

**Introduction to Algebraic Topology (Part-1)**  
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**Lecture Number 21**  
**A Theoretical Application**

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**Module 21**

We shall now give an application of this in obtaining a powerful tool in homotopy theory which 'replaces' any map by an inclusion map which is a cofibration.

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As a student, I was wondering why the several books give definition of cofibration for inclusion maps only, whereas fibration is for arbitrary map not necessarily for quotient maps. The answer was in this result that we are going to do. But there is an even more satisfactory answer. Much much later, only some 10 years back, I read it from Hatcher's book that every cofibration is actually an inclusion map.


So what are we going to do? Because we are studying things up to homotopy, in homotopy theory, this result says that any map can be replaced by an inclusion map. And that inclusion map is a cofibration. So this was so powerful that now you can think up to homotopy, every map is a cofibration.

So this completely justifies the somewhat artificially introduced the notion of cofibration. We are not claiming that every map is a cofibration but up to homotopy it is. So let us go through this one. So, later on, we will see a lot of applications of this and so, this itself is somewhat an hard work you have to do now.

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**Theorem 4.3**

Given any map  $f : X \rightarrow Y$ , let  $i, j, \hat{f}$ , etc., be as in (6). Consider the following homotopy commutative diagram:

We have,


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So here is the statement. Given any map  $f$  from  $X$  to  $Y$  let us construct the mapping cylinder  $M_f$ . Recall what is  $M_f$ .  $M_f$  was obtained as the quotient of  $X \times \mathbb{I}$  disjoint union  $Y$  by the identification namely, each  $(x, 1)$  is identified with this image  $f(x)$ . So I am making this diagram of various functions here,  $X$  is included in  $M_f$ .

This map  $X$  to  $Y$  and  $M_f$  to  $Y$ . There is an  $\hat{f}$  and then  $Y$  is also included in  $M_f$  by an inclusion map  $j$ . So this  $\hat{f}$  is actually an extension of  $f$ , because  $X$  is thought of as a subspace  $M_f$ . So all these things I am writing down carefully namely  $\hat{f}$  composite  $i$  is  $f$ . In the first triangle it is commutative, -- top triangle.

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Given any map  $f : X \rightarrow Y$ , let  $i, j, \hat{f}$ , etc., be as in (6). Consider the following homotopy commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{i} & M_f \\
 \downarrow f & \searrow \hat{f} & \downarrow Id \\
 Y & \xrightarrow{j} & M_f
 \end{array}$$

We have,

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$j$  composite  $\hat{f}$ , in here from  $M_f$  to  $M_f$ .

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(a)  $\hat{f} \circ i = f$ .  
 (b)  $j \circ \hat{f}$  homotopic to  $Id_{M_f}$  relative to  $Y$ , i.e.,  $\hat{f}$  is a strong deformation retraction of  $M_f$  to  $Y$ . In particular,  $Y$  is a SDR of  $M_f$ .  
 (c)  $M_f \cup X \times \mathbb{I}$  is a strong deformation retract of  $M_f \times \mathbb{I}$ .  
 (d)  $i$  is a cofibration.  
 (e)  $j \circ f$  is homotopic to  $i$ .

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Given any map  $f : X \rightarrow Y$ , let  $i, j, \hat{f}$ , etc., be as in (6). Consider the following homotopy commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{i} & M_f \\
 \downarrow f & \searrow \hat{f} & \downarrow Id \\
 Y & \xrightarrow{j} & M_f
 \end{array}$$

We have,

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This map is homotopic to Identity of  $M_f$ . So the second triangle here the bottom triangle---- it is not a commutative triangle, it is homotopy commutative.  $j$  composite  $\hat{f}$  is homotopic to identity. And this homotopy is relative to  $Y$ , that is  $\hat{f}$  is a strong deformation retraction of  $M_f$  to  $Y$ .

In particular  $Y$  is a SDR of  $M_f$ .  $M_f \cup X \times \mathbb{I}$  is a strong deformation retract of  $M_f \times \mathbb{I}$ . So  $i$  is a cofibration, the inclusion map, from (c) to (d)--- you know how to go-- because we have seen this one.  $j$  composite  $f$  finally this  $j$  composite  $f$  coming from  $X$  to  $M_f$ ,  $j$  composite  $f$ , it is homotopic to the inclusion map itself.

So when you put this, this arrow identity, if you do not put this one that diagram is commutative, if you put this one it will be homotopy commutative. If you come from here to here and come from here to here, it is homotopy. Because this is homotopic to the identity, precisely the last thing it says-- $j$  composite  $f$  is homotopic to the inclusion map  $i$ .


So let us prove them one by one. Some of them are obvious. Namely, the first thing that  $f$  hat composite  $i$  is  $f$ , we have already seen. What is  $f$  hat? Each point of  $M_f$  is a class--  $(x, t$  or bracket  $y$ . Remember that means equivalence class of elements from  $X \times \mathbb{I} \cup Y$ . On  $Y$  part, it takes bracket  $y$  to itself. If it is  $(x, t)$ , it is taken to  $fx$ , so it is well defined because whenever there is an identification, namely, at  $X \times \mathbb{I}$ ,  $x \times 1$  is identified with  $fx$  only. So it makes sense. So it is an extension of  $f$  which is defined on  $X \times 0$ . So first part we have already seen.

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**Proof:** We have already seen (a). Consider the maps  $G_1 : X \times \mathbb{I} \times \mathbb{I} \rightarrow M_f$  and  $G_2 : Y \times \mathbb{I} \rightarrow M_f$  defined by

$$G_1(x, t, s) = [x, (1-s)t + s]; \text{ and } G_2(y, s) = [y].$$

These maps fit together and induce a homotopy  $G : M_f \times \mathbb{I} \rightarrow M_f$  from  $Id_{M_f}$  to  $j \circ r$  relative to the subspace  $Y$ . This proves (b).



- (a)  $\hat{f} \circ i = f$ .
- (b)  $j \circ \hat{f}$  homotopic to  $Id_{M_f}$  relative to  $Y$ , i.e.,  $\hat{f}$  is a strong deformation retraction of  $M_f$  to  $Y$ . In particular,  $Y$  is a SDR of  $M_f$ .
- (c)  $M_f \cup X \times \mathbb{I}$  is a strong deformation retract of  $M_f \times \mathbb{I}$ .
- (d)  $i$  is a cofibration.
- (e)  $j \circ f$  is homotopic to  $i$ .

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Secondly, we have to define a homotopy here. So take  $G_1$  from  $X \times \mathbb{I} \times \mathbb{I}$  to  $M_f$ , and  $G_2$  from  $Y \times \mathbb{I}$  to  $M$  defined as follows:  $G_1(x, t, s)$  goes to  $x$ , something.  $x$  is as it is you see, everything is happening in  $\mathbb{I} \times \mathbb{I}$ . So that 'something' is  $1 - s$  times  $t + s$  --- it joins  $t$  and  $1$  ---  $1 - s$  time  $t + s$  times the constant  $1$  --- that is the identity function and the constant function are being joined. That is first one. The second function  $G_2(y), s$  is just  $y$ , --- ignores the  $s$  coordinate.

These maps together induce a homotopy from  $M_f \times \mathbb{I}$  to  $M_f$ . On  $X \times \mathbb{I}$ , when  $t = 1$ , it is identified with  $fx$ . If  $t$  is  $1$  this is just  $x, 1$ . And this is just  $y$ . So these maps fit together to define maps from  $M_f \times \mathbb{I}$  to  $M_f$ . Before we go down to the quotient spaces,  $G_1$  and  $G_2$  are easy to understand. On  $M_f$

$\times \mathbb{I}$ , it is from identity of  $M_f$  to  $j$  composite  $r$  relative to  $Y$ . On  $Y$  always it is the identity function. What is  $j$  composite  $r$ ?

Let just see that, what is this  $r$  I have got? A typo, this is not correct.  $j$  composite  $f$  hat, it should instead of  $j$  composite  $r$ . That is not correct. Identity of  $M_f$  to  $j$  composite  $f$  hat relative to  $Y$ .

Now the proof of (c).  $M_f$  union  $X \times \mathbb{I}$  is a strong deformation retract of an  $M_f \times \mathbb{I}$ . So what do you have to do? You have to do just same thing -- what you have done.

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Let us use the notation  $[x, t], [x, a, b]$  to denote the image of  $(x, t) \in X \times \mathbb{I}, (x, a, b) \in X \times \mathbb{I} \times \mathbb{I}$ , etc., under  $q$  or  $q \times Id$ , etc., where  $q : X \times \mathbb{I} \rightarrow CX$  is the quotient map. The function  $H$  respects the equivalence relation defining the cone because, the  $t$  coordinate of  $S(0, t', t'')$  is always zero. Hence there is a well-defined function  $\bar{H} : CX \times \mathbb{I} \times \mathbb{I} \rightarrow CX \times \mathbb{I}$ , viz.,

$$\bar{H}([x, t], t', t'') = [x, S(t, t', t'')]. \quad (8)$$

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I have to recall that the construction that we have done in the previous module where is it? Module 20 or 19 whatever. So let me go back to the formula (8) here all the way this one. This map will be used again.


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For the proof of (c), the construction (8) comes to help. Define  $H_1 : X \times \mathbb{I} \times \mathbb{I} \times \mathbb{I} \rightarrow M_f \times \mathbb{I}$  and  $H_2 : Y \times \mathbb{I} \times \mathbb{I} \rightarrow M_f \times \mathbb{I}$  by

$$H_1(x, t, t', t'') = [x, S(t, t', t'')]; \quad H_2(y, t', t'') = [y, 1 - t'],$$

with similar notation as in (8). Verify that they fit together to define a map  $H : M_f \times \mathbb{I} \times \mathbb{I} \rightarrow M_f \times \mathbb{I}$  which is a strong deformation retraction of  $M_f \times \mathbb{I}$  into  $M_f \cup X \times \mathbb{I}$ . In view of Proposition 4.1, the conclusion (d) is obvious using (c). Finally, (e) follows from (a) and (b). ♠




	
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(a)  $\hat{f} \circ i = f$ .  
(b)  $j \circ \hat{f}$  homotopic to  $Id_{M_f}$  relative to  $Y$ , i.e.,  $\hat{f}$  is a strong deformation retraction of  $M_f$  to  $Y$ . In particular,  $Y$  is a SDR of  $M_f$ .  
(c)  $M_f \cup X \times \mathbb{I}$  is a strong deformation retract of  $M_f \times \mathbb{I}$ .  
(d)  $i$  is a cofibration.  
(e)  $j \circ f$  is homotopic to  $i$ .

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(a)  $\hat{f} \circ i = f.$   
 (b)  $j \circ \hat{f}$  homotopic to  $Id_{M_f}$  relative to  $Y$ , i.e.,  $\hat{f}$  is a strong deformation retraction of  $M_f$  to  $Y$ . In particular,  $Y$  is a SDR

Exactly similar to that. So,  $H_1$  is from  $X \times \mathbb{I} \times \mathbb{I} \times \mathbb{I}$  to  $M_f \times \mathbb{I}$ . Actually it is from  $Y \times \mathbb{I} \times \mathbb{I}$  to  $M_f$ .  $H_1(x, t, t'' x, S)$  similar to what we have done in the case of arbitrary  $X$ . That is precisely what we are doing here exactly same way.  $S$  part is taken here,  $x$  part is undisturbed  $\times \mathbb{I}$ .  $H_2 y, t', t''$  is--  $y$  remains as it is and we are ignoring  $t''$  here  $1 - t'$ .

Taking the reverse of that, if you do this, just like the same, similar notation as in that one. All that you have to do is verify that they fit together, fitting together means what?  $Y$  is a subspace of  $M_f$ . So on that part, you have to verify that wherever this, in this  $Y$  is instead of  $M_f$  here  $X \times \mathbb{I} f X$  when  $X \times 1$  is identified in  $M_f$  where it is like  $Y$ . So there also you have to verify what happens there.

So then they agree as a function. So then continuity follows because all these, wherever they agree all those spaces are closed subspaces. So they fit together to define a map  $H$  on  $M_f \times \mathbb{I} \times \mathbb{I}$ ,  $M_f$  is the quotient of this part  $X \times \mathbb{I}$  disjoint union  $Y \times \mathbb{I}$  one more factor  $\mathbb{I}$  are also taken--- then the identification. You have to verify. So it gives you a map from  $M_f \times \mathbb{I} \times \mathbb{I}$  to  $M_f \times \mathbb{I}$  which is a deformation retract of  $M_f \times \mathbb{I}$  into  $M_f$  union  $X \times \mathbb{I}$

Once you have these the proposition will tell you that the inclusion map of  $M_f$  into  $M_f \times \mathbb{I}$  is a what, sorry, inclusion  $(X)$  into  $M_f$  is a cofibration, so that is (d). (d) tells you that  $i$  is a cofibration. What is  $i$ ?  $i$  is the inclusion map  $(X)$  into  $M_f$ . Finally, what is a (e)? (e) is  $j$  composite  $f$  is homotopic to  $i$ .  $j$  composite  $f = j$  composite  $\hat{f}$  composite  $i$ .



Because  $\hat{f} \circ i$  is  $f$ . Now you have already shown that  $j \circ \hat{f}$  is homotopic to identity of  $M_f$ . Therefore, the whole thing will be homotopic to identity composite to  $\mathbb{I}$ , this is same thing as the inclusion. So (e) follows easily if we put these two things together, (a) and (b) together.

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**Proof:** We have already seen (a). Consider the maps  $G_1 : X \times \mathbb{I} \times \mathbb{I} \rightarrow M_f$  and  $G_2 : Y \times \mathbb{I} \rightarrow M_f$  defined by

$$G_1(x, t, s) = [x, (1-s)t + s]; \text{ and } G_2(y, s) = [y].$$

These maps fit together and induce a homotopy  $G : M_f \times \mathbb{I} \rightarrow M_f$  from  $Id_{M_f}$  to  $j \circ r$  relative to the subspace  $Y$ . This proves (b).

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(a)  $\hat{f} \circ i = f$ .

(b)  $j \circ \hat{f}$  homotopic to  $Id_{M_f}$  relative to  $Y$ , i.e.,  $\hat{f}$  is a strong deformation retract of  $M_f$  to  $Y$ . In particular,  $Y$  is a SDR of  $M_f$ .

(c)  $M_f \cup X \times \mathbb{I}$  is a strong deformation retract of  $M_f \times \mathbb{I}$ .

(d)  $i$  is a cofibration.

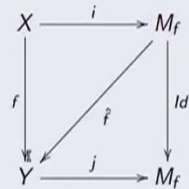
(e)  $j \circ f$  is homotopic to  $i$ .

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**Theorem 4.3**

Given any map  $f : X \rightarrow Y$ , let  $i, j, \hat{f}$ , etc., be as in (6). Consider the following homotopy commutative diagram:



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So this is the essence of all this. It may be difficult to remember. So, let us go through it once again. What you have done? start with  $f$  any map, it is getting replaced by the inclusion map. This  $f$  is getting replaced by the inclusion map. The inclusion map is homotopic to this one ---as if it is homotopic to  $f$ . It is not exactly homotopic to  $f$  because things are not taking place here, but inside here.

So,  $j$  composite to  $f$  is homotopic to  $i$  that is the meaning of saying that  $f$  is replaced by  $i$ , inclusion map up to homotopy. Instead of  $Y$  we have  $M_f$ , but what does it represent?  $M_f$  and  $Y$  they are of the same homotopy type--they are homotopy equivalent to each other. In fact, this is not an ordinary homotopy equivalence. This is a strong deformation retraction.  $Y$  is a strong deformation retract. Up to a strong deformation retract, this  $f$  becomes  $i$ .

We cannot define an inverse of  $f$  here. But the map  $j$  plays the role of homotopy inverse, that is also seen. So  $f$  composite  $j$ , mapped inside  $M_f$ , that is homotopic  $i$ . So, in that sense an arbitrary continuous function has been replaced by an inclusion map into some other space. And what is this space? this is of homotopy type of the codomain  $Y$ .  $Y$  is strong deformation retract, codomain is a strong deformation retract of new one.

The new space contains the codomain also. So it is an enlargement of the codomain of this map.  $f$  has some codomain  $Y$ . Now that  $Y$  we have enlarged and the enlargement does not lose the homotopy information, I mean, homotopy information is not lost. So this is the gist of this theorem and essentially we use the construction of this number (8), namely, going back all the way, of

taking  $\mathbb{I} \times \mathbb{I}$  and constructing a strong deformation retract there. That was the fundamental idea in all these things.

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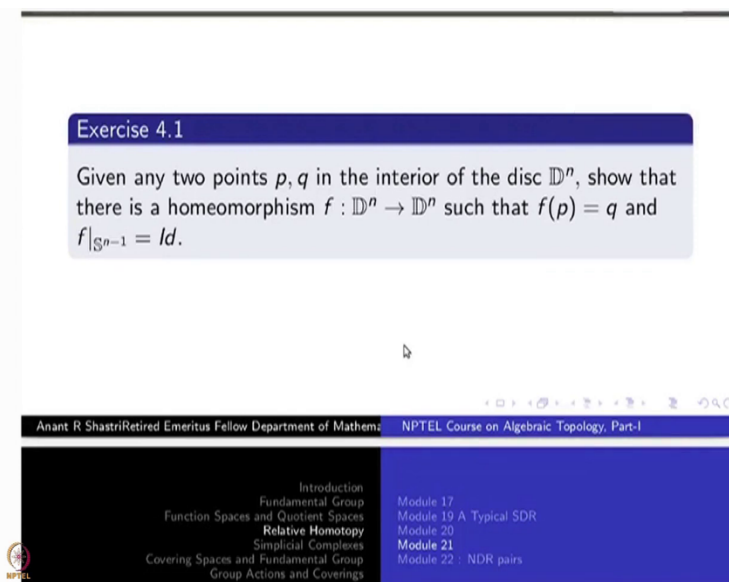
**Remark 4.8**

Thus, the mapping cylinder is a device that enables us to replace an arbitrary map  $f : X \rightarrow Y$  by an inclusion map which is a cofibration up to homotopy. Observe that  $M_f$  contains 'lots' of copies of  $X$  and a copy of  $Y$ . Moreover, (b) of the above theorem tells us that  $Y$  is SDR of  $M_f$ . Also the mapping cone  $C_f$  is called the **cofibre** of the cofibration  $X \hookrightarrow M_f$ . In the next section we shall give a number of applications of mapping cylinder.

Thus the mapping cylinder is a device that enables us to replace an arbitrary map  $f$  from  $X$  to  $Y$  by an inclusion map, which is a cofibration, up to homotopy. Observe that  $M_f$  contains lots of copies of  $X$  and one copy of  $Y$ .  $X \times \mathbb{I} \times \mathbb{I}$ , so for each  $t$ ,  $X \times t$  is there. Except at  $X \times 1$  there are no identifications. From (b) of the above theorem, we have that  $Y$  is a strong deformation retract of  $M_f$ . The mapping cone, there is a mapping cone construction,  $C_f$  is called the cofiber of this fibration.

In the next section, we shall give a number of applications of the mapping cylinder. So, to sum up, the hard work is somewhat over now. So, in next module, we will reap the harvest, lot of interesting applications will be there now for this. So that is all. We will not continue now, we will stop here. So let us take the next thing later. So there are some exercises here.

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The slide contains the following text:

**Exercise 4.1**

Given any two points  $p, q$  in the interior of the disc  $\mathbb{D}^n$ , show that there is a homeomorphism  $f : \mathbb{D}^n \rightarrow \mathbb{D}^n$  such that  $f(p) = q$  and  $f|_{S^{n-1}} = Id$ .

Below the text is a navigation bar with the following information:

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As usually I will just go through a few of them, but I am not giving you any solutions here. The first thing is, you should know all these things, these are all more or less point-set-topological stuffs.

You begin with two points in the interior of the disc  $\mathbb{D}^n$ . There is a homeomorphism  $f$  from  $\mathbb{D}^n$  to  $\mathbb{D}^n$  such that  $f(p)=q$ .  $p$  and  $q$  are both interior points. On the boundary  $f$  is identity.

The boundary is undisturbed, any point  $p$  goes to any other point  $q$ . First prove this one for a closed interval  $a, b$ ,  $a$  and  $b$  are fixed. Give me a homeomorphism which takes any point inside  $a, b$  to another point. And it must be homeomorphism. I think you will be able to do this. Then, for all discs you must be able to do this. For the general case,, once you do it for the unit disc, it will be done for all the discs.

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The screenshot shows a presentation slide with a navigation menu on the left and a video feed on the right. The navigation menu includes: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, **Relative Homotopy**, Simplicial Complexes, Covering Spaces and Fundamental Group, and Group Actions and Coverings. The video feed shows a man with glasses and a white beard, identified as Anant Shastri. The main content of the slide is 'Exercise 4.2' which asks to show that a convex polygon in  $\mathbb{R}^2$  is homeomorphic to a cone over its boundary, and to construct homeomorphisms mapping vertices to points on a circle and to the right-half disc.

Next thing is about how far are these convex subsets related to the standard convex set namely a disc. Let us look at that one with  $X$  be a convex polygon inside  $\mathbb{R}^2$  with  $n$  sides. This is  $n \geq 3$ . A convex polygon is like a triangle or a quadrilateral and so on. I am not assuming any regularity from them, -- any convex polygon. So that  $X$  is homeomorphic to the cone over the boundary of  $X$ , What is boundary of  $X$ ? When you say a triangle, there can be some confusion -- is it the 'full' triangle or the boundary triangle?

So, here I mean by the convex polygon I mean the entire convex set, the whole thing. And the boundary consists of only sides of the polygon (not a convex set!) So, on the boundary, which is a topological space, you take the cone over that. Then you have to show that the cone is homeomorphic to the convex polygon itself. The full triangle is homeomorphic the cone over the triangle. So that is a first exercise here.

Now choose any  $n$  distinct points,  $a_1, a_2, \dots, a_n$ , on the standard circle  $\mathbb{S}^1$ . Now construct a homeomorphism  $f$  from the boundary of this polygon to  $\mathbb{S}^1$ , so that the vertices of  $X$ , all the  $n$  vertices are mapped into  $a_1, a_2, \dots, a_n$ , and here you will have to, you are forced to take the  $a_1, a_2, \dots, a_n$ , in a particular order whichever way, suppose  $v_1, v_2, \dots, v_n$  are the vertices of the polygon written in a particular order but consecutively.

Then  $a_1$  to  $a_n$  must be also consecutive. You cannot shuffle them, you cannot separate them. So that is understood here. So  $v_1$  should go to  $a_1$   $v_2$  should go to  $a_2$  etc  $v_n$  should go to  $a_n$ . And the entire thing must be a homeomorphism of the boundary of  $X$  to  $S^1$ . Next construct a homeomorphism  $g$  from  $X$  to the entire disc inside  $S^1$ , together with  $S^1$ , which extends the given  $f$  in (b). That must be an extension-- on the boundary, it must be your  $f$  and the entire thing must be a homeomorphism  $g$ .

Do the same thing as in (b) and (c) with the right half disc  $G$ , three of the points on the boundary being  $0, 1, 0, 0, 0 - 1$ . So I am taking the half disc here inside  $\mathbb{D}^2$ , inside  $\mathbb{R}^2$ . Instead of taking a convex polygon I am taking another kind of convex set, the half disc is also convex set.

Its boundary consists of a line segment on the  $y$  axis namely  $-1$  to  $1$  and a half circle on the right, right half disc you have taken. So take  $0, 1, 0, 0, 0 - 1$  these three points. Map them to three distinct points on the circle, construct the homeomorphism on the boundary. And then construct the homeomorphism of the half disc to the full disc. That is what you have to do in the (d)

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(e) Assume that  $n \geq 4$  and let  $A_1, A_2, A_3$  be any three consecutive vertices of  $X$ . Let  $Y$  be the quotient space of  $X$  obtained by identifying the points on the edge  $A_1A_2$  with the points in  $A_3A_2$  by the rule  $tA_2 + (1-t)A_1 \sim tA_2 + (1-t)A_3, 0 \leq t \leq 1$ . Show that  $Y$  is homeomorphic to  $\mathbb{D}^2$ .

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Similarly, now this exercise (e). Assume that  $n$  is greater than or equal to 4 now, at least 4. Let  $A_1, A_2, A_3$  be three consecutive vertices of  $X$ . Let  $Y$  be the quotient space of  $X$  obtained by identifying the points of the edge  $A_1A_2$ , with those of  $A_3A_2$ , in that reverse order, - identify them by the rule:  $t$  times  $A_2 + 1$  times  $t$  times  $A_1$ , that will lie on the edge  $A_1A_2$ , should be identified with  $t$  times  $A_2 + 1 - t$  times  $A_3$ .

In other words,  $A_2$  will remain as it is,  $a_1$  and  $a_3$  are getting identified. So this is the only identification the two edges are identified. So for all  $t$  between 0 and 1, we make this identification. Whatever  $Y$  you get out of the entire convex polygon  $X$ , after this identification again is homeomorphic to  $\mathbb{D}^2$ . For this, you are assuming (but don't have to assume) that there are more than three vertices.

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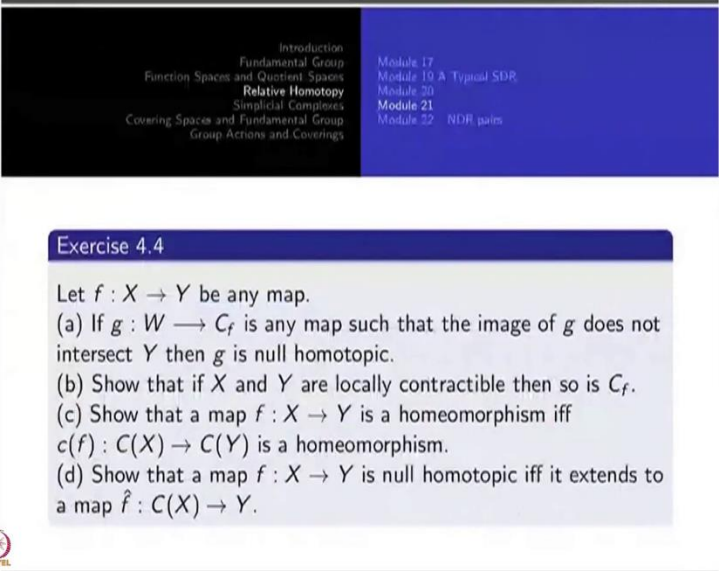
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**Exercise 4.3**

Show that there is a homeomorphism  $\Psi : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I} \times \mathbb{I}$  which takes  $0 \times \mathbb{I} \cup \mathbb{I} \times \mathbb{I}$  onto  $\mathbb{I} \times 0$ .

So like this there are other exercises also here.

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**Exercise 4.4**

Let  $f : X \rightarrow Y$  be any map.

(a) If  $g : W \rightarrow C_f$  is any map such that the image of  $g$  does not intersect  $Y$  then  $g$  is null homotopic.

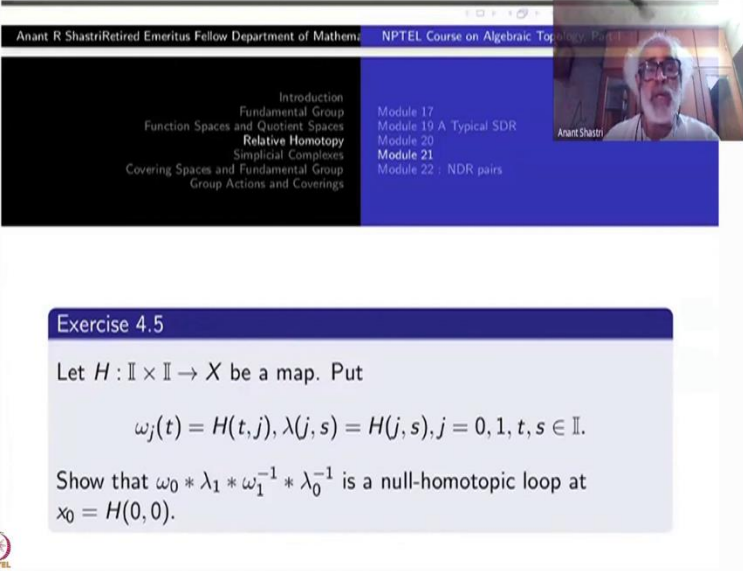
(b) Show that if  $X$  and  $Y$  are locally contractible then so is  $C_f$ .

(c) Show that a map  $f : X \rightarrow Y$  is a homeomorphism iff  $c(f) : C(X) \rightarrow C(Y)$  is a homeomorphism.

(d) Show that a map  $f : X \rightarrow Y$  is null homotopic iff it extends to a map  $\hat{f} : C(X) \rightarrow Y$ .

Then some for the cones.

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**Exercise 4.5**

Let  $H : \mathbb{I} \times \mathbb{I} \rightarrow X$  be a map. Put

$$\omega_j(t) = H(t, j), \lambda(j, s) = H(j, s), j = 0, 1, t, s \in \mathbb{I}.$$

Show that  $\omega_0 * \lambda_1 * \omega_1^{-1} * \lambda_0^{-1}$  is a null-homotopic loop at  $x_0 = H(0, 0)$ .

So later on, these are exercises on loops and homotopies of maps so on. So this is on the first part of the chapter.



(Refer Slide Time: 26:22)

The screenshot shows a video lecture interface. At the top left, a navigation menu lists: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, **Relative Homotopy**, Simplicial Complexes, Covering Spaces and Fundamental Group, and Group Actions and Coverings. To the right of the menu, a list of modules is shown: Module 17, Module 19 A Typical SDR, Module 20, **Module 21**, and Module 22: NDR pairs. A small video window in the top right corner shows the lecturer, Anant Shastri. The main content area features a blue header for "Exercise 4.6" and a text box containing the exercise statement: "If  $X$  is a Hausdorff space and the inclusion map  $A \rightarrow X$  is a cofibration then show that  $A$  is a closed subspace of  $X$ ." At the bottom left is the NPTEL logo, and at the bottom right are navigation icons.

But this one, we have already used, if  $X$  is Hausdorff space and the inclusion map is a cofibration then  $A$  is a closed subspace. I have indicated the proof already. So you can just write down by memory if you have understood it or you have to work it out yourselves.

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The screenshot shows a video lecture interface. At the top left, a navigation menu lists: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, **Relative Homotopy**, Simplicial Complexes, Covering Spaces and Fundamental Group, and Group Actions and Coverings. To the right of the menu, a list of modules is shown: Module 17, Module 19 A Typical SDR, Module 20, **Module 21**, and Module 22: NDR pairs. A small video window in the top right corner shows the lecturer, Anant Shastri. The main content area features a blue header for "Exercise 4.7" and a text box containing the exercise statement: "Establish a bijection between  $\pi_1(X, a)$  and  $[(\mathbb{S}^1, 1); (X, a)]$ , the set of all relative homotopy classes of maps of pointed spaces." At the bottom left is the NPTEL logo, and at the bottom right are navigation icons.

Solution of this also I have indicated namely,  $\pi_1(X, a)$  can be identified with homotopy classes of loops from  $\mathbb{S}^1$  to  $X$ , I mean continuous functions from  $\mathbb{S}^1$  to  $X$ , where 1 goes to  $a$ . Homotopies here keep the point 1 fixed throughout the homotopy. So that is the meaning of  $\pi_1(X, a)$ . This result has been already used. So you write down a detailed proof of that one.

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**Exercise 4.8**

Show that any map  $S^1 \rightarrow X$  is null homotopic iff  $\pi_1(X, x_0)$  is trivial for each point  $x_0 \in X$ .

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So now suppose you have that every map is null homotopic, defined on the circle, then  $\pi_1(X, x_0)$  is trivial. For each point  $x_0$  belonging to  $X$ . What I am saying? Take any loop in  $X$ , suppose it is null homotopic. If any loop in  $X$  is null homotopic then  $\pi_1(X, x_0)$  is trivial for each point of  $X$ . No matter where it is. This should be true for all loops. So this is a straightforward application of this exercise.

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**Exercise 4.9**

Let  $X$  be a path connected space. Show that the set of free homotopy classes  $[[S^1, X]]$  is equal to the set of conjugacy classes of elements of  $\pi_1(X, x_0)$ . Deduce the previous exercise from this.

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So, suppose you have a path connected space. The set of all homotopy classes i.e., without any base points, free homotopy classes is equal to the set of conjugacy classes of elements in  $\pi_1(X, x_0)$ .  $\pi_1(X, x_0)$  is a group. In a group you know what is the meaning of conjugacy classes? That is what you have to show, that conjugacy classes are in one one correspondence with the free homotopy classes of maps from  $S^1$  to  $X$ . Base points are not fixed here. So that is the difference. From this you can deduce this theorem, and this exercise also.

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**Exercise 4.10**

Suppose  $X$  is path connected. Prove that  $\pi_1(X, a)$  is abelian for some  $a \in X$  iff for each  $b \in X$  and for all paths  $\tau$  from  $a$  to  $b$  in  $X$  the isomorphisms  $h_{[\tau]} : \pi_1(X, a) \rightarrow \pi_1(X, b)$  are the same.

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This exercise, actually I have solved it in the theory part itself, roughly, I have given you sufficient hints. So this is easy to work out for.  $X$  is a path connected space. Then  $\pi_1(X, x_0)$  is abelian if and only if what happens? for each  $b$  inside  $X$ , and for any path  $\tau$  from  $a$  to  $b$ , the homomorphism  $h_{[\tau]}$ , which is obtained by conjugating by the path  $\tau$ ,  $h_{[\tau]} : \pi_1(X, a) \rightarrow \pi_1(X, b)$

this map is the same-- for whatever  $\tau$  is. For all paths  $\tau$ ,  $h_{[\tau]}$  is the same map-- same homomorphism, the same bijection whatever.

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**Exercise 4.11**

Let  $x_0 \in X$  be a SDR of  $X$ . Show that given any open set  $U$  in  $X$  such that  $x_0 \in U$ , there exists an open set  $V$  such that  $x_0 \in V \subset U$  such that the inclusion map  $U \hookrightarrow V$  is null homotopic in  $V$ . Use this prove that the point  $(0, 1)$  is not a SDR of the comb space.  $\square$

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This is an exercise which will help you to prove that in the comb space  $(0, 1)$  is not a strong deformation retract. First prove this exercise 4.11. What does it say? Suppose some point  $x_0$  is a strong deformation retract, then for every open subset  $U(X)$  containing  $x_0$ , there is another neighborhood  $V$  of  $x_0$  inside  $U$ , (there is a typo there) such that inclusion map  $V$  to  $U$  is null homotopic in  $U$ .  $V$  to  $U$  you can write a homotopy to a constant function that is the conclusion - starting with the inclusion map to a constant function. That is null-homotopy.

If you use this cleverly then you can show that  $(0, 1)$  is not a SDR of the comb space. No hand waving. For that I already told you what you have to use. The comb space is not locally path connected or locally connected at any point on the  $y$  axis. It fails to be locally path connected. Use this exercise to complete the proof of this SDR.

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**Exercise 4.11**

Let  $x_0 \in X$  be a SDR of  $X$ . Show that given any open set  $U$  in  $X$  such that  $x_0 \in U$ , there exists an open set  $V$  such that  $x_0 \in V \subset U$  such that the inclusion map  $U \hookrightarrow V$  is null homotopic in  $V$ . Use this to prove that the point  $(0, 1)$  is not a SDR of the comb space.

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So let us stop here. Next time we will see many more results. So as I have told you, hard work is more or less over for a while. Thank you.