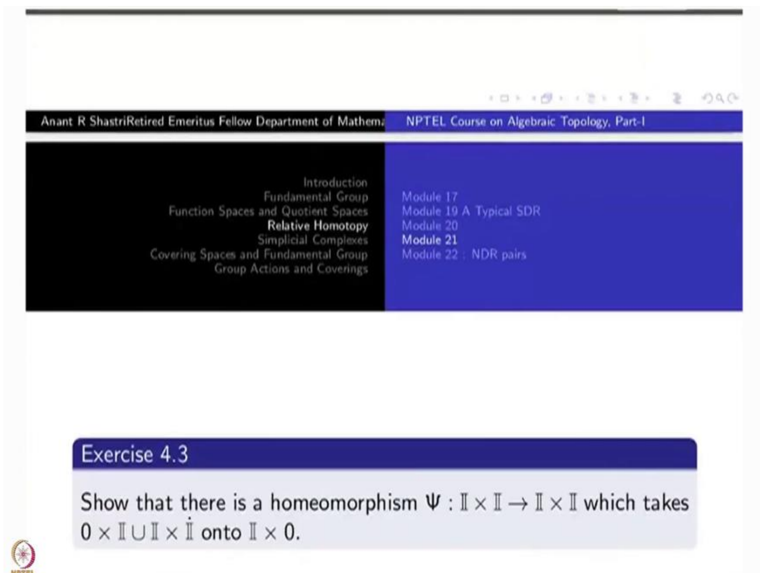

So like this there are other exercises also here.

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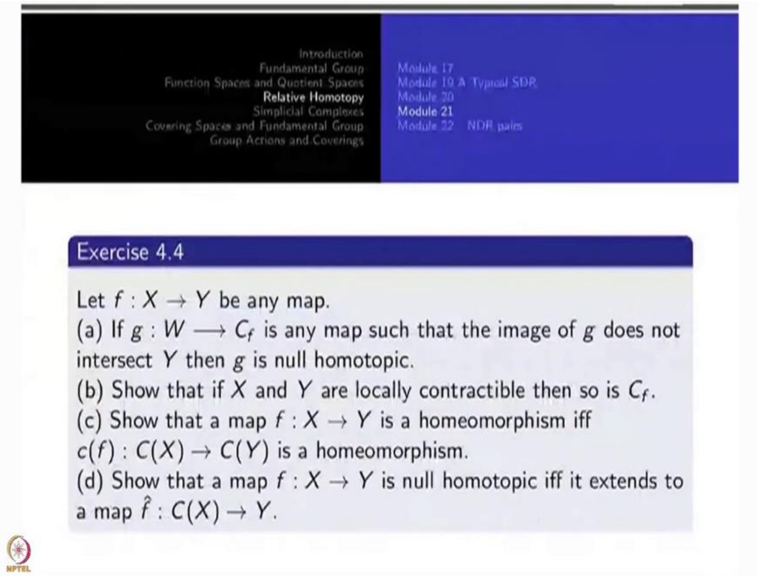

Then some for the cones.

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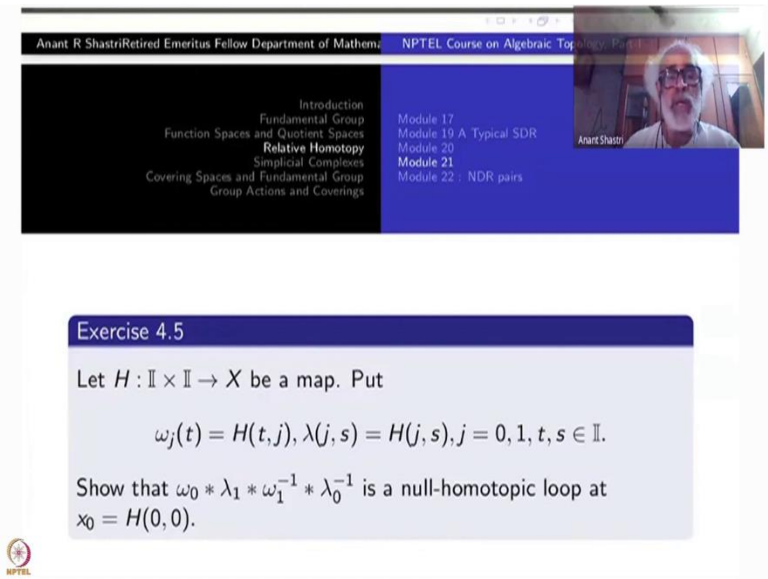

So later on, these are exercises on loops and homotopies of maps so on. So this is on the first part of the chapter.

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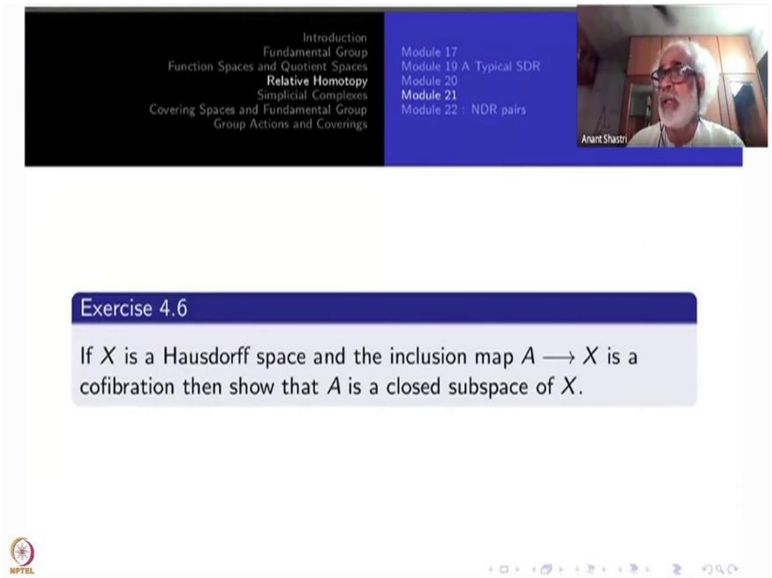

But this one, we have already used, if X is Hausdorff space and the inclusion map is a cofibration then A is a closed subspace. I have indicated the poof already. So you can just write down by memory if you have understood it or you have to work it out yourselves.

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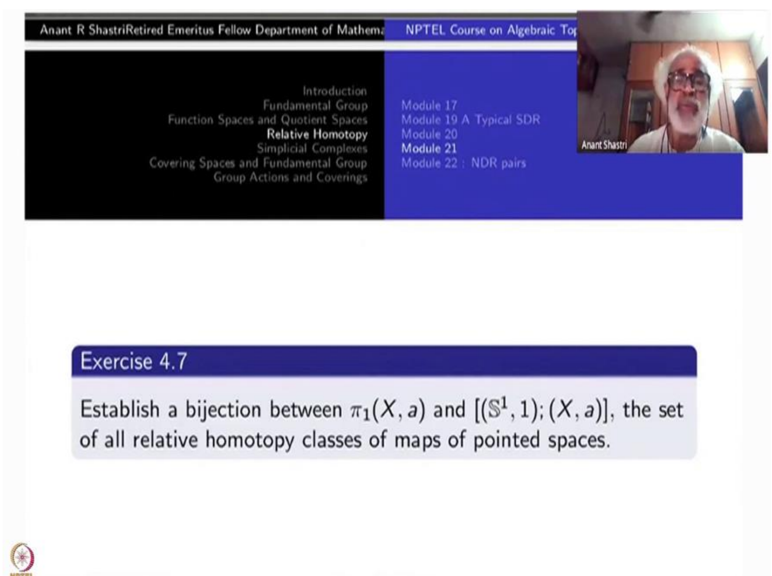

Solution of this also I have indicated namely, $\pi_1(X)$ a can be identified with homotopy classes of loops from $\mathbb{S}^1$ to X, I mean continuous functions from $\mathbb{S}^1$ to X, where 1 goes to a. Homotopies here keep the point 1 fixed throughout the homotopy. So that is the meaning of $\pi\pi_1(X, a)$. This result has been already used. So you write down a detailed proof of that one.

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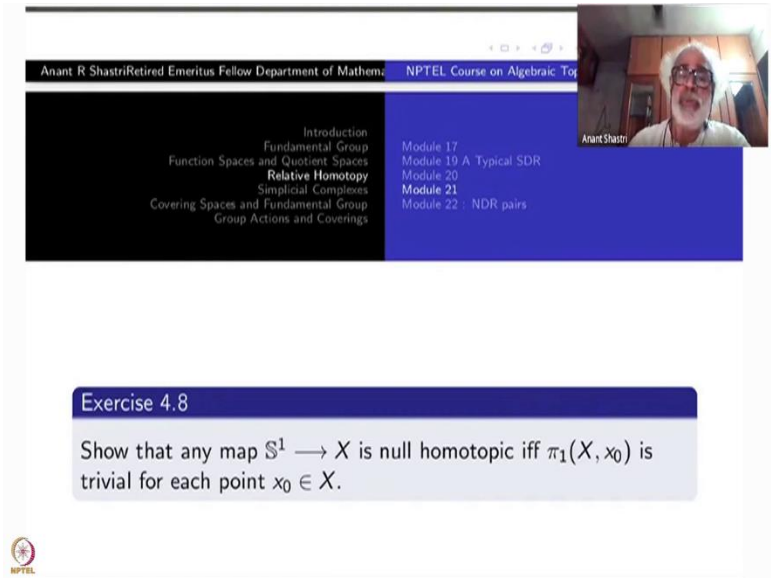

So now suppose you have that every map is null homotopic, defined on the circle, then $\pi_1(X, x_0)$ is trivial. For each point x_0 belonging to X. What I am saying? Take any loop in X, suppose it is null homotopic. If any loop in X is null homotopic then π$\pi_1(X, x_0)$ is trivial for each point of X. No matter where it is. This should be true for all loops. So this is a straightforward application of this exercise.

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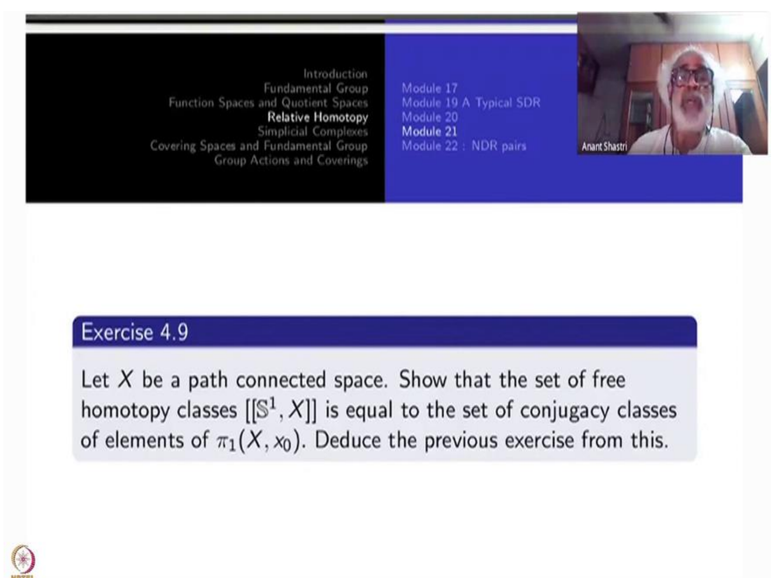

So, suppose you have a path connected space. The set of all homotopy classes i.e., without any base points, free homotopy classes is equal to the set of conjugacy classes of elements in π $\pi_1(X, x_0)$. $\pi\pi_1(X, x_0)$ is a group. In a group you know what is the meaning of conjugacy classes? That is what you have to show, that conjugacy classes are in one one correspondence with the free homotopy classes of maps from $\mathbb{S}^1$ to X. Base points are not fixed here. So that is the difference. From this you can deduce this theorem, and this exercise also.

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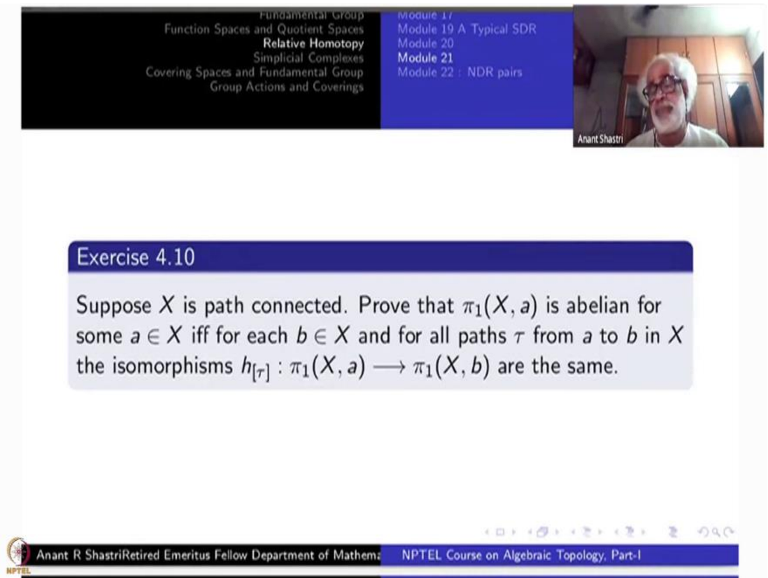

This exercise, actually I have solved it in the theory part itself, roughly, I have given you sufficient hints. So this is easy to work out for. X is a path connected space. Then $\pi\pi_1(X, x_0)$ is abelian if and only if what happens? for each b inside X, and for any path τ from a to b, the homomorphism $h_{[\tau]}$, which is obtained by conjugating by the path τ, $h_{[\tau]} : \pi_1(X, a) \to \pi_1(X, b)$

this map is the same-- for whatever $\tau\tau$ is. For all paths τ, $h_{[\tau]}\tau$ is the same map-- same homomorphism, the same bijection whatever.

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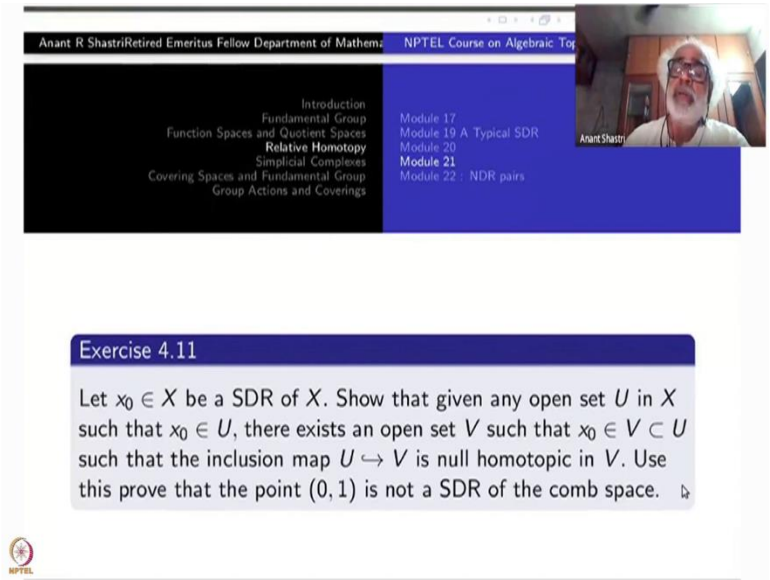

This is an exercise which will help you to prove that in the comb space 1,0 is not a strong deformation retract. First prove this exercise 4.11. What does it say? Suppose some point x_0 is a strong deformation retract, then for every open subset U(X) containing x_0, there is another neighborhood V of x_0 inside U, (there is a typo there) such that inclusion map V to U is null homotopic in U. V to U you can write a homotopy to a constant function that is the conclusion - starting with the inclusion map to a constant function. That is null-homotopy.

If you use this cleverly then you can show that 0 1 is not a SDR of the comb space. No hand waving. For that I already told you what you have to use. The comb space is not locally path connected or locally connected at any point on the y axis. It fails to be locally path connected. Use this exercise to complete the proof of this SDR.

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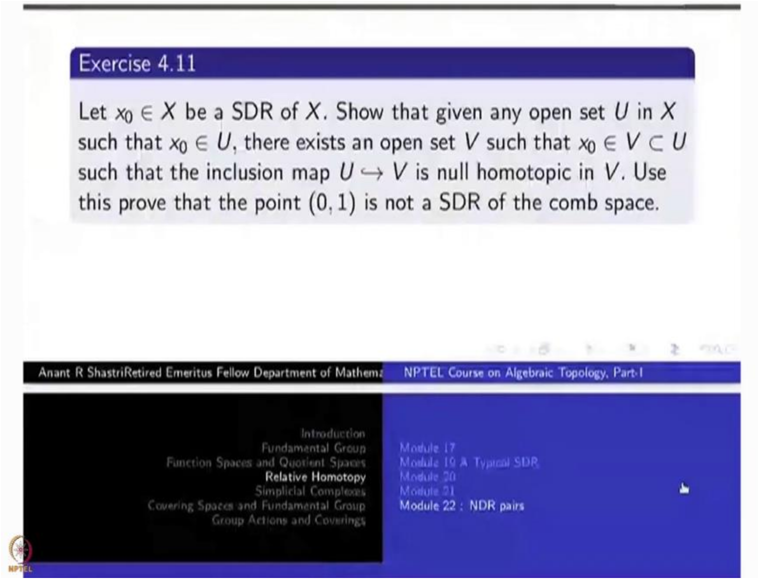

So let us stop here. Next time we will see many more results. So as I have told you, hard work is more or less over for a while. Thank you.