Introduction to Algebraic Topology (Part-I)
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Lecture – 18
Relative Homotopy

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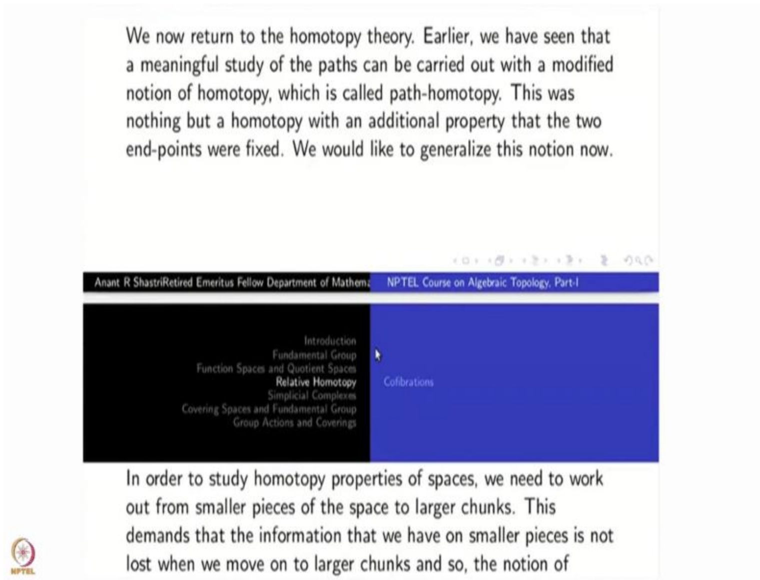

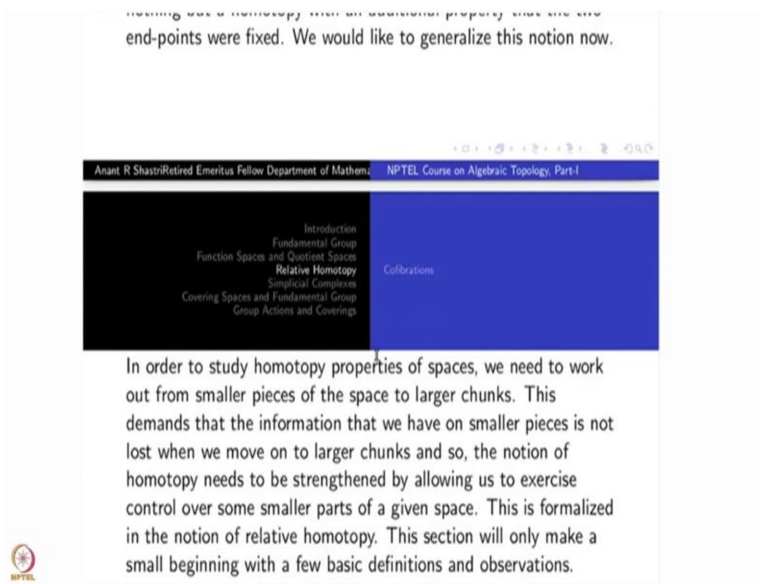

So, with a lot of background from Topology, now we return back to our homotopy theory. Earlier we have defined the notion of homotopy and then while dealing with loops and fundamental group, we modified it. Path homotopy was defined as homotopy which leaves the end points fixed; this idea will now be generalized completely.

Now, in order to study homotopy properties of spaces, we need to work out from smaller pieces of the space to larger chunks of the space. This demands that whatever good work we have done, whatever information we have collected on smaller pieces is not lost, when we move to larger spaces. So, the notion of homotopy needs to be strengthened by allowing us to exercise control over the smaller pieces.

And that is precisely what leads us to what is called a relative homotopy; namely the homotopies which do not change on whatever smaller piece is nice already. There, we have to keep the functions as they are, do not change them. So, this section will make only a small beginning. Yesterday only a few definitions and observations, even those thing have to be developed step by step.

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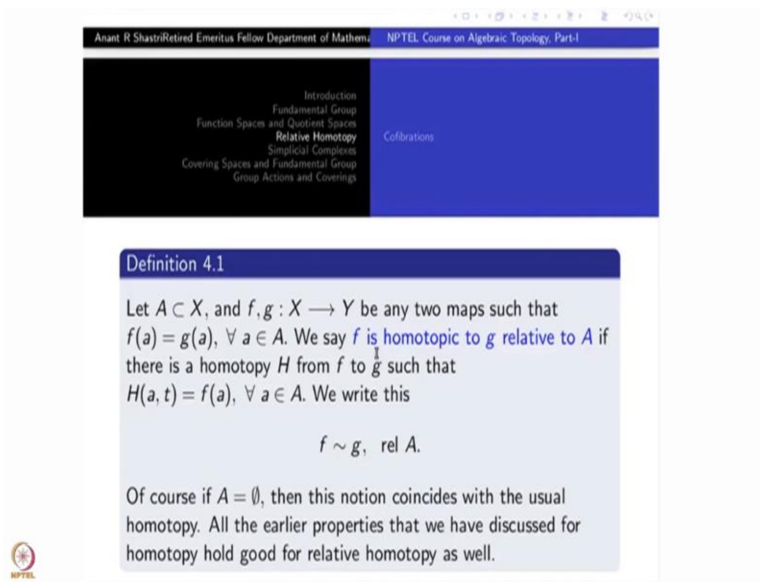

Start with a subset A of X, where X is a topological space; and two functions X to Y, which are continuous functions, such that on A they agree, f a equal to g a, for every a in A. So, on a smaller subset, the functions are the same. Then we say f is homotopic to g relative to A, if there is a homotopy H from f to g such that H of a, t is f a for every a in A.

We write this as f is homotopic to g, without any equality sign there, relative to A; the simplest notation for most useful concept. If A is empty then it is the standard original homotopy that we had. No controls. If X is the interval [0 , 1] and A is 0 and 1, this is precisely the path homotopy that we have.

So, the relative homotopy that we have taken is a perfect generalization of the older concepts. So, all the properties that we have, all the deductions that we have done for older things; they are not lost by this new definition, you do not have to recheck them again; this is one important thing you have to understand.

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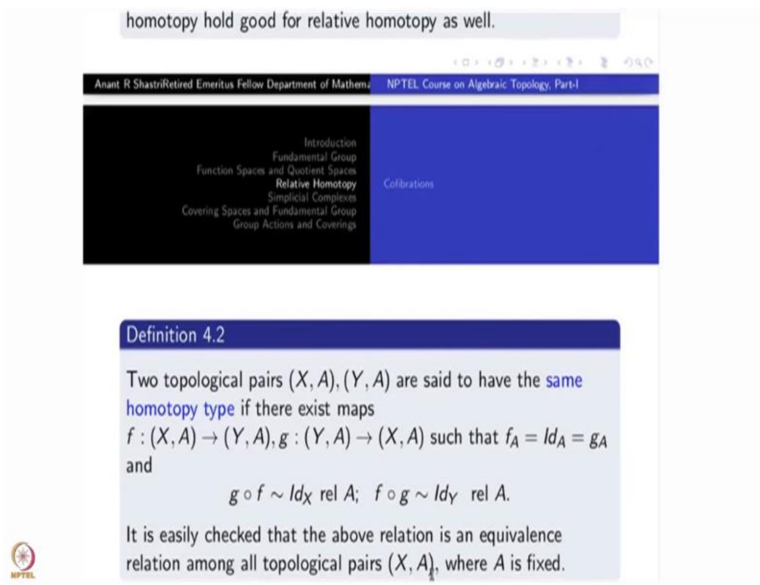

Now, we have defined what is homotopy between X to Y relative to some subspace; we can talk about homotopies of pairs. Here, you take a pair X, A--- means A is subspace of X; and Y should be another topological space, and B should be here instead of A, i.e., B is the subspace of Y. Nothing wrong if A equals B but this will be too restrictive that the same A should be the subspace of both X and Y. We can take both cases.

Generally, by pair of topological spaces, we mean as a topological space; the second entry here must be a subspace. That is all. So, such a pair has same homotopy type, if there exist functions f from X, A to Y, B, g from Y, B to X, A, i.e., such that f of A goes inside B , g of B goes inside A and B and g composite f is homotopic to identity of X relative to A; and f composite g is homotopic to identity of Y relative to B.

So, we keep writing relative to A specifically, if I just write equivalence it may conflicts with the older notation. So, this g composite f is from relative to B it should be. So, it is easily checked that the above relation is an equivalence relation among all topological pairs X, A, where A is fixed. If you take all topological pairs X, A; here you take also Y, A only; topological pairs with A has to be fixed here.

In other words, A to B you can take a homeomorphism. That is the consequence of this one. If g composite f is homotopic to identity X relative to A, this means g composite f is restricted to A is just identity map of A. Therefore, g and f are inverses of each other when you restricted to A or B. So they will be homeomorphic. A and B are homeomorphic is a consequence. and that is one reason why I have written both A here instead of B.

That is why there is nothing wrong, if in the definition you take A and B arbitrary. For topological pairs, when we talking about their are equivalence, then you fixed A and look at all topological spaces which contain A as subspace. Then you have this equivalence relation, whether they are homotopy equivalent to each other with respect to A, i.e., relative to A. So, that will be an equivalence relation; and the classes are called homotopy types.

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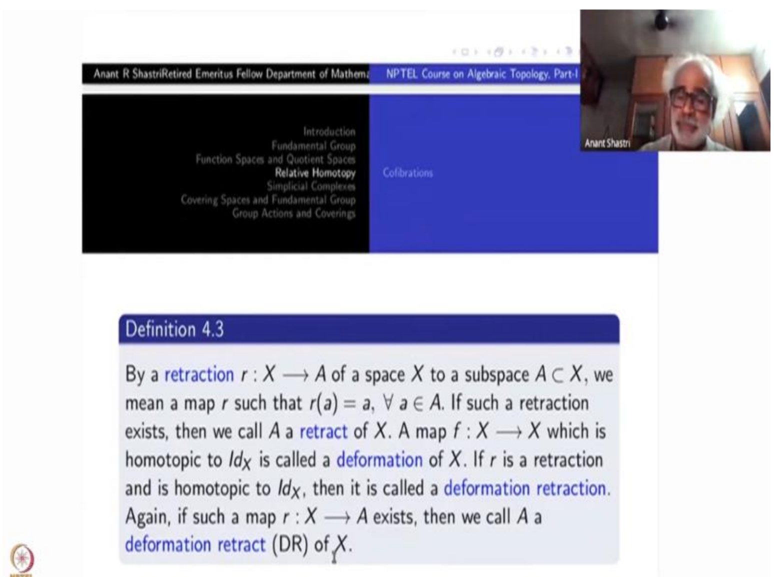

I have already told you what a retraction is, but let us recall: A retraction r is a continuous function from X to A, a subspace, such that on the subspace it is identity, r a equal to a for every a in A. If such a retraction exists then we call A is a retract of X; retraction is an action, it is the function. Retract is the result, A is a retract of X.

A function f from X to X which is homotopic to identity is called a deformation of X. Any function which is homotopic to identity will be called deformation of X. Often some people call the homotopy itself a deformation; so that is an action of the deforming. From identity map, you have taken some other function. If f is a homeomorphism, (we are not bothered so much about homeomorphisms,) maybe f of X is smaller than X, but it is homotopic to identity map, then this is called a deformation.

So, everything is taking place inside X, so that is important; so, any map which is homotopic to identity is called a deformation of X.

Suppose now r is a retraction and it is homotopic to identity; note that retraction is a map from X to A, but A is a subspace of X, therefore, we can view r as an from X to x is space. Then we can talk about whether r is homotopic to identity. If that is the case, r is retraction and is homotopic to identity; then it is called a deformation retraction.

A retraction and it is a homotopic to identity, homotopy is taking place inside X; image of r is A. But, identity map is there from X to X, so it moves slowly and enters inside A; so, that is the meaning of deformation retraction. Again if such a map exists, then A will be called a deformation retract of X; so you have deformation retraction and deformation retract; retraction and retract. Retract is subspace, and deformation retract is a subspace; whereas deformation retraction is the action the homotopy, the map itself.

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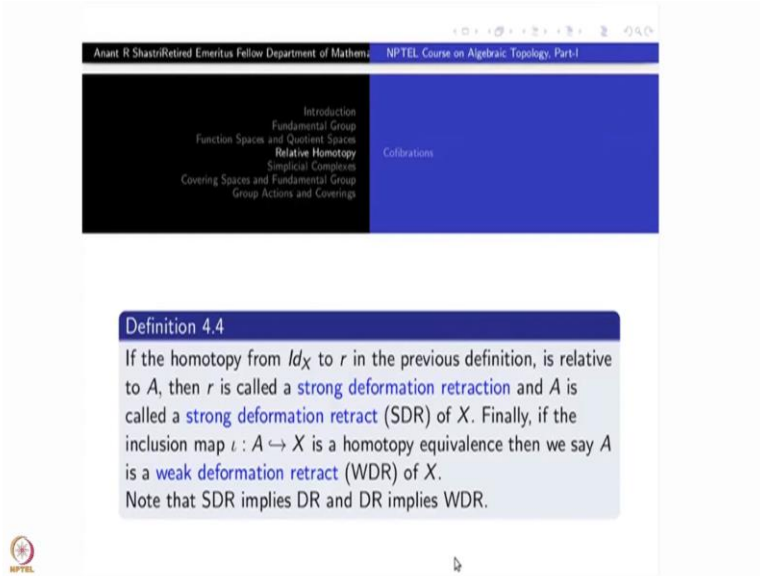

If the homotopy from identity to r in the previous definition, namely, when you defined deformation retraction, if that homotopy is relative to A, the points of A are not moved, then this is called a strong deformation retraction; and A is called strong deformation retract. Finally, one more definition, when the inclusion map A to X is a homotopy equivalence, (say, treat A as some other space, then you know that we have defined homotopy equivalence between A and X; but, if it is an inclusion map itself is a homotopic equivalence) then we say A is a weak deformation retract. Inclusion map has a homotopy inverse means there is a map r X to A, it may not be retract r composite i is homotopic to identity. That is why it is called a weak deformation retract.

One thing is the name should tell you that strong deformation retract implies deformation retract; it is obvious because we put extra condition in a strong deformation retract. During the homotopy, the entire homotopy should be identity on the subspace A. Deformation retract is what retraction map A is homotopy to identity, automatically tells you that inclusion map is a homotopy equivalence; its homotopy inverse is r. The weak deformation retract does not have an inverse, which is a retraction. So, somehow it is equivalence. Its inverse, homotopy inverse from X to A may not be a retraction; therefore, SDR implies DR implies WDR.

Strong deformation retract implies deformation retract implies weak deformation retract. The problem here is: if you are referring to various books, they may differ in the definitions. So take care; so what are the definitions, first thing to check. There are different meanings, the same words may mean different things or different orders.

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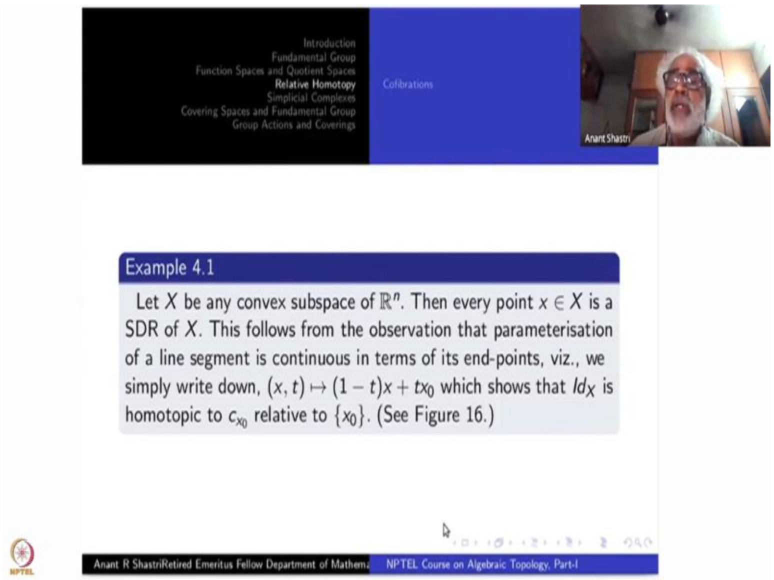

Let us examine some examples. Simple examples familiar to us, first and then later on, we will try to do more and more complicated ones. Take any convex subspace X of $\mathbb{R}^n$ then every point in X is a strong deformation retract of X. While proving that convex subspace is contractible, precisely what are we perceiving. What was the homotopy? $(x, t) \mapsto tx + (1-t)x_0$; whatever the selected point point x_0 is.

So, let x naught be a point in X; the homotopy is t times x, t 1 minus t times x naught. That is the strong homotopy; it will not move the point x naught at all. Everything is joint to this one; so this follows from the observations that parameterization of the line segment is continuous in terms of the end points.

Or we can use the map other way around: x, t going to 1 minus t times x plus t times x naught. This shows that when t equal to 0, it is identity of X, x goes to x, when t equal to 1, it is x goes to x naught. x naught comma t if you take, it is always x naught; no mater what t is; that is the meaning of saying x naught never moves.

Therefore, this homotopy is already relative to x naught. Therefore, what you have got is the constant map x naught, all of x going to x naught; that is a retraction, obviously. And it is homotopic to identity, therefore x naught is a strong deformation retract of X. Identity is homotopic to the constant map. Id of X x homotopic to C x naught; and this homotopy is relative to x naught.

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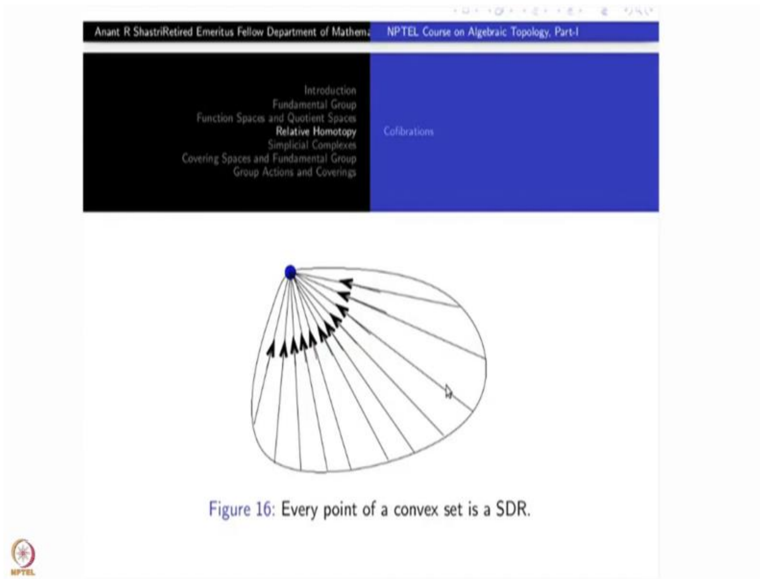

Here is a picture displayed. Suppose you have a convex set like this and take any point. All the points are coming here, this point is not moving. At the end , the result will be a constant map; so this is the picture of a strong deformation retract. You take any other point and then join, this is actually true of a star-shaped set also, with the apex as x naught; that x naught will be what? A strong deformation retract of the whole space. Whenever, a space is star-shaped at a point, viz., the apex point, then that is is a strong deformation retract.

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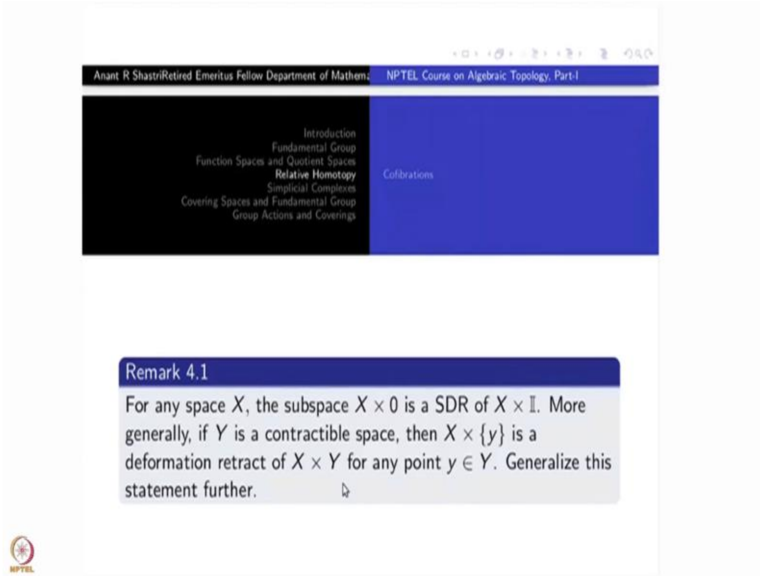

Another remark here for any space X, the subspace X cross 0 is a strong deformation retract of X cross I. Along these intervals for each point, I can just push the whole $X \times \mathbb{I}$ to $X \times \{0\}$. There is nothing special about $X \times \{0\}$ either, you take $X \times \{t_0\}$, then push the entire of interval at each point x of X, to the point (x, t_0).

So, this easy to see that X cross any t, the inclusion of X inside X cross I, is a strong deformation retract of the X cross I. So, more generally take any contractible space Y, then X cross y, where y is a point of Y, is a subspace of X cross capital Y. And if Y is contractible, then X cross little y will be a strong deformation retract of X cross Y. The deformation retract of Y, the homotopy can be used to prove this one, without moving the points of x; x comma whatever point. Maybe you can generalize this one even further.

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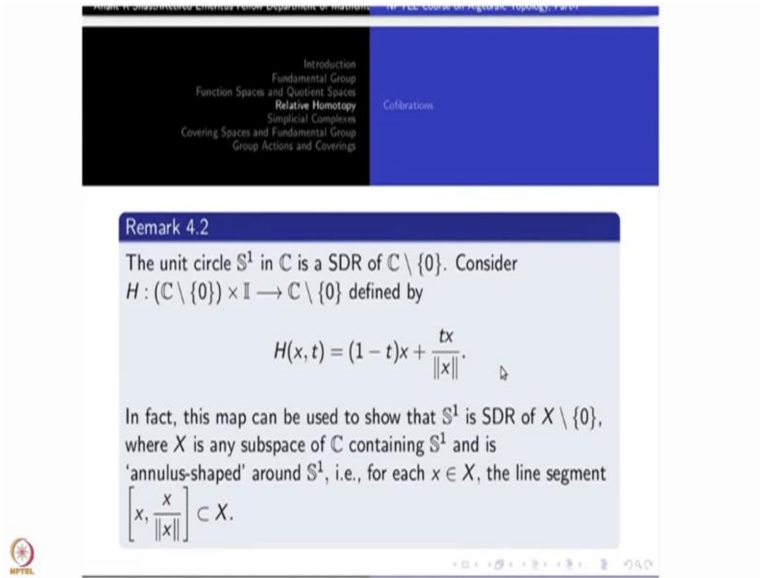

Now, let us workout another familiar example. To begin with, once again we start with $\mathbb{S}^1 \subset \mathbb{R}^2$. So, I say that the circle, the unit circle is a strong deformation retract of $\mathbb{R}^2 \setminus \{0\}$. The origin you have to throw away. How do you do that? Once again the convexity, kind of convexity, whatever, this is not exactly convexity, `convexity about $\mathbb{S}^1$ is what is needed, namely, joining works namely, $(1-t)x + t\frac{x}{\|x\|\}}$ makes sense.

Take any x inside $\mathbb{R}^2 \setminus \{0\}$. Take a real number $t \in \mathbb{I}$. H of x, t, I am defining H. x is a non-zero vector. So, I can divide by its norm, the new norm is 1, I get a unit vector. So, that is the map from $\mathbb{R}^2 \backslash\} \to \mathbb{S}^1$? $x \mapsto \frac{x}{\|x\|}$. Now I am joining that function x by norm x to the identity map of X. 1 minus t times x plus t times x norm x. Putting t equal to 0 this is identity of X, putting t equal to 1, it is the function which take x to x by norm x, which is a retraction. If it is a unit vector already, x by norm x is equal to x. So, it is a retraction and it is homotopic to identity, therefore this is a deformation retraction. During the homotopy if x is already of norm 1; then dividing by x has no effect. 1 minus t times x plus t times x is x; so no point of S1 moves during the homotopy.

So, this is a relative homotopy, therefore it is strong deformation retraction. Now, the same argument will work even if you take not the whole of C minus 0, but some kind of an annular regions and so on, like you can take just all the vectors of length bigger than half, and less than 3 by 2; so, it is an annulus. You should not include 0. Not only that now instead of $\mathbb{S}^1$ and $\mathbb{R}^2$ this whole thing generalizes exactly ditto to $\mathbb{S}^{n-1}$ and $\mathbb{R}^n$. Is that clear?

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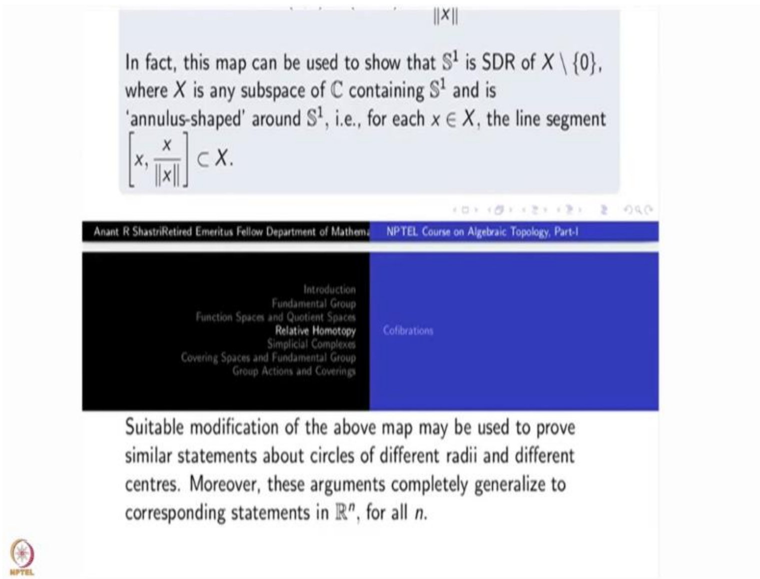

Not only that. What we can do is why 0 itself, you can take any disc, you can take a sphere centered at any other point. Then you have to just translate the whole thing, you choose your coordinates so that the center of that sphere is 0; so same arguments will work.

Not only that you do not have to even throw away the center; suppose you throw away any point inside the sphere, inside the disc, the interior of the disc. Then Rn minus that point can be push it back to the sphere; the sphere will be a strong deformation retract of Rn minus that point. So, you should try to see all these variations of this theme; but this is the simplest theme. But, this can be modify to get many other easy examples.

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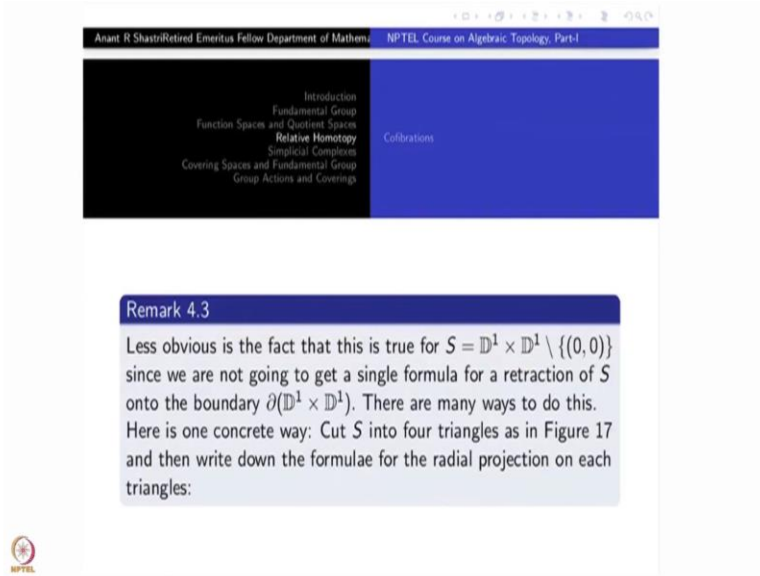

Less obvious is the following. Let us now look at minus 1 plus 1 cross minus 1 plus 1, D1 cross D1, the square. Now, you throw away the origin 0, 0; so that is my X. I want to say that the boundary of the square is a strong deformation retract of this X. What I have done? I have taken the square, then I have thrown away the centre. This space can be pushed out, or you can say the boundary of this one will be a strong deformation retract of entire space; there are many ways to do this.

Instead of the circle I have a square now. The only difficulty is -- geometrically it is very easy-- we have to just push the whole thing, which just means that you are taking stereographic projection. But, writing down a formula is not that simple as in the case of the round sphere; geometric idea is the same. So, I will give you one simple solution here, you are welcome to do whatever you like, different types of maps.

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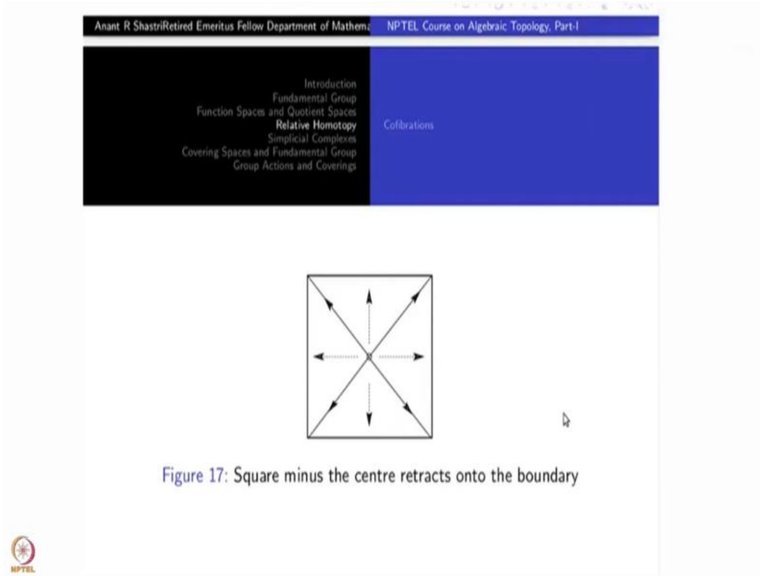

This is the simplest solution. Because the square has four corners, so I divide the whole thing into four equal parts, the four triangles. Then for each triangle I can write it down a simple formula. On the intersection of two nearby triangles namely on this line, the two formulas will agree. The two formulas will agree here, they will agree here, they will agree here and so on. So, they will give you one single continuous deformation, a homotopy from this entire square minus the center point here, to the boundary. During this homotopy, the boundary will not move.

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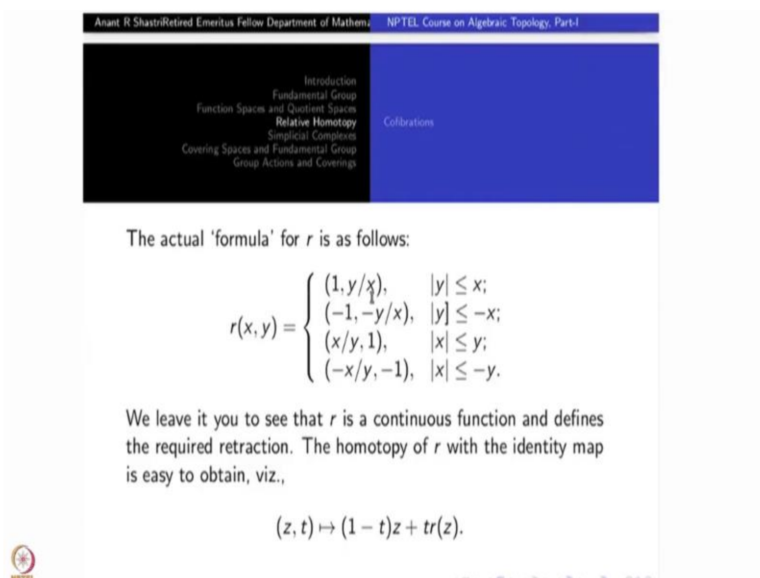

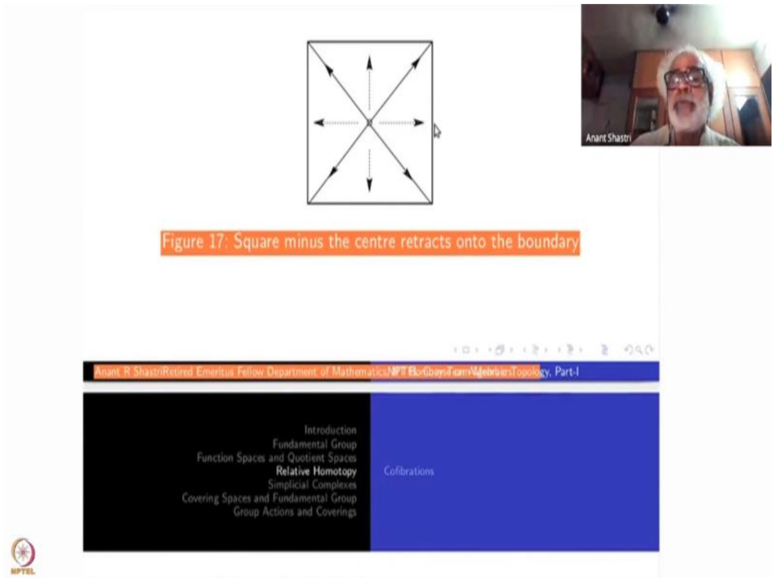

So then, let us write down formula, so here I have written down the formula. For each triangle, you have to put a different formula here. For example, x coordinate is 1, y coordinate is changing; the first formula is for this triangle. So, take any y here x coordinate is 1, y coordinate is changing; send it to y by x, so that is the formula, send it to y by x. X is not 0 because x is 0 here, x is half two one here; so I can divide y by x.

When x is 1, it is y the identity map; so those are the projections here. So, other thing you can work out, once you have written r, which is a retraction. The homotopy we can write down in a single formula; take the retraction and the identity and join them. Writing down the retraction itself will be truncated here; after that there is one single formula like x by norm x.

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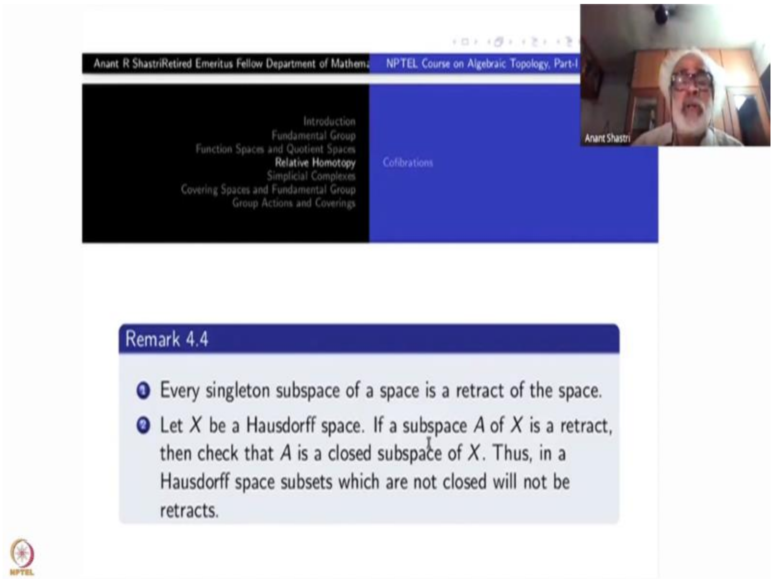

So, here are a few remarks, every singleton subspace of a space is a retract; the retraction being the constant map at that point. But, deformation retract, weak deformation retract, they are the homotopic concept; they are not always true. If X is a Hausdorff space, and if A is a subspace which is a retract; then A must be a closed subspace.

So, this is one of the reasons why in constucting adjuction space, we assumed the subspace, on which the adjunction is taking place be closed. Yesterday one of you asked why do you want to assume that it is closed? This is one of the reason. Thus, in a Hausdorff space subsets which are not closed, they are not retracts; very easy to get examples which are not retracts.

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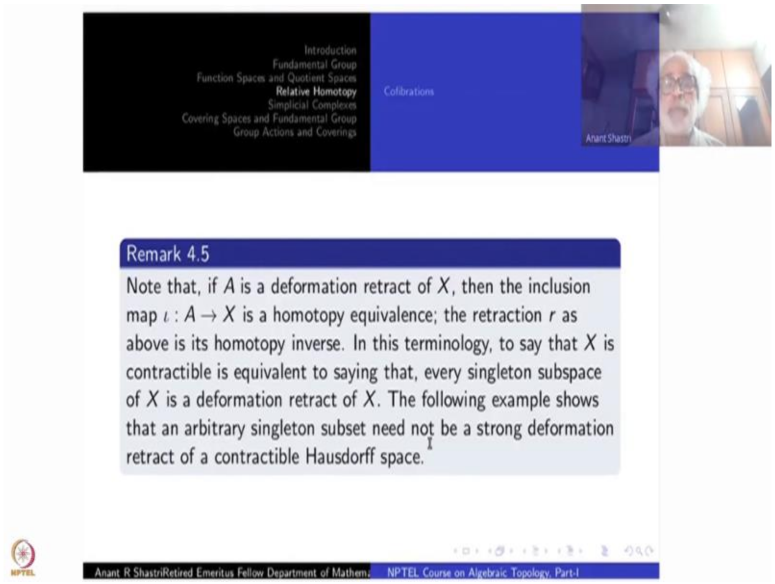

Note that, if A is a deformation retract of X, then the inclusion map is a homotopy equivalence, this I have already told you. The retraction r as above is its homotopy inverse. In this terminology, to say that X is contractible is equivalent to say that every singleton subspace of X is a deformation retract.

In the convex set, it is actually a strong deformation retract, this is what we have shown; but, that is not necessary in general. If you take an arbitrary contractible space, every point is a deformation retract; so that is so, more or less from the definition. But, if you want to have strong deformation retract that may not be true; so I will show you an example and that will be the end of today’s lecture.

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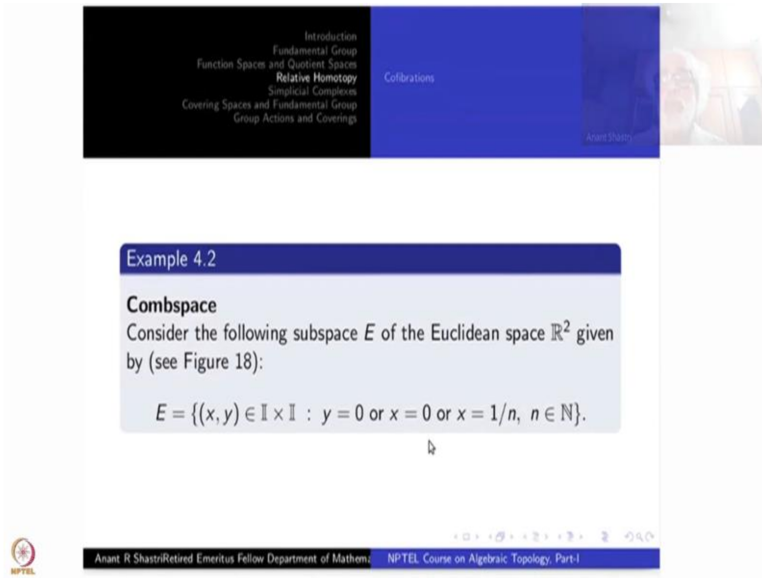

So, the combspace is an example, which is just to warn you that you have to be careful with the definitions here. So, what it is this combspace? It is the subspace of I cross I, the square; consisting of the line y equals 0. That means the x axis is there, or x eqauls 0, the y axis is there; or x equals one by n, y could be anything. 1, 1 by 2, 1 by 3, 1 by 4 and so on; so those are the teeth of the comb. So, there are infinitely many teeth for this comb, and they are all clustering, they are coming closer and closer; at the tooth x equal to 0. So, this is the picture of the combspace.

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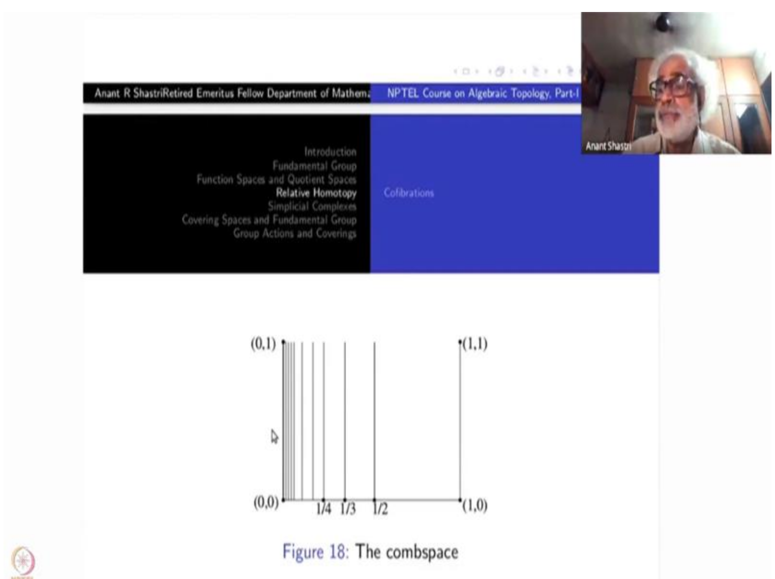

So, in the usual topology what happens? Around this point, around 0 , 0, and along the y axis is here, the topology of this space is -- it is not locally connected, not locally path connected and so on; now, that creates the problem.

First, you can push the whole thing down to the x axis; once you have pushed that, then you can push the whole thing to whichever point you want along the x axis. So, that shows that every point on the x axis is a strong deformation retract, without moving that point. I repeat. I push everything down to $\mathbb{I} \times \{0\}$ first, then everything down to 0 along the x axis and then all these points pushed up to any other point on the y-axis. So, each point on the x axis is a Strong deformation retract of this one. But, points on this line, namely the y axis; other then 0, none of these points is a strong deformation retract.

You can get a homotopy to that point because every constant function is homotopic to identity here; because this is a contractible space. But, the homotopy will not be identity on that point; the point itself has to move all the way to the x axis and then comeback. So, this is because of local nature here, the topology is complicated; this is not locally path connected. For a rigorous proof you have to work harder. That is an exercise. Thank you.