Introduction to Algebraic Topology (Part I)
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Lecture 2
Concepts of Homotopy

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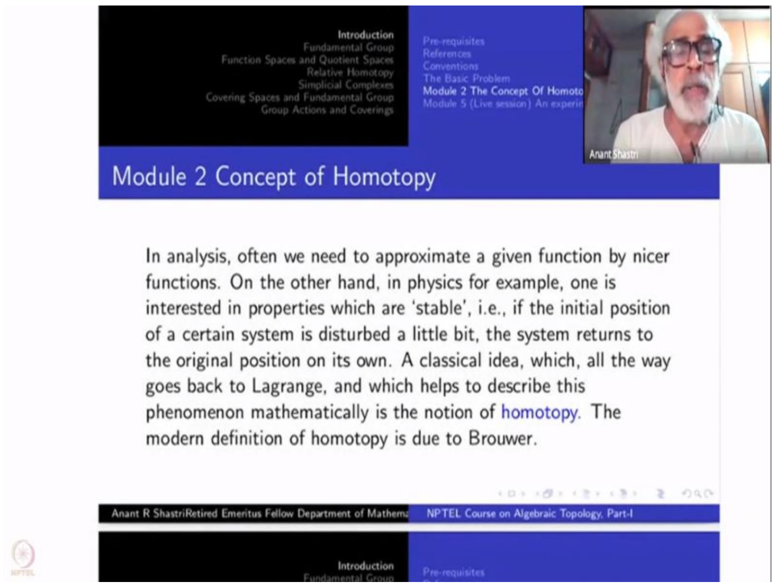

Hello everybody, last time in module one, we told you about the course and about algebraic topology in general. We also showed how a fundamental problem in algebraic topology is shown to be not solvable by solving a corresponding problem in group theory. This was due to Novikov. (00:49). Today we will introduce the concept of Homotopy.

Last time, the two problems that we have stated namely lifting problem and extension problem. So, let us see what homotopy is and how homotopy is going to affect this thing. So, let me tell you something about homotopy itself. In analysis, often we need to approximate a given function by nicer functions, maybe sometimes linear maps or smooth maps and so on.

You see, on one side this approximation is something that we have to keep in mind. On the other hand, in physics, there are certain properties which are more interesting for physicists, namely stable properties. For me who does not know much physics, what it means is the following.

Suppose you take a system and disturb it a little bit, the system should return to the original position on its own. That `little bit' is to be made precise and that precise way is nothing but Homotopy. We will come to that one. The classical idea which goes all the way back to Lagrange, which describes this perturbation as it is called, mathematically is the notion of Homotopy. But the modern definition of Homotopy that we are going to study is due to Brouwer.

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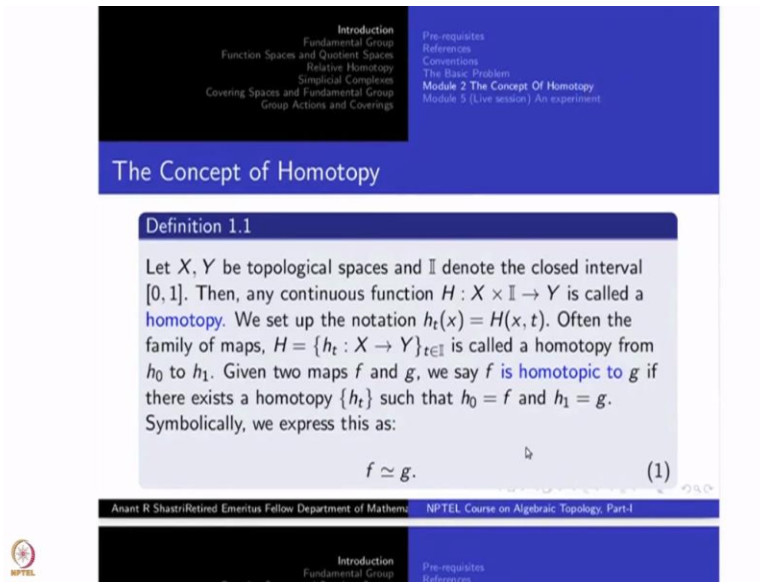

So, let us make a definition. Take two topological spaces X and Y, let $\mathbb{I}$ denote the closed interval [0,1]. Then any continuous function H from $X \times \mathbb{I} \to Y$ will be called a Homotopy. Whenever you have a function like that, the first variable is the space variable. The second variable you can think of as the time-variable. That is why we denote it by (x, t).

So, when you fix a time variable t, you get a function from X to Y at the time t. So, these functions are called, or denoted by little h_t. So, often people in the past, classically, they used to think of this as a family of functions from X to Y, the family being indexed by the set $\mathbb{I}$ It is very important, it is not just a set, it is the space time space namely an interval in $\mathbb{R}$. That continuity of the time is very important.

So, the continuity of H with respect to t also it is not a separate continuity, continuity in x and continuity in t that is not the correct thing. What we need is the joint continuity from the product space $X \times \mathbb{I} \to Y$. Such a thing is called Homotopy. So, this is the modern definition. Symbolically, whenever such a Homotopy exists, the Homotopy from h_0 which you may call it f to h_1 which you may call g. So, whenever such a Homotopy exists we say f is Homotopic to g and write it f ≃ g in this way. So, that is the definition.

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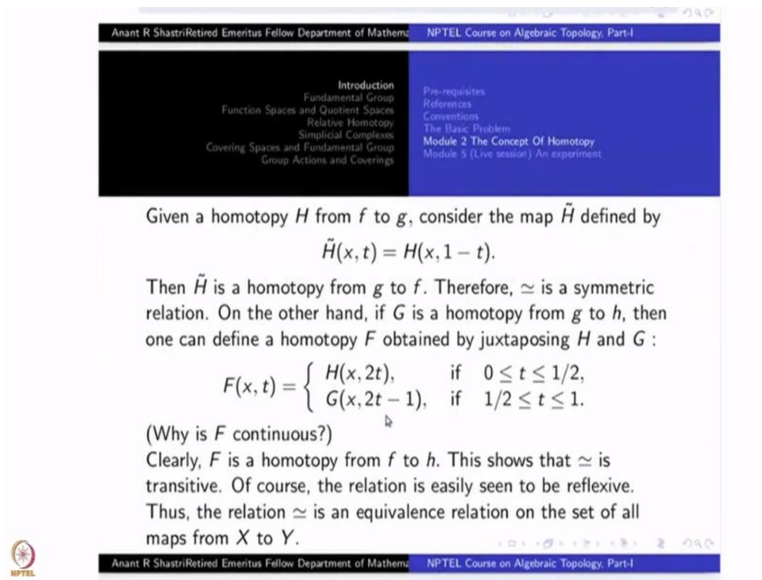

Now, if you have a Homotopy H, you can define another Homotopy $\tilde{H}$ by just reversing the time, namely t being replaced by 1- t, then at time 0 it will be g first, then when time t equal to 1, it would be f. So that will be a Homotopy from g to f. So, what this shows is, whenever there is a Homotopy from f to g, there is a Homotopy from g to f.

So, the relation \`f ≃ g' , whatever this symbol means, this relation is symmetric. That is the meaning of this one. So, on the other hand, it is very easy to see that f ≃ f all that I have to do is to take H(x, t) , ignore t, H(x,t) as f(x). So that will be the identity Homotopy itself from f to f.

So, the relation is reflexive. Finally, to see that it is transitive--- suppose you have a homotopy G from g to h, in addition to the homotopy H from f to g, then you can put these two things together, it is called juxtaposing one homotopy with another homotopy.

So, what is the meaning of that is explained here technically, completely rigorously. Namely, in the first half of the interval you define it to be H, only thing is to fill up the interval by speeding up--- the you double speed, t going to 2t and $F(x,t) = H(x,2t)$. Exactly similarly, you have to do in the other half of the interval. Only thing is now, the 0 is taken by half and 1 is taken by 1. So, you have to do this shifting also, this will become G(x, 2t-1), in the interval ½ $\leq$ t $\leq$ 1.

Such a function is automatically continuous because in the X × [0,½] it is H that is continuous and in X × [½ , 1] it is G, that is continuous and on the intersection which is X × ½, the two functions agree. So there is one single function, though there are two formulae. So, this F becomes a homotopy from H(·, 0) which is f to G(· ,1) which is h. This shows that the relation, whatever we wrote, is an equivalence relation---- reflexive symmetric and transitive.

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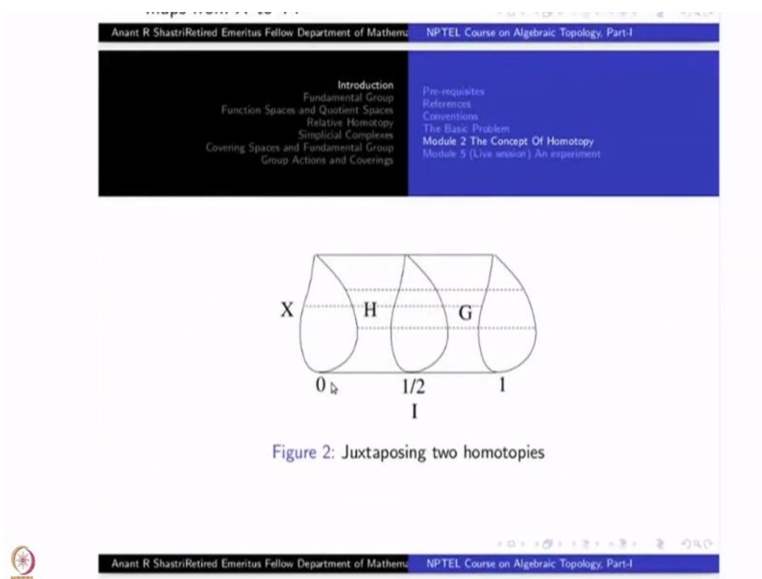

So, here is a picture which tells you this. In the half part X × [0, ½], I have taken H, then in the second half I am putting G. So, they patch up because on X × ½, what is the function?--- the function H is equal to little g here, and the function G is also little g. So, f $\simeq$ g, g $\simeq$ h means f $\simeq$ h.

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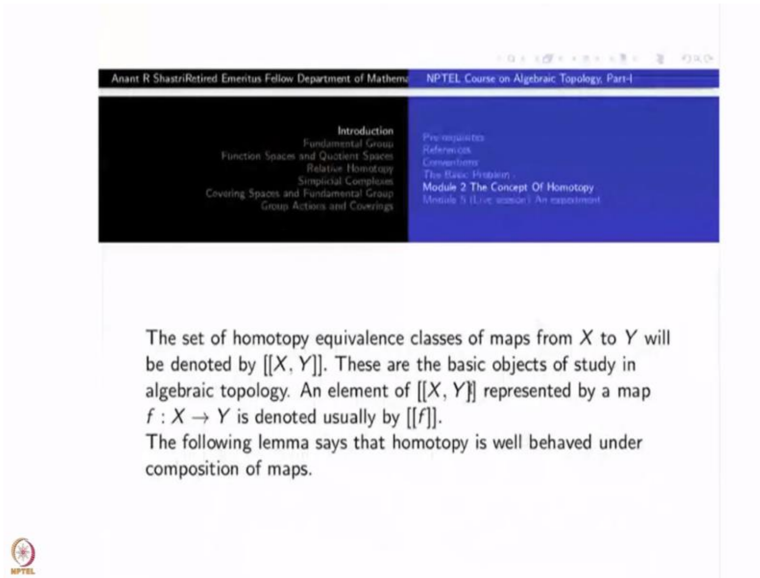

The set of homotopy classes of maps from X to Y, it will be temporarily denoted by [[X , Y]]. This is just a set now. You see, already we have started doing algebraic topology here, starting with spaces X and Y and maps from X to Y, we have constructed purely a set of equivalence classes of functions here.

These are basic objects of study in algebraic topology, these sets. Elements of this [[X Y]], are always represented by some maps f from X to Y and that class will be denoted by [[f]]. Now, we have to study what happens under compositions and so on and that is the next lemma.

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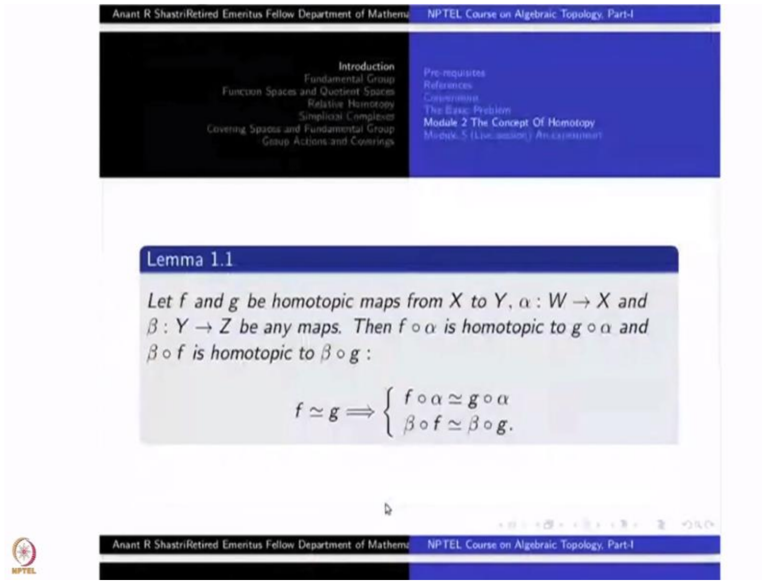

The next lemma says that suppose f and g are homotopic from X to Y. Suppose there is a map $\alpha\alpha$ from W to X, then you can first take $\alpha\alpha$ α and follow it by f, or first take $\alpha\alpha$ andα follow by g, then you have two different maps, these maps will be homotopic. Similarly, suppose there is a map $\beta\beta$ from Y to Z. Now you can take first f from X to Y, then $\beta\beta$ or first take g and then β, so those two will be also homotopic. All these follow if f homotopic to g.

Pre-composition and post-composing of homotopies is a homotopy. So, this is the message of lemma 1.1, a fundamental lemma. Proof is very easy. All that you have to do is to take corresponding compositions of homotopies.

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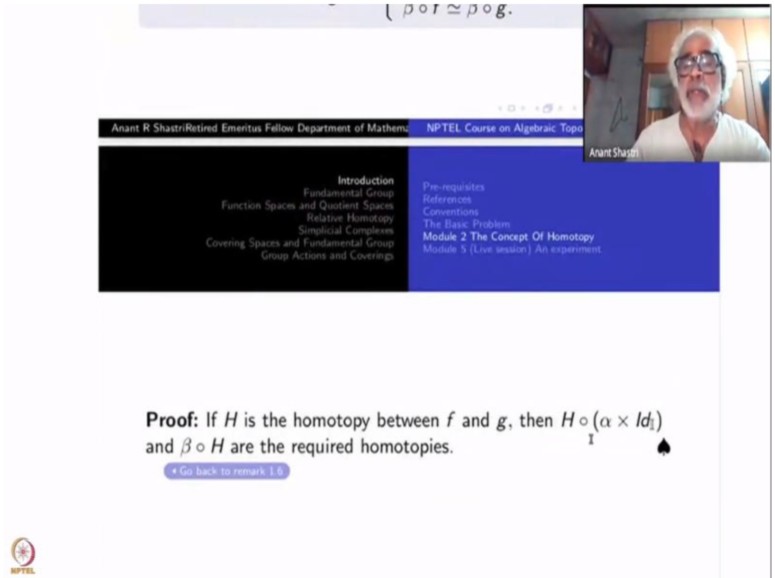

Starting with a homotopy H from f to g, take H composite $\alpha\alpha \times$ identity. What do you mean of $\alpha\alpha \times$ identity? On the W factor, (remember $\alpha\alpha$ is a map from W to X,) on the W factor, it is $\alpha\alpha$ and on the I-factor, it is the identity. So $\alpha\ \alpha \times$ identity of (w, t) is nothing but (α α(w), t).

Then apply H. That will give you a homotopy between f $\circ\ \alpha\alpha$ and g $\circ\ \alpha$. Similarly, if you take $\beta\circ$ H that will give you a Homotopy of β $\beta\circ$ f with β $\beta\circ$ g. So those are the proofs which are very straightforward.

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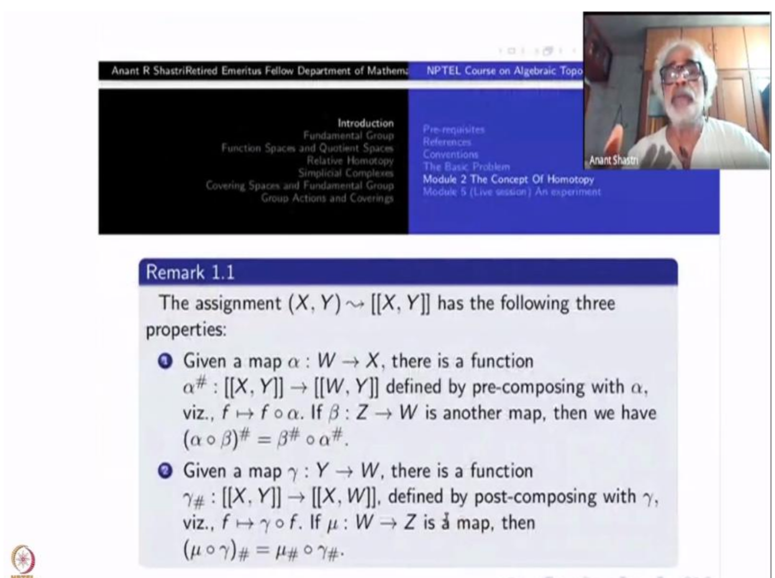

Now, we can sum up the effect of this lemma on these sets. Starting with a pair of topological spaces, we immediately wrote down this set, what is this set?--- take a continuous function from here to here and take its homotopy class. So, this is a set of such homotopy classes.

This assignment has some wonderful natural properties with respect to compositions. If you pre-compose a function in a class in [[X,Y]]α a function from W to X it will become a function from W to Y. So, what happens to the class? The class will become a whole class, there is no change that is the whole point here. So, $\alpha\alpha^{\#}$ from [[X,Y]] to [[W,Y]], (these are now sets,) will be well defined.

A homotopy class here goes to a homotopy class here, because, if two functions f, g are homotopic here, f $\circ\alpha$ α and g $\circ$ $\alpha\alpha$ will be homotopic here. So, that is the whole idea. Similarly, you can compose once more. If $\beta\beta$ is from Z to W, then what you get is

($\alpha\circ\beta$)$^{\#}$ is equal to $\beta\beta\beta^{\#}$ $\circ$ $\alpha^{\#}$

So, this is our pre composition. Exactly the same thing happens with the compositions on the right also. Then we get $(\mu \circ \gamma)_{\#}$. Now, this time we have a notation change. The checks are suffixes here, they are superscripts in the first case. Now the suffixes are subscripts, And $(\mu \circ \gamma)_{\#}$ is equal to $\mu\mu_{\#}$ composite $\gamma\gamma_{\#}$. Moreover one more important property is that if you take identity map,

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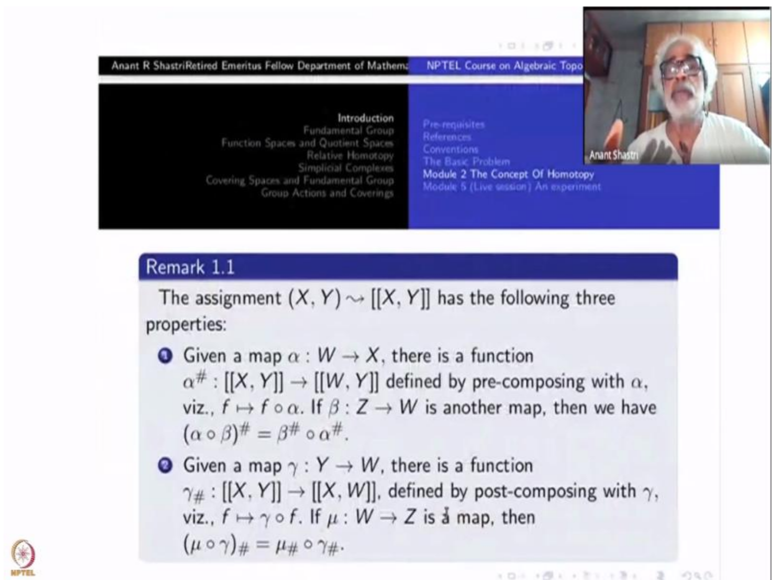

then identity$_{\#}$ is nothing but identity whichever way you take either pre composite or post composite. How to get this one I have told you, it is just lemma 1.1 that we have just seen along with one simple property namely associativity of functional compositions (f ○g)○h, you can put the brackets on the other way round, f ○ (g ○ h). So that is Associativity. That is all you have to keep using.

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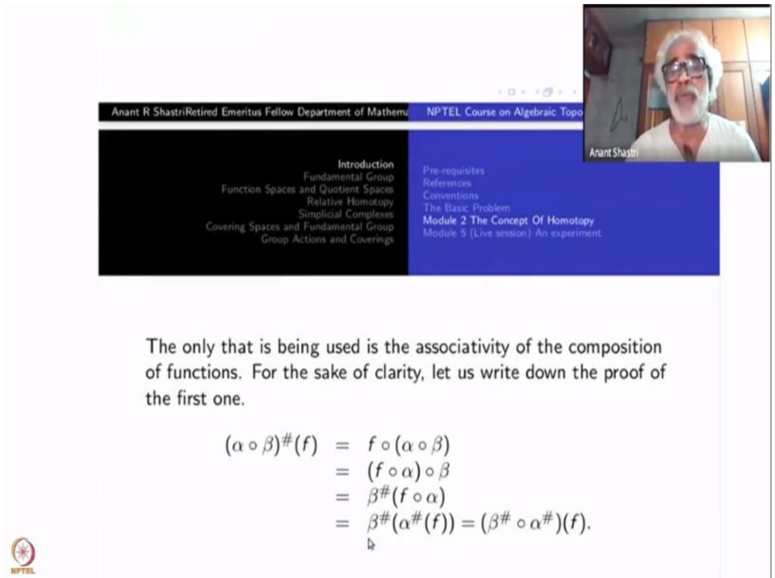

In fact let us just clarify this, that there is no hanky-panky being done ---just to be very sure. So, let us verify one of the properties in the first one, namely, start with $(\alpha\alpha \circ \beta)^{\#}$, I want to show, is equal to $\beta\ \beta^{\#} \circ \alpha^{\#}$, it interchanges the slots, that is very important. Whereas, if you take the lowers check, then it will be $(\mu \circ \gamma)_{\#}$ is equal to $\mu\mu_{\#}$ composite $\gamma_{\#}$, in the same order.

So, let us verify this. So, what do I have to do? Operate on f, (f is a function from X to Y) on both sides and verify that they are the same. So, start with this one, what is the definition of ($(\alpha\alpha \circ \beta)^{\#}$ on a function f? It is $f \circ (\alpha \circ \beta)$. Now use the associativity of this composition. This is the same thing as bracket $(f \circ \alpha\alpha) \circ \beta$. But then this is equal to $\beta\ \beta^{\#}$ of the function $(f \circ \alpha)$. But $f \circ \alpha\alpha$ is $\alpha\ \alpha^{\#}(f)$. Now $\beta\ \beta^{\#} \circ (\alpha^{\#}(f))$ is just bracket $(\beta^{\#} \circ \alpha^{\#})(f)$. Exactly the same way you can verify the other one also.

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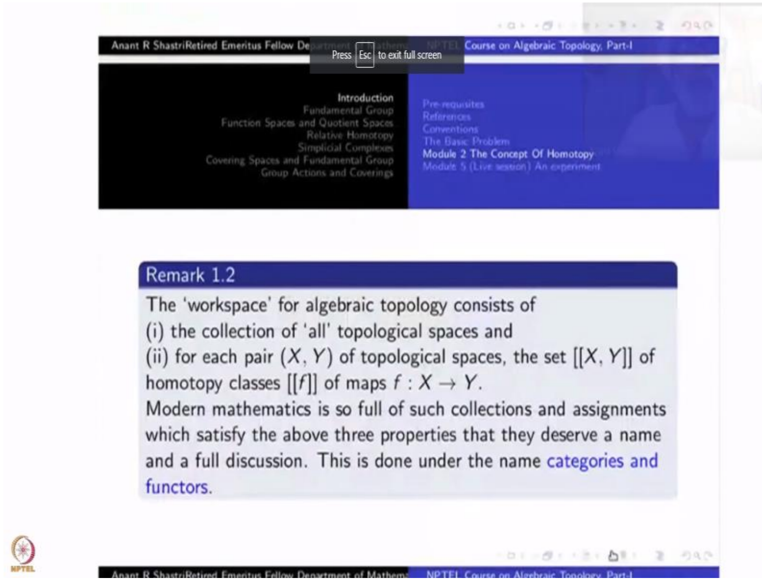

As I told you, I keep repeating this, the workspace of algebraic topology is, in the whole, (later on, we can make it smaller and smaller), the whole is all topological spaces and for each pair of topological spaces, the set of homotopy classes of maps from X to Y. This is the one which you are interested in, not exactly the maps but the homotopy classes.

Modern mathematics is full of such collections of assignments. So, we have assigned---starting with a pair of topological spaces we have assigned. We will now study these sets. This set, when we take special cases, will have lots of additional structures, different kinds of structures.

Yes, I know we will not study this step, this set when you take special cases will have lots of structures, different kinds of structures and that is what we have to study.

So, the property of this assignment which we have seen just in the previous remark, so those properties are themselves called, you know, they are categorized, you can say they have been put inside a kind of discipline. It is called categories and functors, which is a part and parcel of algebraic topology. The categories and functors give you the basic language required to express so many complicated ideas in simple language.

This has been, this is a very modern discovery I would say. People like Euler and Gauss (18:31), they never had this kind of language, even Riemann did not have, even Hilbert did not have this. So, if we try to read their papers, it will be very, very difficult for us; it is an entirely different kind of language.

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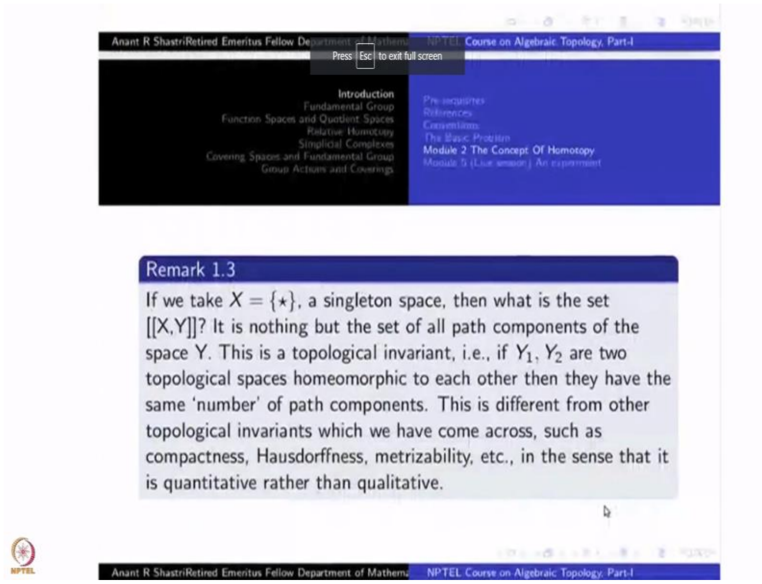

Let us take a closer look at these classes. Suppose I take a special case namely the domain X is just a single point that I denote by {*}---that is a single point, singleton space. Then what is the set of homotopy classes of maps from X to Y. To understand this first of all we must see what all the maps are from a singleton space. Any function from a singleton space will be automatically continuous, there is only one topology there.

So, what are all functions on a singleton space? What is the meaning of a function? You have to just mention one point in the space Y. As soon as we mention that, the function is well defined--- the domain singleton goes to that point.

Therefore, this is nothing but the set of points in Y. Now what is a homotopy on $\{*\} \times \mathbb{I}$? This space is nothing but homeomorphic to $\mathbb{I}$ itself. So, they are functions from $\mathbb{I}$, of course, continuous functions, so these maps you must have studied,-- they are called paths. The starting point is some point that is one function, the end point is another function.

So, these things are connected by a path. Equivalence classes of such things are nothing but path components of Y. This is one of the simplest algebraic topology invariants. This is a homotopy invariant which we have constructed now. This was already there in point-set-topology. The path connected components and number of path connected components is something different from all other concepts, like compactness, separation axioms, T1-ness T2- ness, various things.

So, this is a quantitative invariant as compared to qualitative invariants. So, all topological invariants in algebraic topology will be of this type. They will be quantitative not qualitative, quantity just does not mean just the cardinality. There is much more than that. So, you can say it is algebra. There are structures there of course, like groups, abelian groups, rings and so on.

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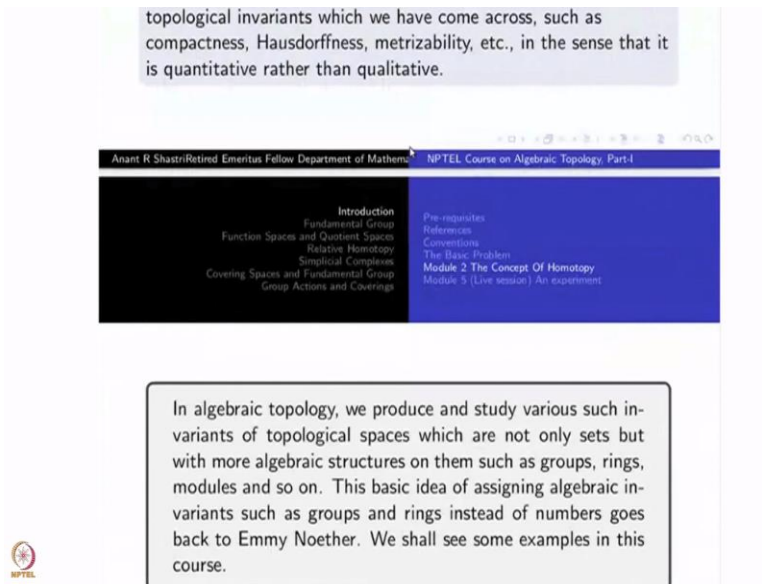

So, we produce in algebraic topology and study various such invariants of topological spaces which are not only sets but have more algebraic structures on them, rings, modules and so on. This basic idea of assigning algebraic invariants such as groups and rings instead of numbers goes back to Emmy Noether. Before that, people were just counting the numbers, path connected components, number of. If not, the number of `arcs' needed to cut down a given something. For example, look at an arc, you remove one point, it gets disconnected. Look at the circle, you remove one point, it is still connected. So, you need to remove two points. So, this is the kind of thing they were studying.

So, Emmy Noether says do not worry about such things, the best thing is to assign group structures to them. So, this was one of the landmark contributions of Emmy Noether. Emmy Noether has done lots of physics also.

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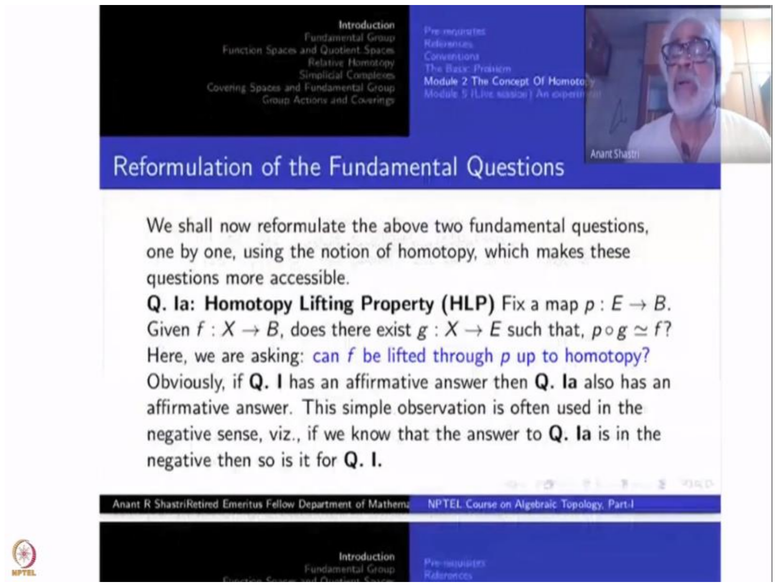

Now it is time for us to reformulate our fundamental topological questions in terms of homotopy. First question we are going to change was the lifting problem. Remember that. We are going to make it into a homotopy lifting problem then things which have satisfied certain properties will become homotopy properties. So, fix a map p from E to B, given a function f from X to B, the question is: does there exist a map g from X to E such that p $\circ$g $\simeq$ f instead of equality.

Everything else is the same instead of equality here, replace it by homotopy equivalence. So, here we are asking: can f be lifted through p up to homotopy. This is the meaning of--this is another way of saying the same problem. Obviously the original question when you have equality here, has an affirmative answer then the changed question Q1a, I am talking about, it also has an affirmative answer because equality implies homotopy equivalence also.

But the converse is not correct. Even if this equivalence problem is solved, there may not be any function g such that $p \circ g = f$. But this simple observation that the homotopy equivalence problem will be assured if there was actually a lifting is useful in a negative sense. If this is not true, namely you cannot lift your homotopy then of course, you cannot lift the map at all, therefore a negative answer to the homotopy version would give a negative answer to the original one. Even this much service given by this problem is of importance to us.

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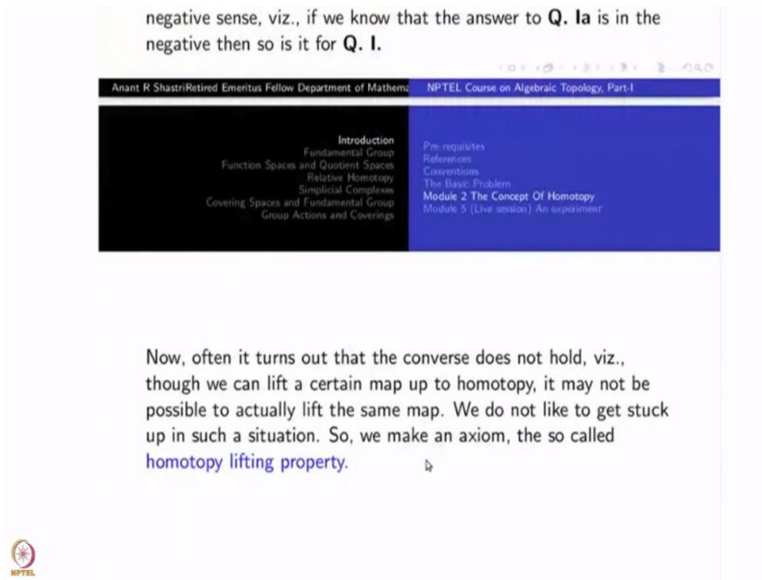

But what we want to use is, we will ignore this portion, we will only solve or answer whether something can be homotopically lifted or not and we are satisfied by that. So, we will pretend as if the problem is over, when you have lifted a function up to homotopy.

What does that mean? We just make a hypothesis, namely assume that if something can be lifted up to Homotopy then it can be lifted. Okay? So, this is called the homotopy lifting property. This is an axiom now, on the function on the map from E, p from E to B. So, let me state this one correctly, rigorously.

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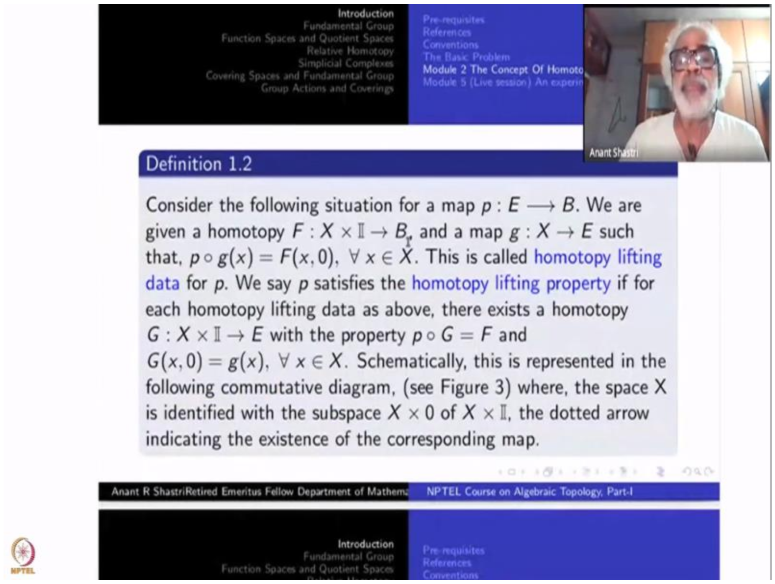

So, start with a map p from E to B. We are given a homotopy F from$X \times \mathbb{I} \to B$, okay? and a map g from X to E such that, when you take p $\circ$g that is a map from X to B, it is the function F(x,0), that is, the starting point of this homotopy. The starting point of this homotopy, namely, suppose F (x, 0) is equal to small f(x), then that map has been already lifted, that is, g.

So, this is a part of the hypothesis. All this is called homotopy lifting data- this much is given. We say p satisfy the homotopy lifting property if each such homotopy lifting data will give rise to an actual homotopy from g, Now, it is from $X \times \mathbb{I} \to E$,see the given homotopy was X× I to B. Now the lifting of this whole thing has gone into E such that when you compose G with p, it comes to F and the starting point of this homotopy G on X × {0} is the given function g from X to E. Okay?

If this happens for every X and every F and every g, there must be a capital G like this, then we say p satisfies homotopy lifting property. Understand? So, let me put this in a figure, in a picture, which will be just easy to remember.

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This is the picture. So, here you have g from $X \times 0$ to E and here your $X \times \mathbb{I}$ and this is your inclusion map eta. $X \times 0$ goes to $X \times 0$ under this inclusion map. F is the homotopy. At the starting point, it is $g \circ p$ it is F on $X \times 0$. This much is given. As soon as you are given this, you must get this dotted arrow, capital G from $X \times \mathbb{I} \to E$, such that when you composite with p this must be F. So this triangle is commutative and this triangle is also commutative. Same thing as saying that if you restrict it to $X \times 0$ it must be equal to g.

So, this is the conclusion and this is the data. For every such data if you can come from here, from this part to this part that means p has homotopy lifting property. So, this is the definition.

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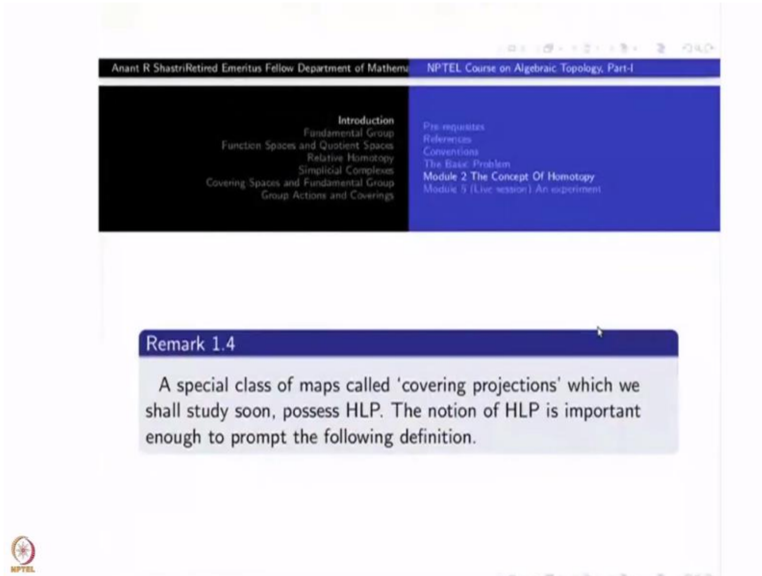

So, this will take care of all our botherations about the point-set-topology and we can do only algebraic topology. This is the whole idea. Such special classes of maps are, you may say, very rare. No! That is not true. There will be plenty of them and one such class is the one we are going to study very rigorously in this course, namely, covering projections. The notion of homotopy lifting property is important enough to make another definition.

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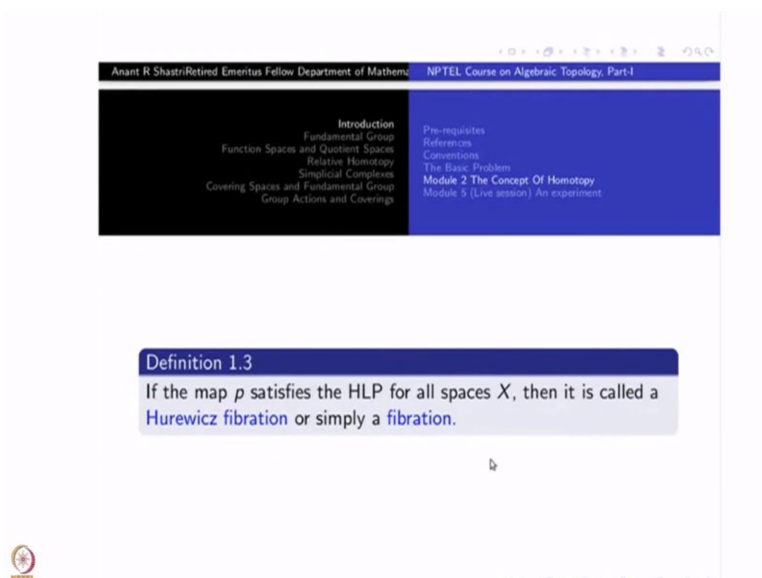

Namely, such things are called Hurewicz fibrations or just fibrations. You see, because Hurewicz was the first one to notice it and study it quite deeply. So, people call it Hurewicz fibration. What is the meaning of a fibration? It is a map from one space to another space, which has the homotopy lifting property.

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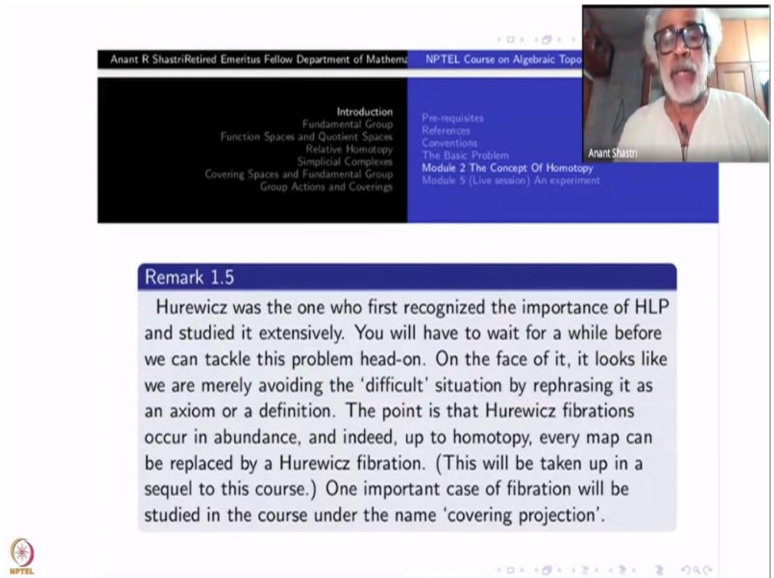

So, this, what I am telling you, Hurewicz was the first to recognize this one. So, one important case of fibration will be studied in this course and that is covering projection. We will take up the second question a little later, namely in the next module. So, today this is enough. Thank you.