Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture No. 17
Quotient Construction continued

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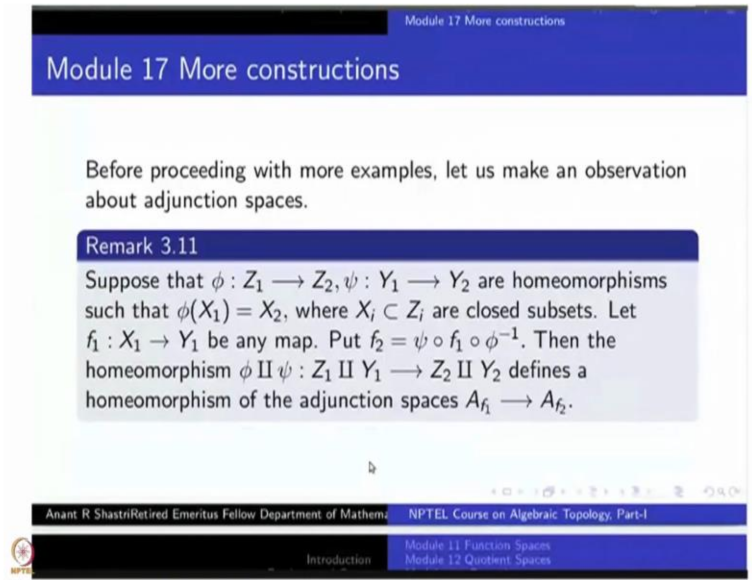

I would like to present a few more examples of Quotient Constructions, which are useful in differential topology. Before that let me make one remark, suppose you have a map ϕ from Z1 to Z2, which is a homeomorphism. Similarly, another homeomorphism ψ now Y1 to Y2; X1 is a subspace of Z1, X2 is a subset of Z2, and under ϕ, X1 goes to X2. So, I am doing adjunction space constructions with Z1, Y1 and over X1; similarly with Z_2, Y2 over X2, through maps f1 and f2 respectively. I assume that f2 is exactly equal to: you start with X2 come to X1 via phi inverse, then take f1, then go back via psi, i.e., $f_2 = \psi \circ f_1 \circ \phi^{-1}$.

If you have this data, then what you have is: two disjoint union Z1 and Y1; and Z_2 and Y_2. And you have a homeomorphism; phi disjoint union psi,-- the patch up of these two homeomorphisms, because they are disjoint. But, now because of this equation that is: f2 is precisely f composite psi composite phi inverse, in the two relations, there is compatibility. Whatever, identifications you do here, corresponding identifications will take place there. Therefore, out of this homeomorphism from phi disjoint unit psi, you get a homeomorphism of the quotients. First of all you get a map from the adjunction space A_{f_1} to adjunction space A_{f_2}.

Once this is stated, the verification of it is totally obvious. First of all you should see that there is surely a map which is a bijection. Continuity of this one will be totally obvious; because the corresponding functions are continuous on the mother spaces. So, similarly if you take inverses that map will be from Z2 disjoint in Y2 to Z1 disjoint union Y1. Same argument will apply; so, inverse will also be continuous.

So, this is the topological invariance of adjunction space construction. Later, we will see similar version for homotopy invariance of adjunction spaces, which will require some additional hypothesis in place of homeomorphisms.

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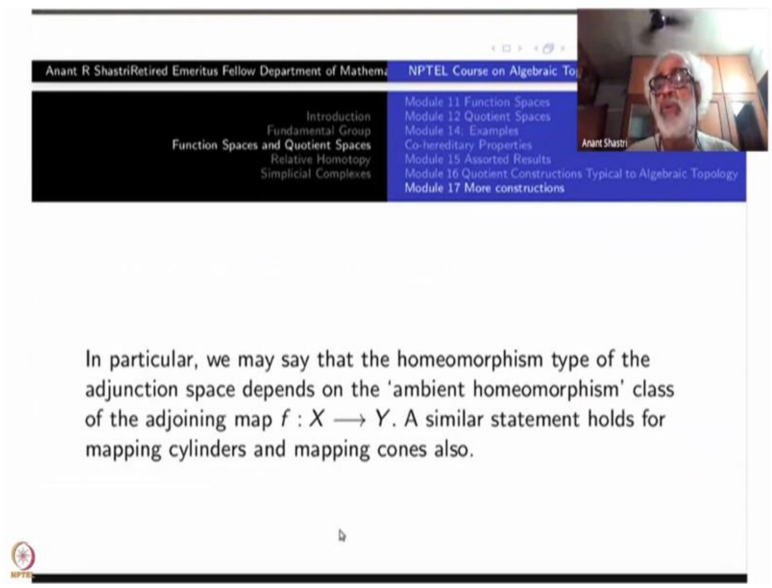

In particular, we may say that homeomorphism type of the adjunction spaces depends on the ambient homeomorphism class of the adjoining map f from X to Y. Note that $f_1 : X_1 \to Y_1, f_2 : X_2 \to Y_2$, X_i, Y_i are just related by a homeomorphisms, that is not enough. The homeomorphism should take place on the domain at the top level namely Z1 to Z2; and corresponding homeomorphism Y1 to Y2 so that the two maps are correspond under that under those homeomorphism; that the ambient homeomorphism class.

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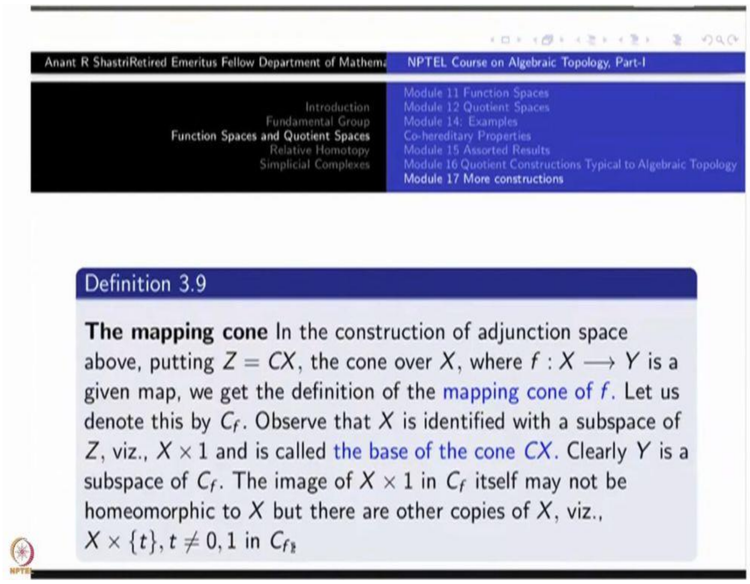

So, here is another construction now, which is a slight modification of the mapping cylinder. Remember that mapping cylinder of a function f from X to Y defined as a quotient space of $X \times \mathbb{I}$, X cross interval 0, 1 disjoint union Y; by the relation: $(x, 1) \sim f(x)$. So, you can think of this as adjunction space of $X \times \mathbb{I} \sqcup Y$ via the map which is defined on X cross 1 to Y. This we have seen. Instead of that instead of X cross I, you take the cone; namely when X cross 0 is collapsed to a single point.

Then, X is identified X cross 1 as the subspace; so you can think of Y as defined on a subspace of this Z =CX, and then take the adjunction space. That is called the mapping cone of f, which is nothing but --- another way of looking at it-- the quotient of the mapping cylinder, in which the first coordinate X cross 0 is also identified to a single point. So, recall that X here, you usually identify it with X cross 1 in the ordinary cone; but itself. But, when you go to the mapping cone, the X cross 1 is need not be a subspace; you will have to take X cross half or some other t.

Any $X \times \{t\}$, t not equal to 0 or 1, will do. Those will be the copies of X. Whereas, Y is always a subspace of this mapping cone. The image of X cross 1 in Cf itself, may not be homeomorphic to X; but the other copies of X namely X cross t, where t is not equal to 0 or 1. So, this is the modification. Do not make the mistake that X cross 1 is still a subspace of this mapping cone of the function f.

So, there are various kinds of constructions like this. The next construction is also very useful, which is just doubling the cone.

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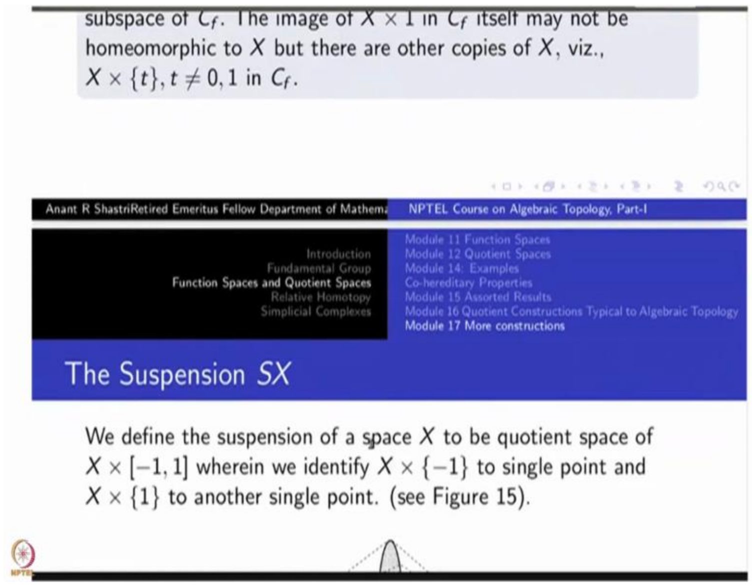

The cone is defined as the quotient of $X \times \mathbb{I}$; wherein you have identify X cross 0 to a single point. This time what we will do? We will take two copies; so what we will do? We will take instead of $X \times \mathbb{I}$, we will take $X \times [-1, 1]$, so that to begin with we have a line of double length here. Now, you identify at both the ends: X cross minus 1 as well as X cross 1 to two different single points. So, this is the picture.

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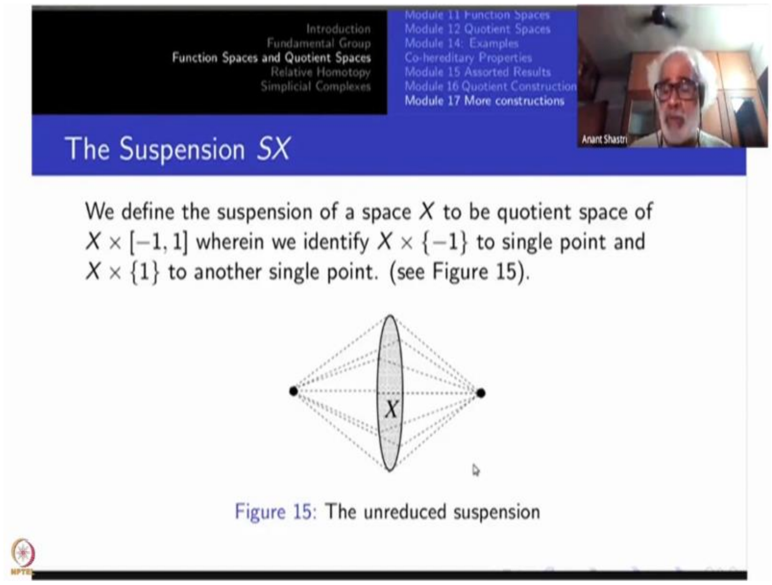

We have X here and X cross minus 1 to plus 1 like this in the cylinder. The X cross 1 goes to a single point here, X cross minus 1 comes to a single point here; so this is called the suspension of X. There is another terminology here, what are called as reduced suspension which I am not going to define right now. That will be needed when you are deep into the homotopy theory; right now you do not need it. So, in the light of that these are called unreduced suspensions. For example, suppose X is a single point, then what is the suspension of X? There is no identification at all; we have to just take X cross minus and plus 1.

So, the cone over a single point was the interval zero to one itself. Similarly the suspension over a single point is also an interval. More interesting thing happened when you take a space with two points, with discrete topology and not indiscrete topology. Then what will be the suspension? Now, X cross J will consist of disjoint union of two copies of the interval. One end point consists of minus 1 cross two points; the other one will be another plus 1 cross two points. These two things are identified to single points. So, what you get is like a diamond shape now, which is nothing but a space homeomorphism to a circle.

So, the two point-space can be thought of $\mathbb{S}^0$. Then we have minus 1 plus 1 cross the interval minus 1 plus 1, $\{-1, 1\} \times [-1, 1]$, and identify the end points of this; then you get the circle. Actually you better identify it with a diamond shape, but diamond shape is also homeomorphic to the circle. So, the suspension of $\mathbb{S}^0$ is $\mathbb{S}^1$; what happens to the suspension of $\mathbb{S}^1$? Here is a picture.

Forget about the shaded part in this picture-- if you take just the boundary, that is a circle. When you take the suspension what is this space?

It is nothing but union of two copies of discs; because the cone over circle is a disc, identified along the common circle, the base, which will be nothing but the 2- sphere; suspension of a one circle is a 2-sphere. Exactly the same argument will tell you that suspension of any n sphere is n plus 1 sphere. When I say n plus 1 sphere, I will just mean homeomorphic; so you can directly write down a neat homeomorphism. If you use cos theta, sin theta and so on; it will become neat instead of the corner here.

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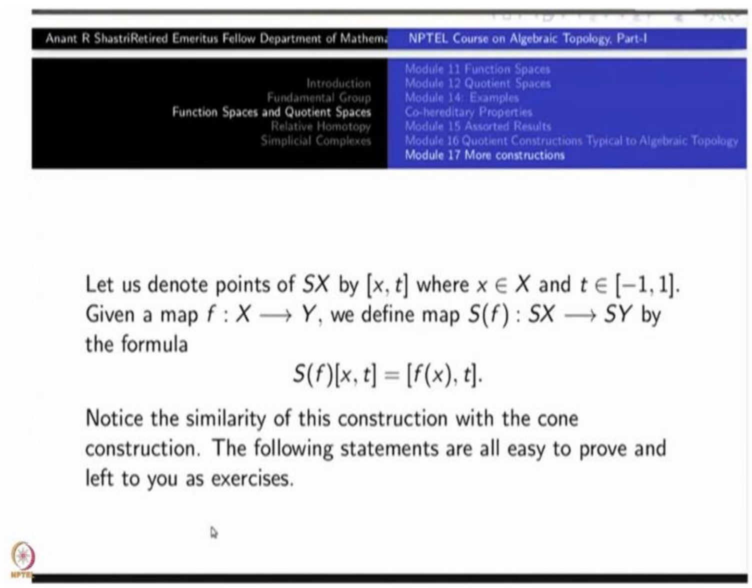

There is a nice property of this suspension just like the cone at the function level. Suppose you start with a function from X to Y; let S of X, S of Y be the suspensions. Then there will be S f from SX to SY, just defined by the formula S f of a class x, t goes to the class of f of x comma t. To begin with if you do not take class here, this is a map at the level of X cross minus 1 plus 1 to Y cross minus 1 plus 1; which is just f cross identity. Whatever, is identified in the domain, all of them go to points in which get identified in the codomain. Therefore, this map will factor down to give you the continuous function at the suspension level.

By an argument similar, you have seen CX to CY you have Cf, whenever f is a continuous function from X to Y Cf is continuous. Similar to this one, you have at the suspension level also. Thus, suspension has all that functorial property, if you take f to be identity, S of identity is the identity function of the suspension SX; and if we have X to Y and Y to Z, two functions and if you take g composite f , then the suspension of that will be suspension of g composite suspension of f. Because all those things are happening in the first coordinate here; g composite f cross Id is is g cross Id composite f cross Id, that is all. The t coordinate is not disturbed, the functional composition is take on the first coordinate here.

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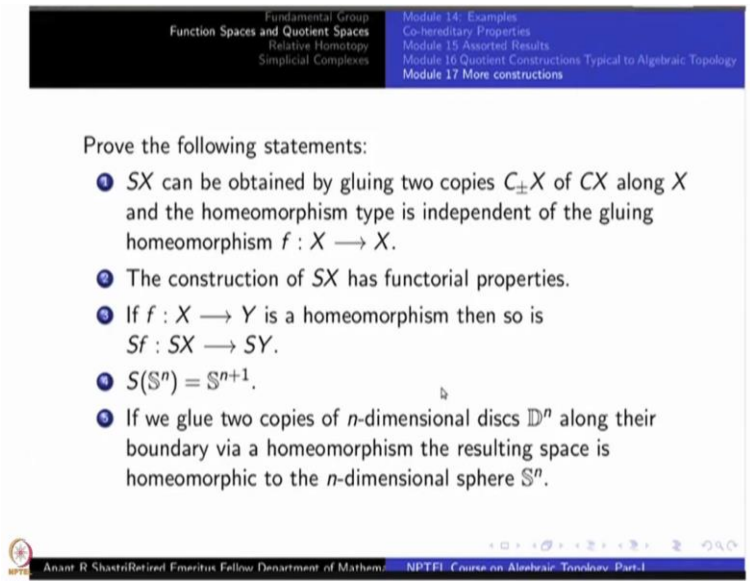

Now, here are a few observations which I will leave it to you to think about. I will go through them slowly, so that you may think about it. Finally, you have to come out with a proof. SX can be obtained by gluing two copies of CX, the cone; which you can write as C plus X and C minus X. Along a copy of X they are identified. The homeomorphism type of the result is independent of the gluing homeomorphism f from X to X. You can take any homeomorphism X to X, and identify C plus with C minus of X; the resulting space always will be homeomorphic to SX.

Construction of SX is functorial property; this I have already explained--- the meaning of this statement. If f is a homeomorphism, then Sf will be also a homeomorphism; if g is the inverse of f, then Sf inverse will be Sg. For the the last statement, which also I have explained, you have to write down some rigorous proof of it. That is all. The suspension of Sn is 1-dimensional higher sphere namely Sn plus 1. To help this one I am giving you the topological result here namely, if we glue two copies of n-dimensional discs along their boundary, via a homeomorphism, the resulting space is homeomorphic to n-dimensional sphere $\mathbb{S}^n$.

The simplest case n equal to 1. You take two intervals close intervals, identify the boundary via a homeomorphism from 1 to the other-- just a bijection. a goes to a prime, b goes to b prime-- identify; what you get is a circle. So, this generalizes to all the dimensions. So that is how S of Sn becomes Sn plus 1. If you have done this one before, then there is no more exercise. Because you would have done this exercise already, and you would have also learn that the cone over is a sphere is a disc of 1-dimension higher. If you put them together you will get this one. So, now the exercise is only to write down the details. Next, I would take one more example.

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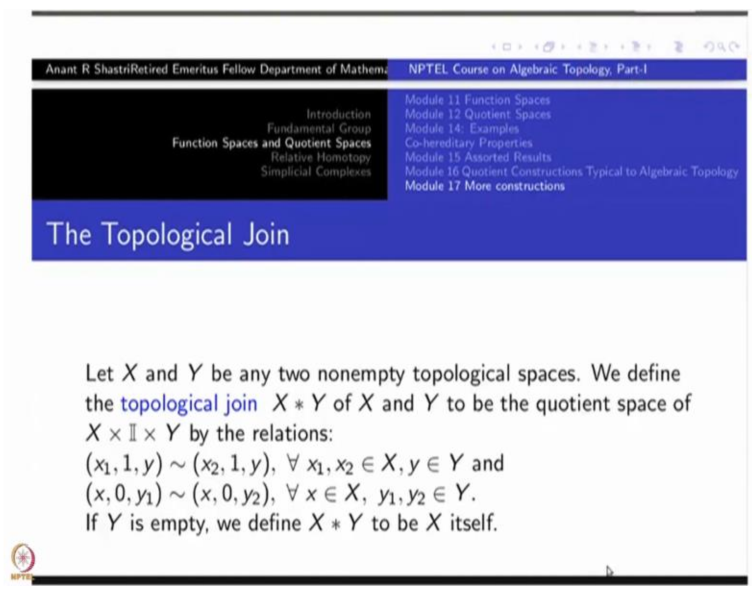

This is called topological join; this is actually a far more generalization than cone or suspension and so on. You can get suspension as a special case of this one, again we will see how. So, let X and Y be two nonempty topological spaces. (If empty space, then product is empty and so on, we do not want to get into that.) You define the topological join X star Y of X and Y to be the quotient space of X cross I cross Y. Note that it is not exact X cross I, it is not I cross I; it is X cross I cross Y. If both X and Y are I, then this would be a cube I cross I cross I; that is the simplest picture you will have.

Now, what I am going to do? identify each x1 comma 1, y, I mean this t parameter is 1-- with x2 comma 1 comma y. The third coordinate must be the same, then first coordinate is any arbitrary x; they are all identified to a single point, for each y in Y. So at one end I am identifying X times y to a single point for each y. You can write X cross Y cross 1 instead of X cross 1 cross Y. For each y in Y the X \times y is identified single point; so that is one thing. Same thing I want to do the other way around at 0: for each fixed x; I identify y1 with y2, for all y1 and y2, x is the same. So, for every x in X and y1, y2 in Y, we identity.

I started with nonempty spaces. if Y is empty what is X cross Y; it will be empty. So, you do not want that you would like to have X star Y as X itself and not empty. Maybe you would like to define it as empty that maybe be better; so I am not very sure of this one. If you need some change, you can define like this but you may run into other problems.

Intuitively, what is this topological join? X cross I cross Y; it has union of lines parameterized by x comma y this is the line segment. Union of lines parameterized by x comma y. But, when you identify all the points in X to a single point, what you have is just one single line from each y in Y to that point. Similarly, from each x\in X you have a line segment form x to the single point which represents all of Y and so on. So, I will explain this one with a better example here. No, not by an example--- trying to explain this in the abstract sense way only.

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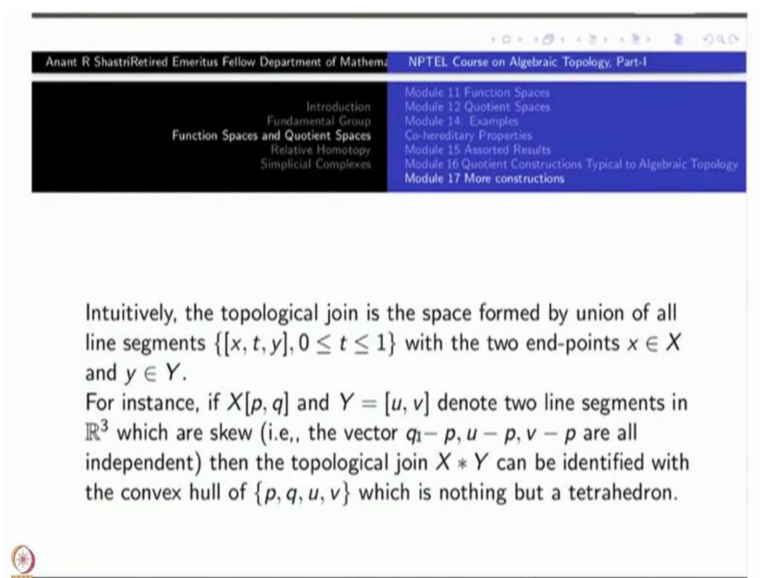

So, look at the class of x comma t comma y. So, t is never identified to anything; so as t range from 0 to 1, you will get a line. This line you can think of as starting at x and ending at y. In other words, you see in abstract sense, the join   is creating lines from x to y. In the cone what it has created? An extra point was taken and there are lines from every point of x to that extra point--that is a cone. Here what I have done? Here instead of just one extra point, a space is already given, X is given and another space Y is given. Now, for each point y of Y , you draw a line from x to y.

Where will you try to draw that? that is the point. It is not happening in any $\mathbb{R}^n$ or in any Euclidian space or any vector space, it is abstractly happening. So, that is why you take X cross I cross Y and do the identification. So, to justify this explanation, let me give you a simple example; there is a typo here. Start with X to be a line segment p, q and Y to be another line segment u, v, let these denote two line segments in $\mathbb{R}^3$ which are skew; the entire line must not intersect, nor , parallel that is called a pair of skew lines. So, how to ensure that?

There are many ways of ensuring it; namely you can fix any one point say p, as the origin. Then look at q minus p, u minus p and v minus p; so you get three vectors, these three vectors must be independent. So, that will ensure that p minus q and u minus v are not parallel vectors; so lines spanned by the segments are not parallel. Neither they intersect because if they intersect, there will be only two vectors here. The third vector will be dependent,--- either 0 or equal to the difference of these two or sum of these two. That is all.

So, the topological join of this X and Y the two line segments, which are skew is nothing but the convex hull of this. Take the convex hull of all these four points, that is called a tetrahedron. The convex hull of 3 independent points would have been a triangle; so there is a fourth point which is not in that plane; so what you get is a tetrahedron which is same thing as taking arbitrary point in p, q from p to q, join it to some arbitrary point in u and v, between u and v.

Take the collection of all these, so that will be the join of X and Y; it is also equal to just the convex hull of p, q, u, v which is the tetrahedron. So, in some sense the join is generalizing the construction of convex hull.

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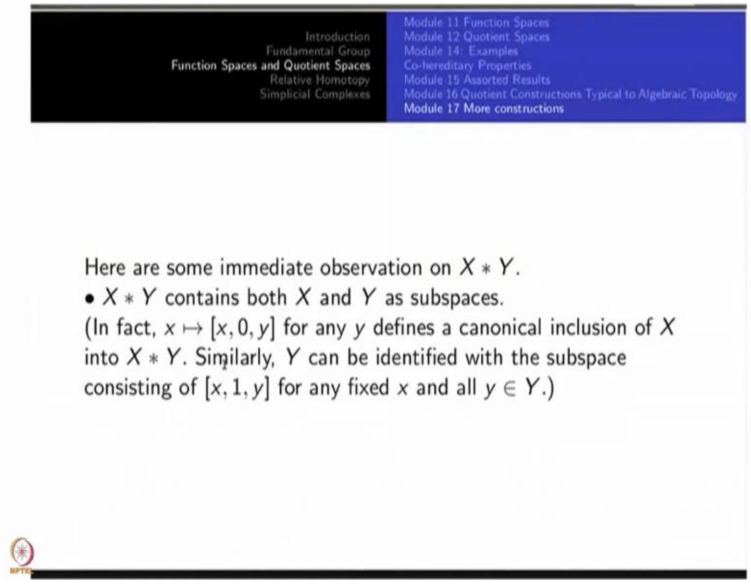

Now, we can do some elementary algebra of sets on this star; the star denotes join X and Y. X star Y contains both X and Y as subspaces; only at one point the points of X are identified namely when the t coordinate is 1. Similarly, when t coordinate is 0 points of Y are identified. If you take t coordinate not equal to 0 or 1 the whole X cross Y will be there, so both X and Y are subspaces. In fact, you take x going to x comma 0 comma y; there is no identification of x here the y coordinates gets identified; so this is a nice copy of X on the zeroth level. Similarly, Y going to any x comma 1 comma y going be a inclusion of y; x comma 1 comma y. These are subspaces of X star Y, just similar to the space of the cone or suspension and so on.

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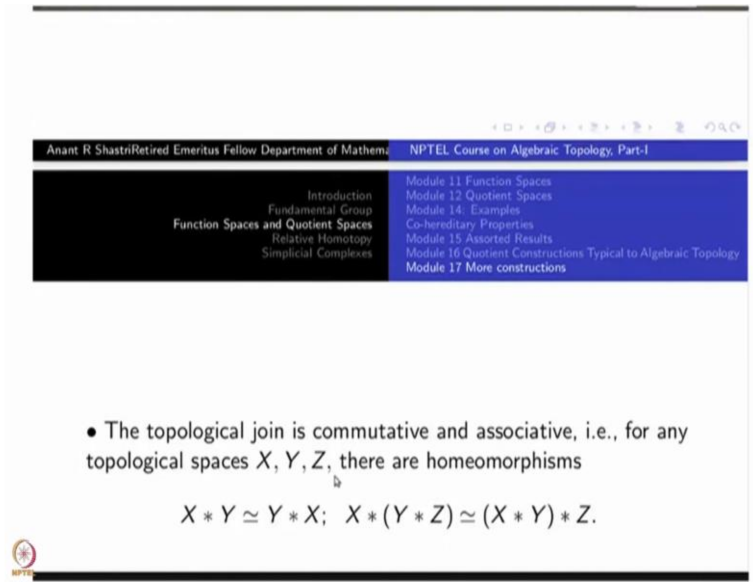

The topological join is both commutative and the associative; what is meaning of this? Up to the homeomorphisms X star Y is same thing as Y star X. So, this follows by the commutativity of the product X cross Y is homeomorphic to Y cross X; similarly, it is associative. X star Y bracket star Z is same thing as X star bracket Y star Z. All that you have to do is first have this law in the product space; X cross I cross Y cross I cross Z; you put the bracket wherever you like.

So, when you take quotient space that X star Y star Z; or X star Y star Z. But, the identifications are exactly same; exactly same whether you do it this way or this way. Therefore, these two spaces are homeomorphic. Similarly, actually easier, to see that this is commutative.

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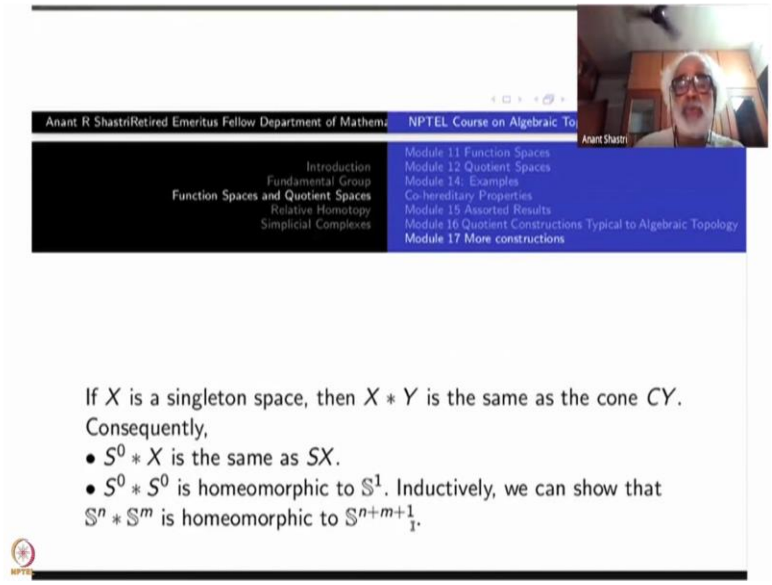

Suppose Y is a singleton space. then what I have? I have X cross I cross a single point, this is the same as X cross I; then I make identification. What I what it is at 0, I am defining all the X to be single point; so that will become like a cone. Then I come to 1, there is already one point there;, singleton y. There is no identification. All the points of Y should be identified, but no other points. Therefore, this is just the cone. So, this is one of the reasons why the operation join does not have a identity element – when Y is singleton it gives CX not X. So, if we want the operation to have an identity, then you may have to take Y as empty set?

That is one of the reasons why I have proposed: If Y is empty, then you define X star empty space as X; this is just a forced definition, it does not follow logically. Because if you take the product space, with an empty set, it is empty already. So, what we know is if you take one of the things as single point, then this is the cone. If you take one of them as a two point spce, then what you get is S naught X. This will be exactly same as the suspension. So, these are kind of explanation I have given, it should be clear that why this is true.

S naught cross I cross X will consist of two copies of $X \times \mathbb{I}$. When you perform identifications around X we will get two cones; and they will get identified because of identifications for S naught; so, you get $\mathbb{S}^0$ star X is SX. This is a special case of Join, I already told you.

So, S naught star S naught. We discussed this. What it is? It is S of $\mathbb{S}^0$ which is $\mathbb{S}^1$; and S $\mathbb{S}^0 \star \mathbb{S}^1 = S(\mathbb{S}^1) = \mathbb{S}^2$. And so on. So, S naught of S naught, i.e., S naught star of S naught is homeomorphic of S1. Inductively we can show that $\mathbb{S}^m \star \mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{m+n+1}$.

We can write Sn star Sm inductively S naught star Sn minus 1 star with Sm. By associativity, we take the first star that will be equal to, by induction, S of n plus m. But, now I am taking S naught of that, by this S naught of that will be a one more dimension here; S n plus m plus 1. The induction starts at n equal to m equal to 0; so that is a proof. So, we can stop here.