Introduction to Algebraic Topology (Part-1)
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Lecture Number 20
Generalized construction of SDRs

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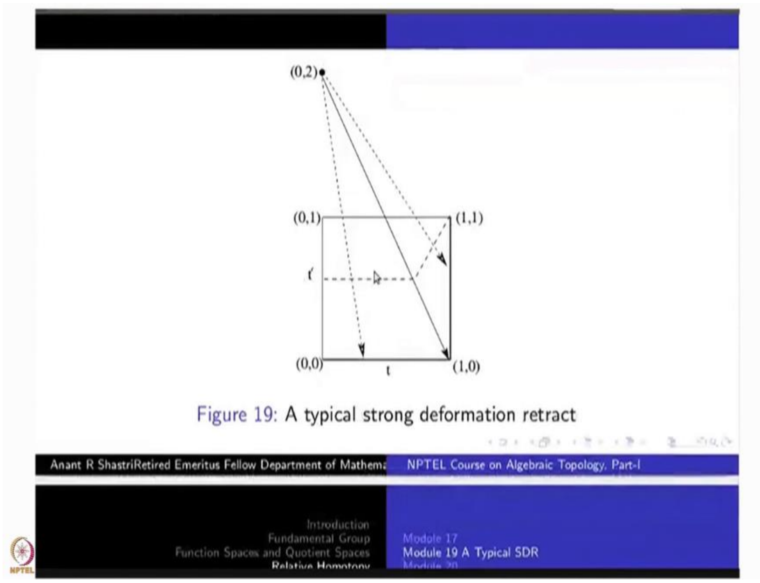

Last time, remember this picture in which we showed that the union of the two segments here on the boundary of the square is a deformation, it is a strong deformation retract of the entire square $\mathbb{I} \times \mathbb{I}$. Just reflect this picture along the y axis. What do you get? You will get - 1 × $\mathbb{I}$ here, $[-1, 1]$, the interval here, and 1 × $\mathbb{I}$ here. As a subspace of $[-1, 1] \times \mathbb{I}$ and it will be a strong deformation retract.

There is no need to write down another proof here. Now imagine you are inside this is x- axis and this is y axis and take another third axis horizontally in $\mathbb{R}^3$, then along the y axis, instead of reflecting, now you rotate this picture. What do you get? You will get the disc here of radius 1× $\mathbb{I}$ strongly deformation retracting to the tub, namely $\mathbb{S}^1 \times \mathbb{I}$ union the bottom disc.

Do you follow this geometric argument? Once I have written down this one, I do not have to write formulas for the other two-- it is obvious. So today, we are going to make use of such ideas and do a lot of more generalizations or constructing deformation retracts.

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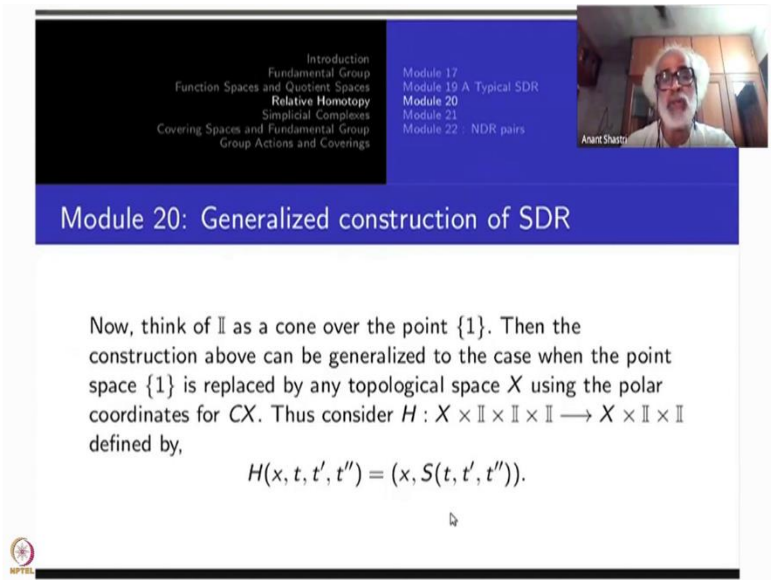

So that is the topic today, generalized constructions. Think of the interval, close interval 0 to 1 as a cone over 1. What is a cone over 1? What is a cone over anything? By definition it is $X \times \mathbb{I}$ and then $X \times 0$ being identified to a single point. So if X is singleton 1. What is $X \times \mathbb{I}$? It is a interval. There is no further identification because endpoint is just a singleton 1 only. So think of this as a cone or, sorry interval as a cone over singleton 1.

Then the construction above can be generalized to any space X instead of the singleton 1. So what do you do? you use polar coordinates for the cone over X. So, let us look at this map $X \times \mathbb{I} \times \mathbb{I} \times \mathbb{I}$ to $X \times \mathbb{I} \times \mathbb{I}$.

See in the earlier case when X was singleton, we had this homotopy capital $\mathbb{S}$ from $\mathbb{I} \times \mathbb{I} \times \mathbb{I}$ to $\mathbb{I} \times \mathbb{I}$. Which is a strong deformation retract. OS, the singleton space X was not written. But now I want here, write the whole space X so you have to write $X \times$ all that. Now the X Factor is a dummy. What we are interested in is the homotopy capital $\mathbb{S}$ which was defined last time from $\mathbb{I} \times \mathbb{I} \times \mathbb{I}$ to $\mathbb{I} \times \mathbb{I}$. So therefore let us define H $H(x, t, t', t'')$ going to x as it is, comma $\mathbb{S}$ of the three variables t, t', t''.

𝕊 is obtained by stereographic projection from 0,. So this was the map. So what will be the image, namely when t″=1. All these you have to see.

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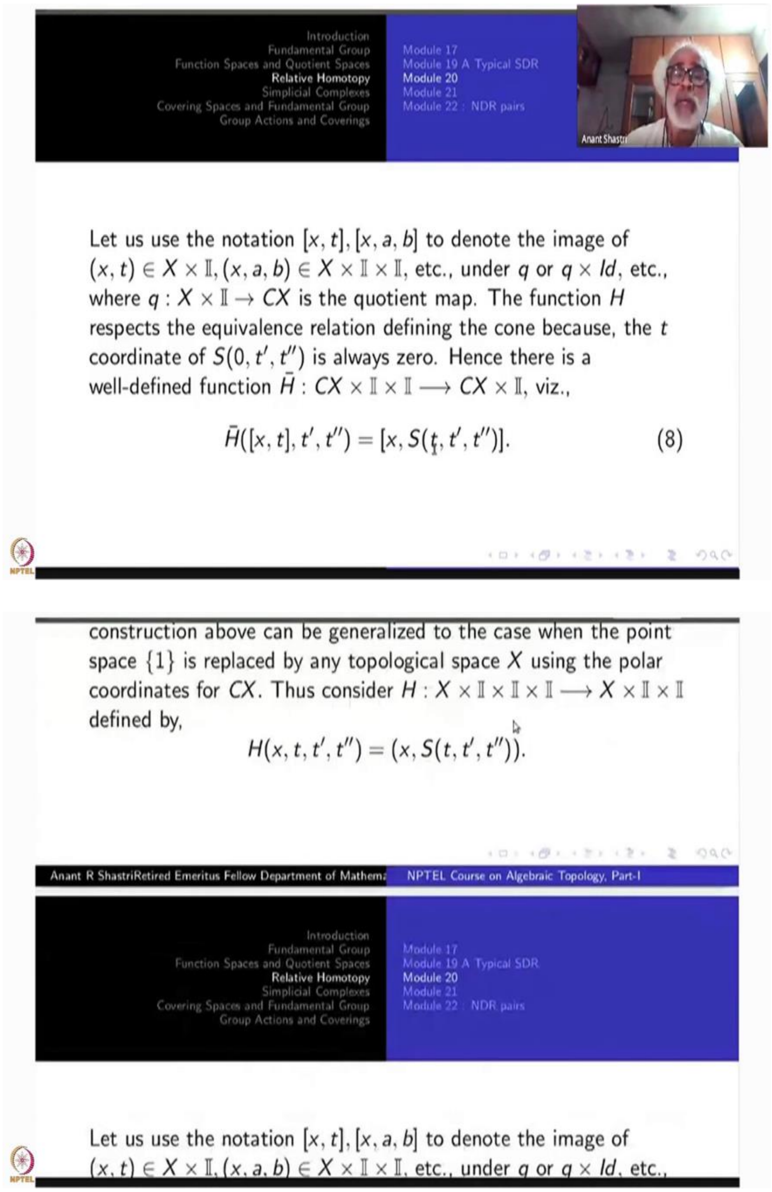

Let us fix some notation. Round bracket (x, a , b) round bracket etcetera to denote the image of (x, t) or (x, a, b) etcetera inside the product space. Under the quotient maps where quotient maps correspond to cone construction. There is one cone quotient map from X × 𝕀 to CX. Then another one from X ×𝕀 × 𝕀 to CX × 𝕀. Let us have one uniform notation instead of too many notations.

Whenever I write round brackets they are in the original product space, and whenever I write square brackets they are corresponding to quotient space, $CX \times \mathbb{I}$. Starting with a map from $X \times \mathbb{I} \times \mathbb{I} \times \mathbb{I}$, the last two factors are not affected at all. But $X \times \mathbb{I}$, on this factor, there will be a quotient map to CX, wherein $X \times 0$ has been identified to a single point.

Now, the point of this one is when the first coordinate here, this t is 0, then this entire map

$S(0\ t, t', t'')$ is the single point 0. Therefore it will factor down from the quotient space to, the quotient space here, $CX \times \mathbb{I}$, using the property of quotient spaces. You see that is the whole idea here. So, under the quotient map, $q \times$ identity, the function H respects equivalence relations defined by a cone, because the t coordinate of $S(0\ t, t', t'')$ is always 0. So actually H(x , something $\times$ something $\times$ something goes to $x \times 0 \times$ something, so gets collapsed to $0 \times$ something.

The t coordinate of this part is always 0. So here a t coordinate going to 0. So is this fine. Hence, there is a well-defined quotient map, I mean the induced map, or induced map H bar from $CX \times \mathbb{I} \times \mathbb{I}$ to $CX \times \mathbb{I}$. What is the formula? H bar of this equivalence class. $H(x\ t, t', t'')$ has the first coordinate as x , $S(t, t', t'')$. So, first of all, you need to verify why H bar is well defined.

Because when t is 0, there may be different x here. But when t is 0, look the image, x , some number but that is also 0. Therefore, this is well defined function on $CX \times \mathbb{I}$. This is two coordinates here. $S(t, t', t'')$ may look like some $\alpha(a, b)\,\beta$ where a, b are elements of $\mathbb{I}$. When t is 0, αa is always 0. Therefore, H bar is well define function. That is what I wanted to say.

Instead of writing S of this one into different formula and so on you just write in this way. So you have to just check this S is the function that we have worked last time. So let look at this one again.

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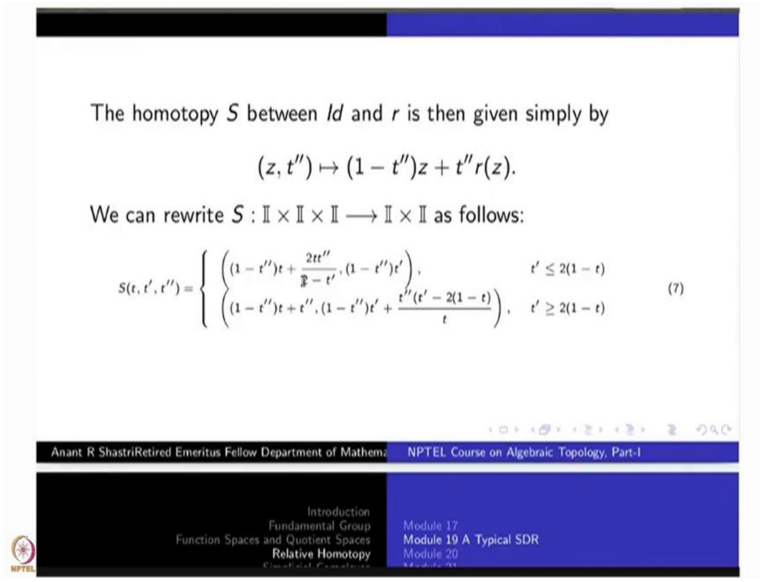

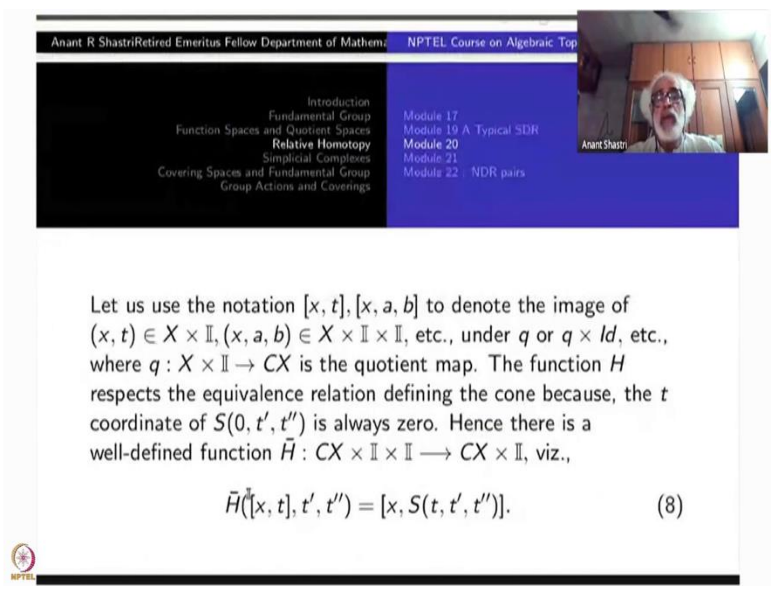

When t is 0, so what happens to the first coordinate here? That is why I have written down fully in terms of t, t′, and t″. If I write like this, it is not clear what happens here. When t is 0 this is the formula which is valid, other one will not occur at all. Never t′ > 2. So use this one,-- t′ less than equal to 2, is possible, so look at this formula when t is 0,   the first coordinate is 0.

A t here, there is a t here too, so this part is also 0. The first coordinate here is 0, the second coordinate is 1 -t″times t′, is independent of t, that does not matter. This part is 0 is all that I need. So we have a map like this, now you can verify that this actually a homotopy of the identity map, ie., when this last coordinate t″ is 0. And when I put t″ =1, it takes values inside the tub, all that you can verify exactly same way as in the case 𝕊. The x- part has no role to play here.

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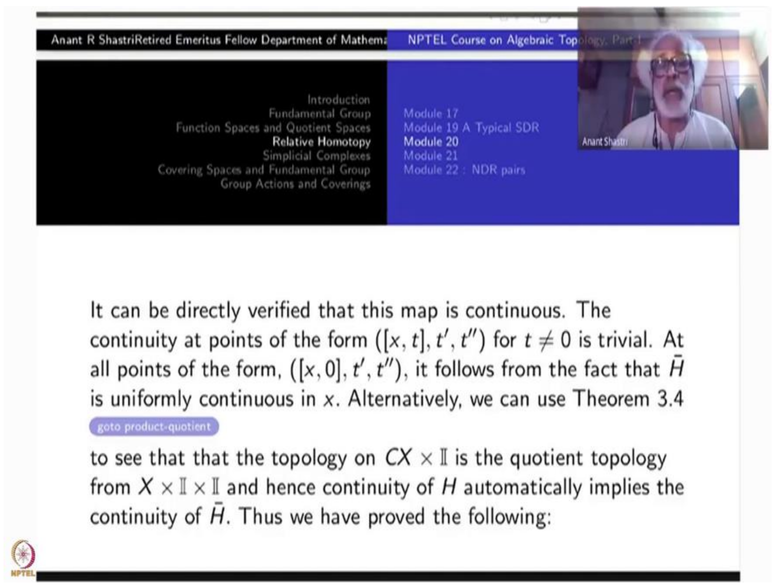

So it can be directly verify that bar H is a continuous map also. But you do not need to do that because of the quotient space theory. The continuity can be deduced because we have a formula there. Alternatively, we can use theorem 3.4 which says that the product of quotients is a quotient. Product of the quotients, here the second factor is actually identity. Why it is a quotient map? because 𝕀× 𝕀 is compact.

So in particular it is locally compact, we have proved that for locally compact space X, and q from Y to Z is a quotient map, then X × Z to Y × Z, q × identity is also a quotient map. If you use that theorem, continuity of H bar is obvious--- you do not have to verify. But if you do not want to use that theorem, you can directly verify this by looking at the formula: at each part you just see that it is given by a certain formula, projection maps x , something, that something is continuous.

So that is what it is. Too show that CX × 𝕀 × 𝕀 is a quotient topology you can use this theorem and then you are done.

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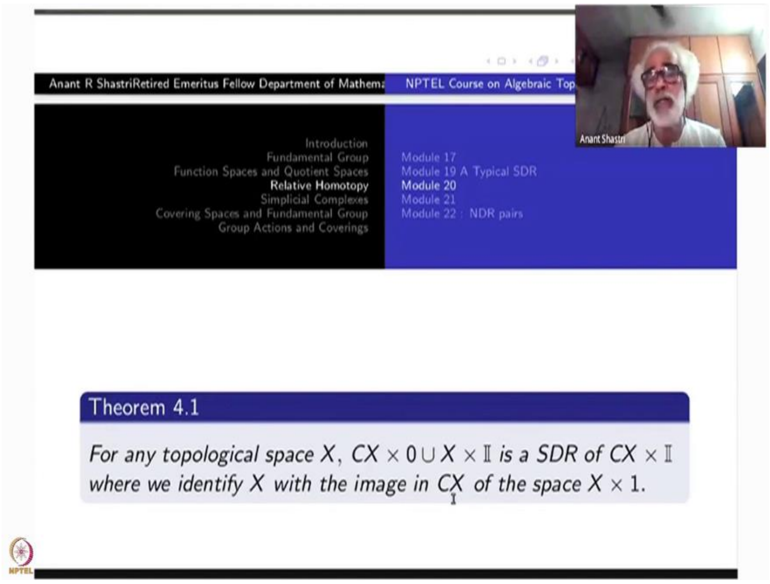

So the conclusion is that for any topological space X, CX × 0 union X × $\mathbb{I}$, is a strong deformation retract of CX × $\mathbb{I}$, where we identify X with it image in CX, viz., x goes to (x, 1).

This is the starting point, the bottom line segment and a vertical segment parallel to the y-axis. This is a strong deformation retract of $\mathbb{I} \times \mathbb{I}$ --- that was the starting point. But instead of that $\mathbb{I}$ I am replacing the single point 1 , 0 by X. Two such examples I gave-- one is got by reflection, but the whole idea is that the interval $[1-,1]$ can be thought as the cone over $\mathbb{S}^0$ the second one is cone over $\mathbb{S}^1$ is got by rotating the whole picture along the y axis.

Why bother? That is all geometrically you can do. But you can do this by just simple argument. Just parameterise over X, you get a cone over X. So the bottom CX × 0 and X × $\mathbb{I}$, that is a subspace of CX × $\mathbb{I}$ which is a strong deformation retract of CX × $\mathbb{I}$.

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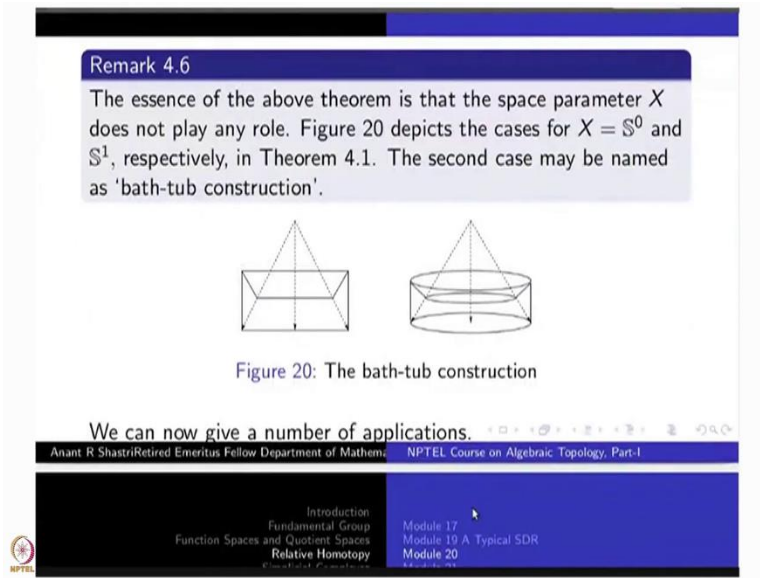

So this figure I have already shown you. Now I am drawing those two figures you see the reflection here of the earlier picture and this is obtained by rotating. So what do you have got here is now a bathtub. The solid thing, whole this, is like a piece of ice, the ice melts down, and only the container is left out.

There will be a number of applications now, one by one.

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Figure 20: The bath-tub construction

We can now give a number of applications.

Proposition 4.1

Let A be any closed subspace of X. Then (X, A) has HEP with respect to every space, (i.e., $A \hookrightarrow X$ is a cofibration) iff the subspace $Z := A \times \mathbb{I} \cup X \times 0$ is a retract of $X \times \mathbb{I}$. (See Figure 21.)

Let A be any closed subspace of X. Then the pair X, A as homotopy extension property with respect to every space, if the subspace $Z = A \times \mathbb{I}$. union $X \times 0$ is a retract of $X \times \mathbb{I}$.. What is the meaning of homotopy extension property with respect to every space?

That is the inclusion map is a cofibration. So this is what-- the inclusion map is a cofibration if and only if $A \times \mathbb{I}$ union $X \times 0$ is retract of $X \times \mathbb{I}$. So this is the direct application to homotopy theory of that simple construction of SDR we have done.

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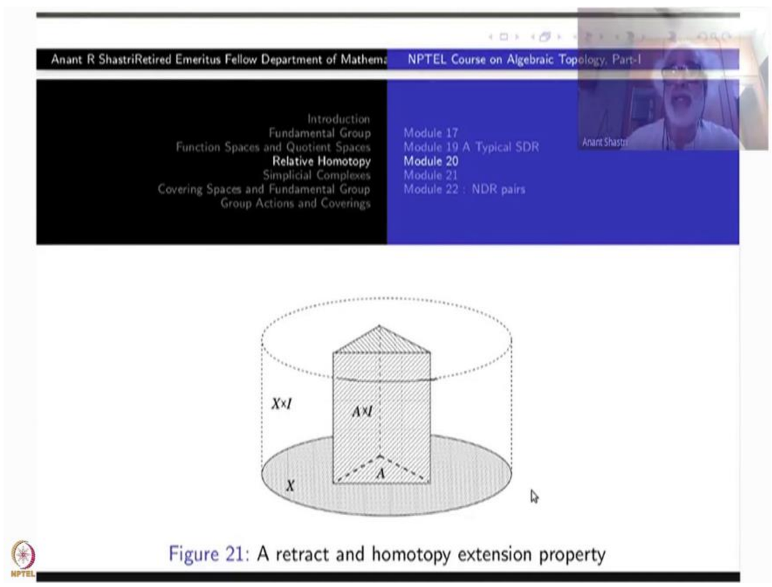

Figure 21: A retract and homotopy extension property

This a picture, bottom thing is X, $X \times 0$. In this picture A is the triangle, is a subspace of X there, $A \times \mathbb{I}$ is standing here. You have to take $A \times \mathbb{I}$ union the bottom X, $X \times 0$, this is a strong deformation retract of the entire $X \times \mathbb{I}$. The hypothesis is that A to X, the inclusion must be a cofibration. The stronger conclusion is that it is `if and only if'. That is a fantastic conclusion.

Why? Because to verify something is cofibration, you have to verify something for every space, every map. Verifying any property for all spaces is an impossible task. Here you have to only verify that this particular subspace, whether it is a strong deformation retract of $X \times \mathbb{I}$ or not; over. So this is going to be extremely useful theorem. Looks like small results. But these are going to build up the whole theory, what is called a homotopy theory.

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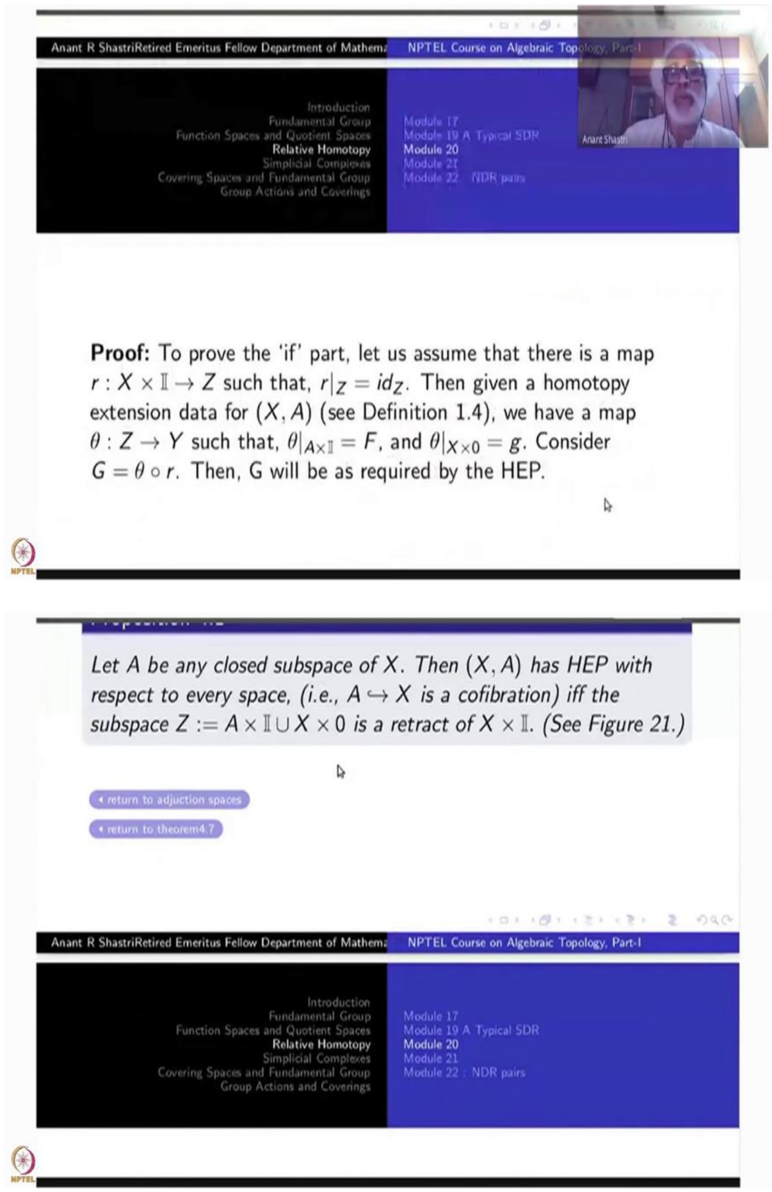

So let us work out. How is it done? First part, suppose you have the deformation retract and we want to show that A to X is cofibration. Cofibration means what? So let us assume that there is a retraction r from $X \times \mathbb{I}$ to Z this is just a retract, sorry, it is not even a strong deformation retract, suppose you have just retract. Retract means what? Restricted to Z, it is identity. It is continuous function from the whole space $X \times \mathbb{I}$ to subspace to Z.

What is Z? Z is, I introduce this notation namely, $A \times \mathbb{I}$ union $X \times 0$. Suppose there is a retraction. Now suppose there is a homotopy extension data as in definition 1.4. What does this mean? This means that you have a map F from $A \times \mathbb{I}$ to Y the restriction of it to Across 0 can be extended to a map on $X \times 0$. Then I want to have an extension on the entire of $X \times \mathbb{I}$, that is a homotopy.

So suppose you have such a homotopy data. So, I have obtained from it a map θ which is the union of these two things, namely, θ restricted to $A \times \mathbb{I}$ is the homotopy F and θ restricted to $X \times 0$ is the map g which is an extension of F on $A \times 0$. So, define $\theta\theta$from Z to Y by putting these together, namely restriction to $A \times \mathbb{I}$ is F and on $X \times 0$ it is g. So what you have is a map θ from Z to Y.

Then there is a map from $X \times \mathbb{I}$ to Z which is r. take the composite, G equal to $\theta\theta$composite r. All that I want to say is G is the required homotopy extension. Because r is identity on this part, then you take r composite θ, when you restrict it to $X \times 0$, it will be the required map g, similarly on $A \times \mathbb{I}$. That is all. And it is an extension it is defined on whole of $X \times \mathbb{I}$. So this comes as if by a magic. Now. this retraction we want to construct if the inclusion map is a cofibration-- that is the converse.

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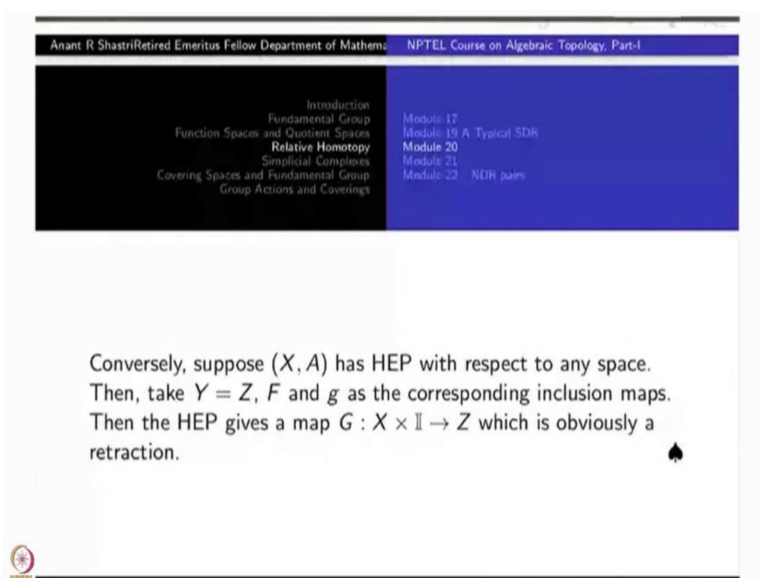

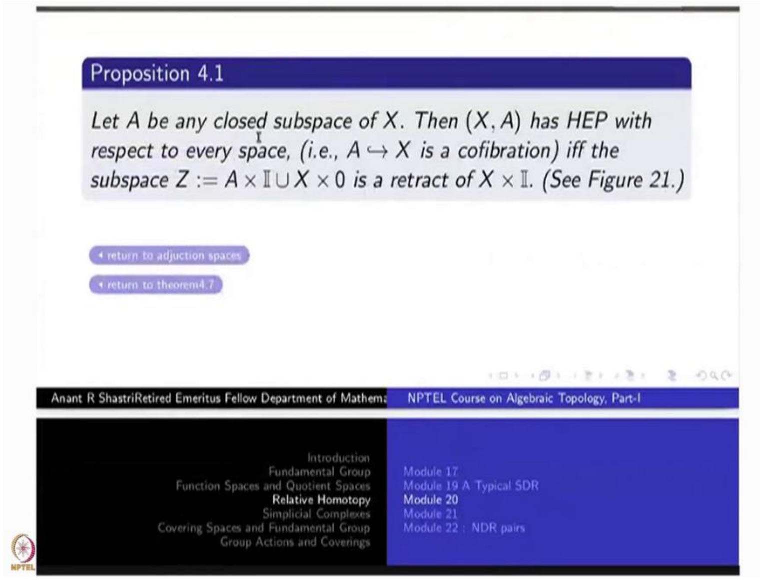

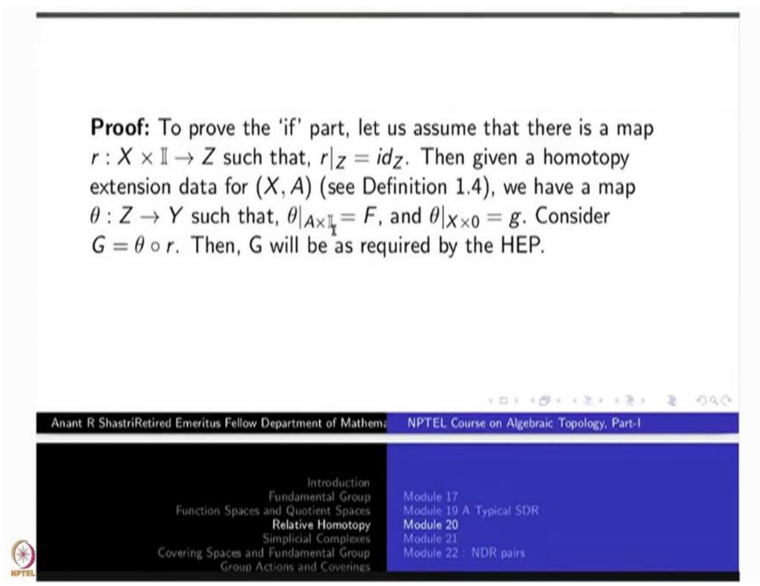

Suppose the pair (X,A) has homotopy extension property with respect to every space. Every space Y. In this case now, I take Y to be Z itself and F, and g be the corresponding inclusion maps. What is F, $A \times \mathbb{I}$ to Z it is the inclusion. What is g, $X \times 0$ to Z is the inclusion. So that is the data. So with this data there will be a capital G from $X \times \mathbb{I}$ to Z, it is a continuous function which extends this one that just means that on Z, it must be identity.

So it is a retraction. So that is the statement. It says that starting with A any closed subspace of X, then being a cofibration or Z being retract-- these two things are equivalent. Inclusion map is cofibration, or this $A \times \mathbb{I}$ union X cross 0 is a retract,-- these two are equivalent. Where did we use that A as closed subspace? Why do we have that hypothesis?

This one I defined, it's θ, by patching two different functions. On $A \times \mathbb{I}$, it is F, so continuous. Restricted to $X \times 0$, it is g, so continuous. these are hypothesis. But why $\theta\theta$ is continuous on Z? On the intersection $A \times 0$, this F agrees with g restricted to $A \times 0$ because g is an extension of that $A \times 0$. So, as a function, θ θ is well defined. But why it is continuous?

You need some hypothesis namely, if A is closed, then the intersection in $A \times 0$ here also will be a closed subspace. Look at $A \times \mathbb{I}$ and $X \times 0$, what is the intersection? it is $A \times 0$. That must be a closed subspace. This is the same thing as A must be a closed subspace of X. So the hypothesis is required there, to say that $\theta\theta$ is continuous.

It is not a very costly hypothesis because, finally what you get is the converse. Namely A to X is a cofibration then what happens, Z becomes a close subspace of $X \times \mathbb{I}$, if X is Hausdorff. Any retract of a Hausdorff space is closed, that is what we know.

If Z is closed, then A cross 0 will be closed subspace, it is very easy because you just intersect it with $X \times 0$, sorry Z intersection $X \times t$ say, $X \times 1$. That is actually is $A \times 1$. $A \times$ one is closed is the same thing as A is closed in X. So under Hausdorffness, the closeness of A is a must. So we are not assuming too much.

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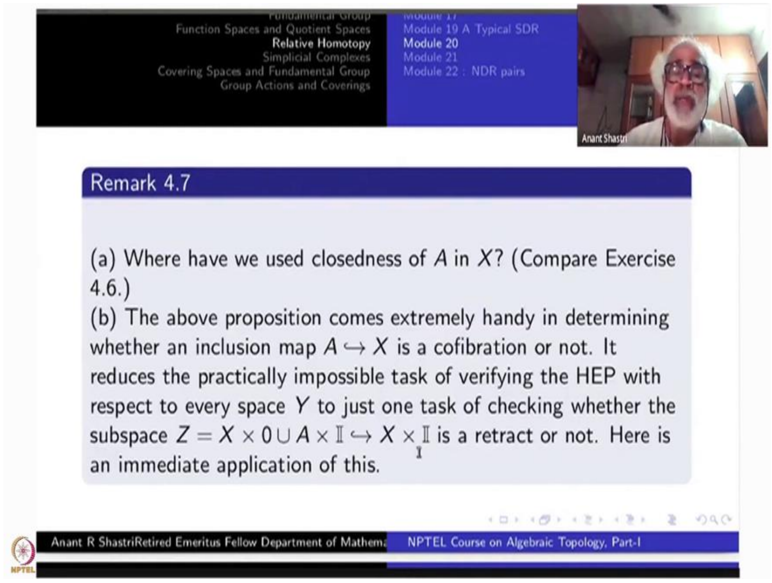

So I have told you already that this proposal comes extremely handy in determining whether the inclusion map is a cofibration or not. It reduces the practically impossible task of verifying the homotopy extension property with respect to every space to just one a task of checking whether the subspace X × 0 union A × 𝕀 is a retract or not. This itself has many big applications.

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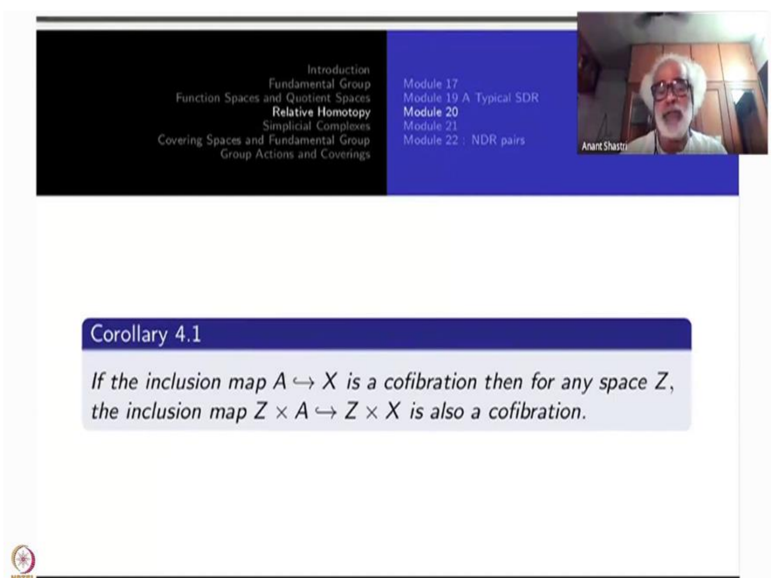

So let us take a look at some of these applications now. If the inclusion map is a cofibration, then for any space Z, the inclusion map Z × A to Z × X is also a cofibration. So Z × A to Z × X what is the map, it is the inclusion map again because A to X is an inclusion map. How do you do that?

Of course, here all I have this for A, I assume that is A is a closed subspace now. Because that is used. To apply that theorem you need a closed subspace.

So why is it a cofibration? I have to just verify that, $Z \times A \times I$ union $Z \times X \times 0$ is a retract of Z cross $X \times \mathbb{I}$. From the hypotheses that for $X \times \mathbb{I}$ the corresponding things are true. So everywhere you just take $Z \times$... So this is very straightforward.

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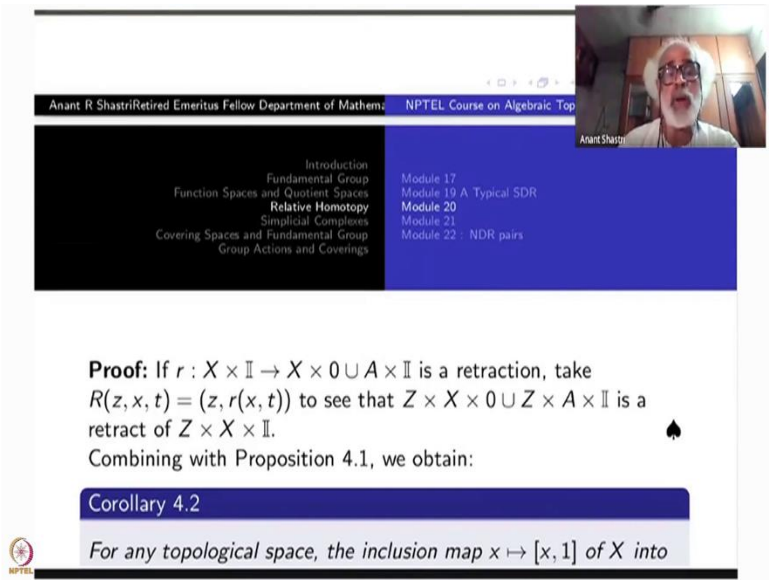

This what I have done with a retract where X × 𝕀 to X × 0 union A × 𝕀. R(z , x , t) keeps z as it is. Look at the second factor, r (x, t). Just taking identity cross r, $Id_Z \times r$. That will give you the corresponding retraction for the product.

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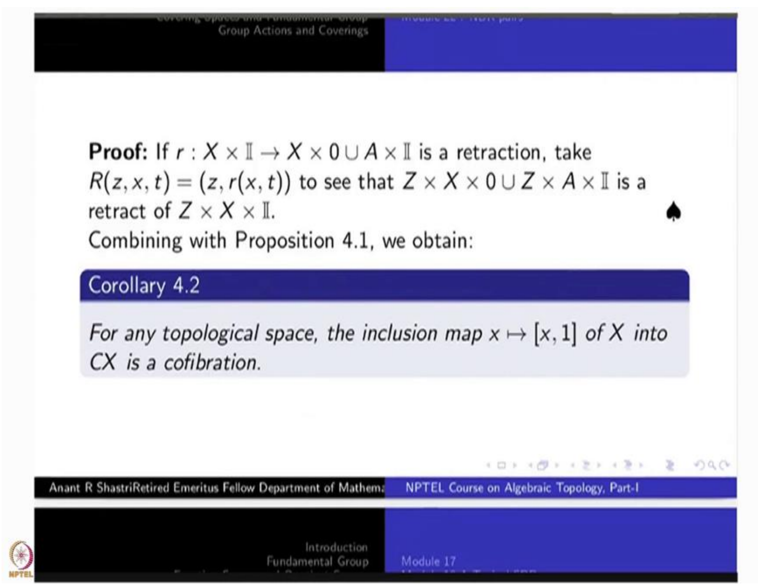

Next next corollary is also as easy. Take any topological space, and inclusion map of X, x maps to x × 1, of X into CX. Remember in the cone, X was identified with X × 1 as a subspace. So now I am thinking of X as a subspace of the cone. That inclusion map is always a cofibration. At least here, you do not have to assume something is closed etc. Automatically X is a closed of space of the cone. So now, X to CX itself is a cofibration, the inclusion map is a cofibration.

Can you see how to get this corollary? Just like in this case of product, now we can take the cone over that one. In the previous example you got a product. You can parameterize once you have this r. Parameterize by any other another factor. So I will leave it to you to think about these things. Maybe unless you write down something by yourself you may not realize it fully I think. CX the cone over X contains the base space X, as X × 1, the inclusion map will be a cofibration. Verify this is the statement.

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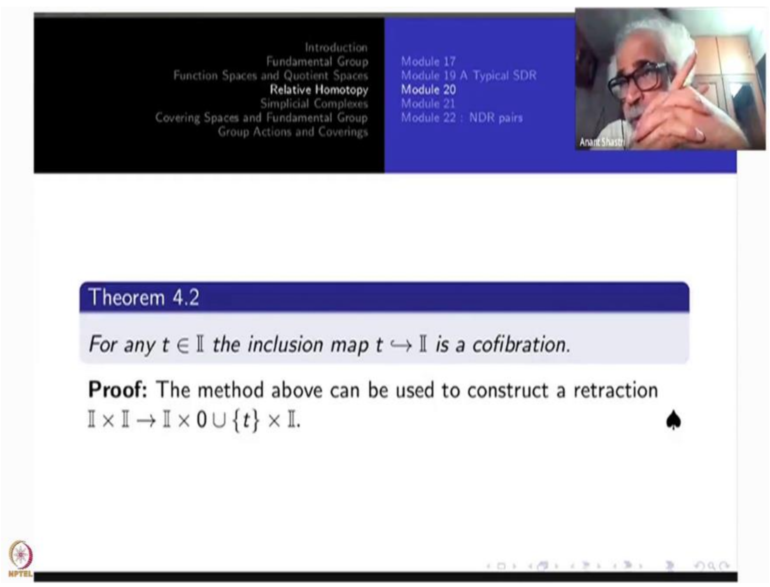

In particular, we have seen that 1 is the bottom of the cone over one which is the interval 0 1. The singleton 1 inclusion into the interval is a cofibration. You can directly prove this. Also but, actually the first picture itself proves it at the starting point. But now I have something more here, a corollary. Take any point in between also. This is also a cofibration. I tell you how to get it. Take reflection, and move the center and so on.

So this is the trick. You reflect it on the other side. So one is a retract, you get a retract on the other side. So instead of an end two end points you will get a central one an edge here. So above construction can be used to get a picture for this one also. Same method of construction can be used to prove that any inclusion map of singleton into $\mathbb{I}$ is a cofibration. Go back to that theorem that proposition and try to prove that retract is possible, that is all.

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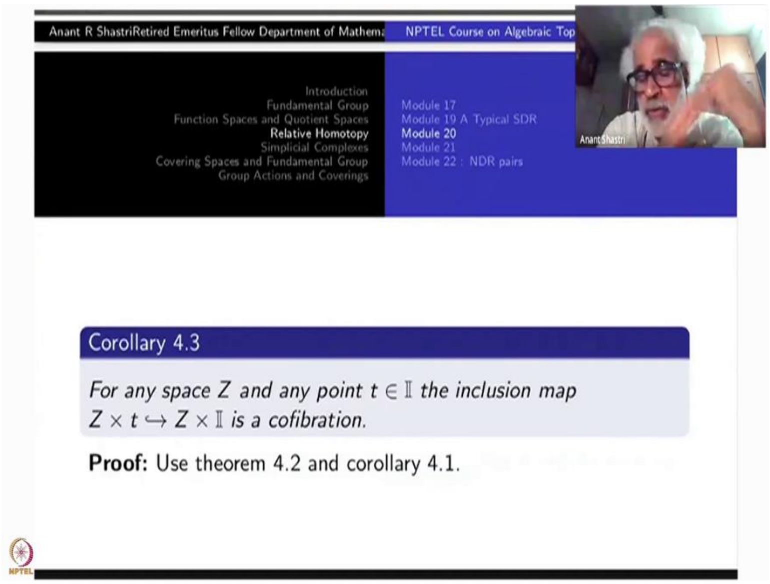

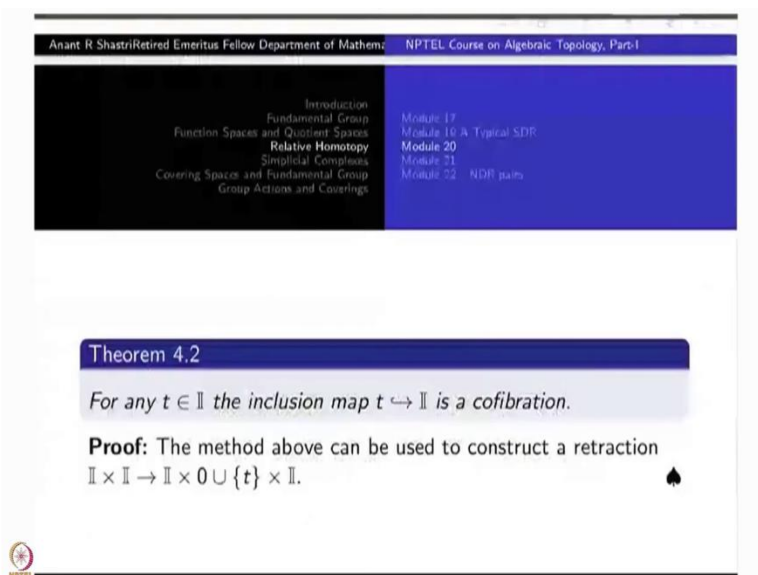

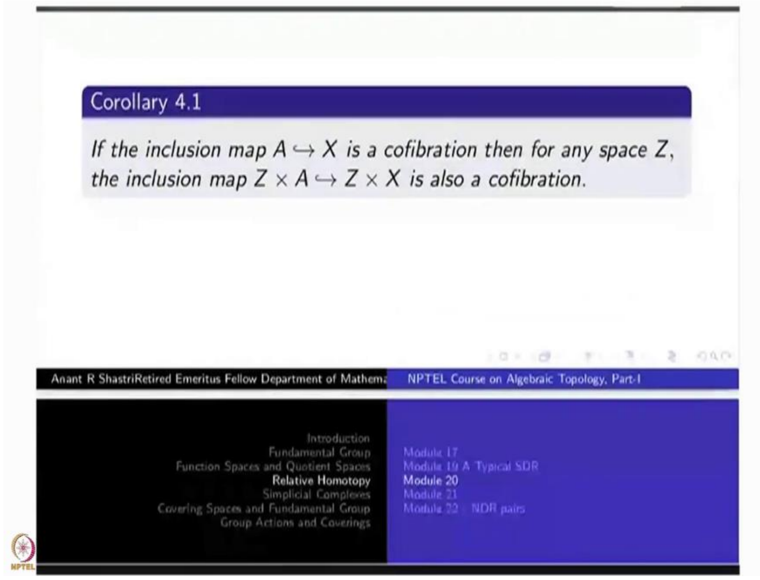

Once we have this, for any space Z and any point t in $\mathbb{I}$ the inclusion of map $Z \times t$ into $Z \times \mathbb{I}$ is a cofibration. This is a direct consequence of two of the previous theorems, first we can use this one, and then we can use this product theorem, combine these two corollaries, you get this corollary. For any point t, $Z \times t$ inclusion into Z across $\mathbb{I}$ is a cofibration. So, let's stop here and take up further applications in the next module. Thank you.