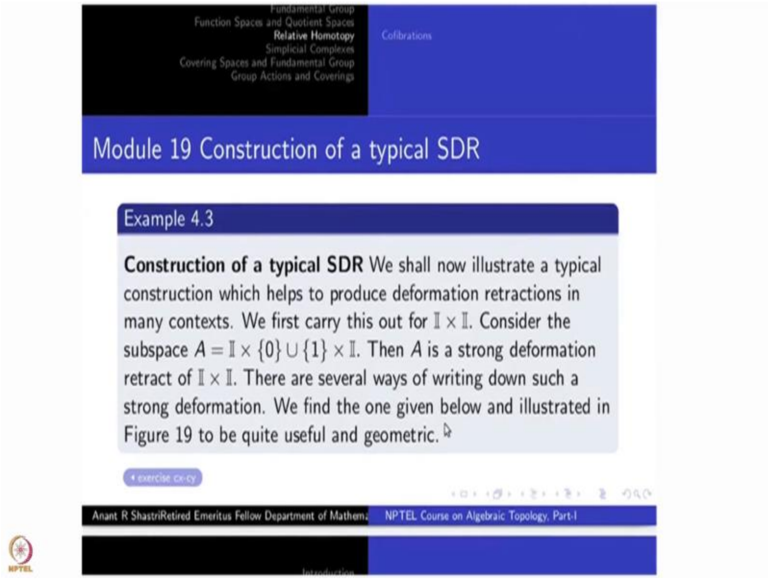
Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture – 19
Construction of a Typical SDR

(Refer Slide Time: 00:16)

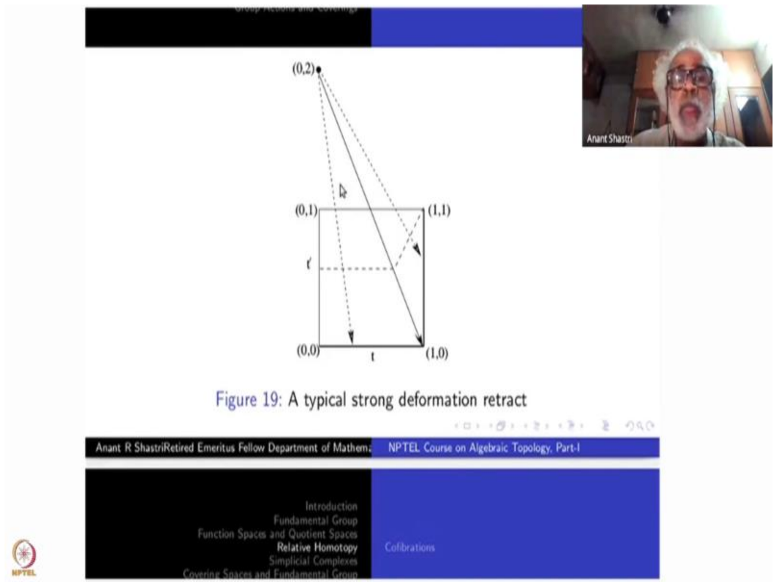


Welcome to module 19, so now we will discuss a typical construction, a typical SDR strong deformation retract; from which you can construct almost all well known deformation retracts. As such there are not many ways of even constructing homotopies; so they all fall into a certain pattern. Almost always things are happening in the domain of the function itself.

So there are that kind of consideration is what I am going to do now, the construction of a typical strong deformation retract. So, all these things we will carry out first of all in $\mathbb{I} \times \mathbb{I}$. Then slowly we will make it slightly complicated. But, that is it, after that that can be used with a slight modifications and so on in many situations. So, what I do? In $\mathbb{I} \times \mathbb{I}$, I take the x axis part and on y axis, I take $1 \times \mathbb{I}$; alternatively, you may take the other two sides also.

So, instead of the entire boundary I am taking two of the sides, not opposite, adjacent sides; so, $\mathbb{I} \times 0$ and $1 \times \mathbb{I}$. Then this subspace is a strong deformation retract of $\mathbb{I} \times \mathbb{I}$; it is intuitively clear. But that is not where we live; we will write down a specific strong deformation retract. Even to do that there are many ways. So, what I am going to tell you is a geometric way of doing this, and this can be generalized to construct many other things; so here it is.

(Refer Slide Time: 02:52)



What do I do? In $\mathbb{I} \times \mathbb{I}$, I have chosen this $\mathbb{I} \times 0$ and $1 \times \mathbb{I}$; this is my subspace A. I want to get a strong deformation retract of the entire square onto subspace A. What I do? I take some point on the y axis above this square somewhere; so here just for certainty, I take the point (0, 2). That is not needed; I could have chosen this point or this point or far awayfrom 0, some (0,200)... no problem. So, that point I am taking and take a view of the whole thing. So first I see this one from here.

Then I can just keep looking-- if this is a transparent stuff, my eyes will see all the way to bottom here. So, this line you tell me what I should do? So, take a point here; I am going to push this point slowly to this point. This entire line, namely $0 \times \mathbb{I}$ is going to be pushed down to this point (0,0). Then from a point here we are going here; this point will go to this point.

So, this is one part, you will have one single formula for this patch But, when you go further, a point here will be going to this point here; namely on the segment $1 \times \mathbb{I}$, the vertical par. Then this line. Then the formula will be different, we have seen that. So, I am going to push this one, this portion here, this point remains here of course; there is nothing to push that to.

So, this way pushing means what? You do it at time t; at time 0, it is identity nothing has happened. At time t say t′, this part is already pushed up to here; this point has come here, this point has come here, this point has come here, this point has come here and so on. Only this much is left out, so keep doing it. At time t = 1, only this segment and this segment will remain; so that is the homotopy.

During this homotopy, points of A do not move here, therefore is a strong deformation retract, so this is geometric proof. After this many books stop here, they do not write down the formula. Writing down is good for your own sake, you get more confident; so I have done that, you can write down yourself. But, you may not know how to write down such a formula.

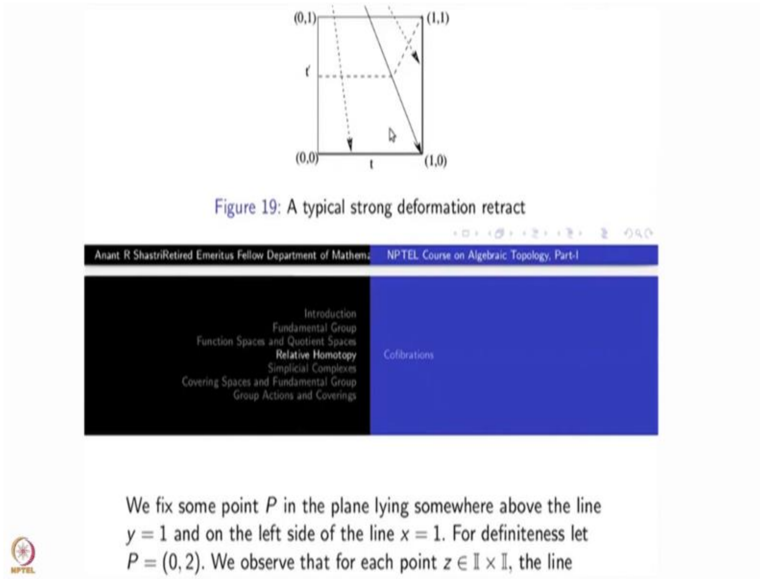
So, these are just actually linear projections; so, it is linear algebra. So, I have explained this with the point 0, 2. You have to parameterize, take a point here parameterize this portion from here to here; this portion, this line segment. Write down parameterization for this line; use that parameterization as a homotopy from this point to this point. At t = 0, it should be this point and t =1, this should be this.

You have to write a continuous function say, one formula, up to here from here to this point. Can you write down the coordinates of this point? What is this point? This point is intersection of this horizontal line here; and the line joining 0 to 1, 0. So, you can write down; so, this is just plane coordinate geometry; that is precisely what I have done here.

(Refer Slide Time: 07:26)

Figure 19: A typical strong deformation retract

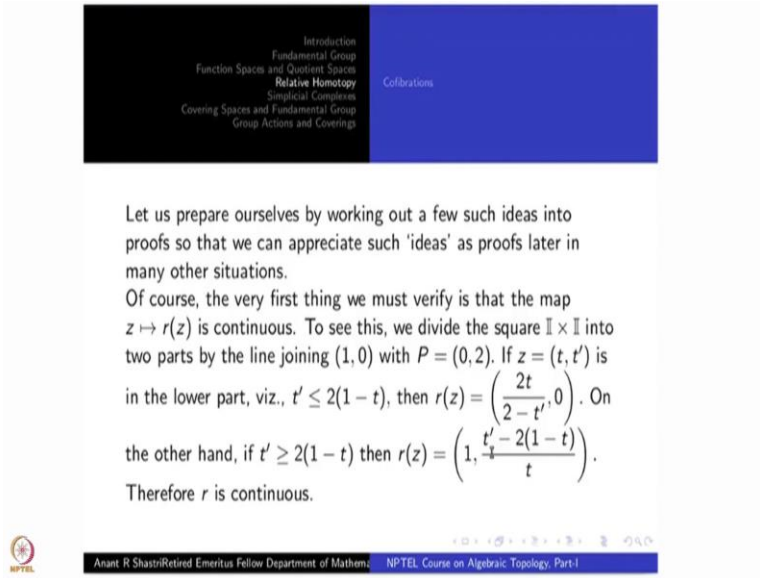
We fix some point P in the plane lying somewhere above the line $y = 1$ and on the left side of the line $x = 1$. For definiteness let $P = (0, 2)$. We observe that for each point $z \in \mathbb{I} \times \mathbb{I}$, the line through P and z meets A in a unique point $r(z)$. We can then simply take the map which 'pushes' the point z to $r(z)$ in a unit time as our homotopy. The idea is over. Often many expositions in algebraic and differential topology stop at this stage. A student of topology is expected to appreciate this as the final proof and others are expected to swallow it in good faith.

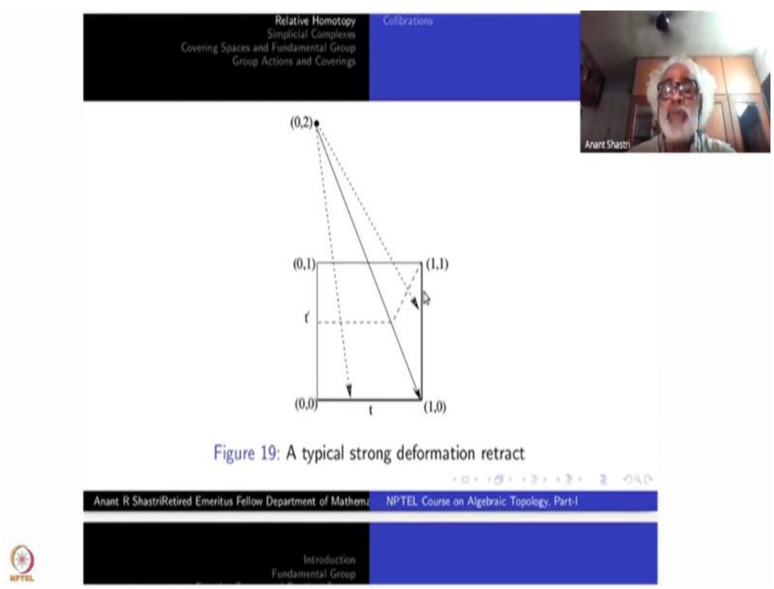


The map pushes the point arbitrary z whatever z, to r (z); r (z) is a retract at the end, the end result here. Any z here comes to r (z), the tip of the arrow will be the point r(z); point here will come here. Point here will also come here, so that is r (z). The student of topology is expected to appreciate this as the final proof and others are expected to swallow it in good faith.

This is the way many books explained it, the maximum explanation they would offer. So, if you want to feel that, if you want to do this kind of things on your own in a different situation; what is the way to learn? This is an easy method. Here you have to be very sure that you can write down the formula; so you have to write down.

(Refer Slide Time: 08:40)



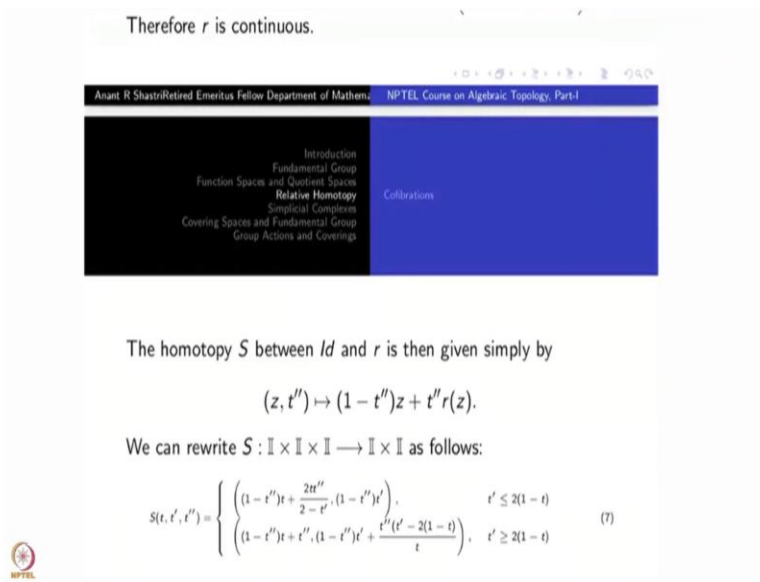


So, let me go through this for a while, I am doing this just for you. Let us prepare ourselves by working out a few such ideas into proofs; so that we can appreciate such ideas as proofs later in many other situations, this my intention. Of course, the very first thing to verify is that the map z going to r(z) is continuous.

To see this, we divide the square $\mathbb{I} \times \mathbb{I}$ into two parts, by the line joining (1,0) to (0,2); that is what I told you. Join (1,0) to this (0,2), so this is one quadrilateral; that is a triangle, so it is divided into two parts. So, one formula r is this one, it pushes everything to the horizontal axis, other formula pushes it to vertical axis on $1 \times \mathbb{I}$.

So, here I have two different formulas. If z =t, t′ is in the lower part; same as saying that t′ is less than equal to 2 into 1 - t; then r (z) is given by this formula. On the other hand, if t′ is bigger = 2 into 1 - t, then r (z) is this one; the first coordinate is here is something, the second coordinate is 0. So, it is on the x axis, here is the first coordinate is 1; the second coordinate between 0 and 1. So, why this inequality, this inequality is giving you the two divisions; this one will give you what I have told you as the quadrilateral. This one gives you the triangle. So, on that line itself both these formulas will agree, the entire line goes to (1,0). Therefore, it is a continuous function.

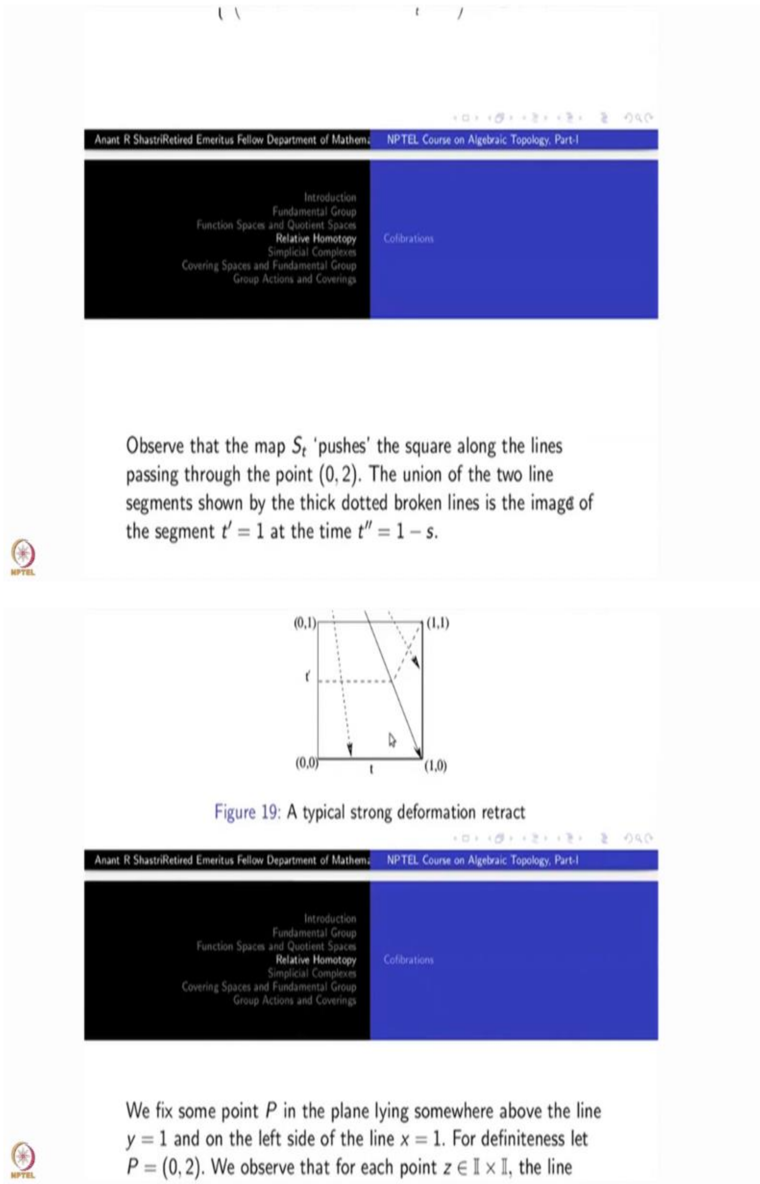
(Refer Slide Time: 10:58)



After that writing homotopy between identity and r is easy; take 1 - t″ times z + t″r (z) itself. Now you can appreciate why I did it for the case of the square minus centre. I had to write down four different formulas. Once I have written down r (z), the homotopy is easy. There is only one way to get the strong homotopy 1 - t″z + t″ r(z).

So, t″ because I have t and t′ already used. So, full formula for those t’s in terms of r (z) is written down that is not necessary; check this yourself.

(Refer Slide Time: 11:57)



Observe that each map S_t, S_t, pushes the square along the lines passing through (0,2); so this is what I told you. The union of two line segments shown by the thick dotted broken lines is the image of the segment at time t'; all that I have told you. Once more to go back. At $t = t'$, this line, the top line becomes the broken line like this. Because this point is pushed here and this line segment is pushed to this slanted line here.

At the final stage this entire line is broken up into this line union this line segment; so, this point goes here. Any questions here, because now I am going to leave it here; so that you can verify all these formulas. Then use this one to construct more complicated examples. So, next time when you come before this one, you better check this one properly; and understand this homotopy very carefully. We will stop here.