Introduction to Algebraic Topology (Part-I)
Professor Anant R. Shastri
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture No. 16
Quotient Constructions Typical to Algebraic Topology

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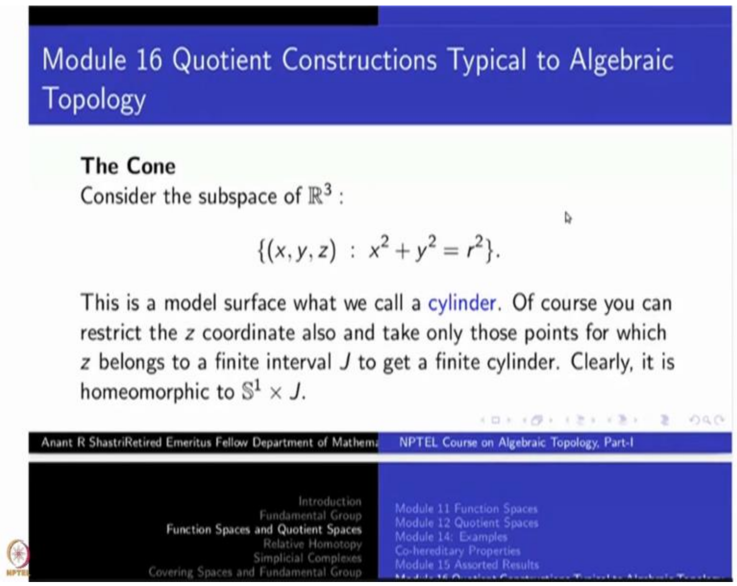

So, welcome to module 16. The topic today is the Quotient Constructions which are Typical in Algebraic Topology. So, let us begin with the so called cone construction. The subspace of $\mathbb{R}^3$ which is given by the equation $x^2+y^2= r^2$; no condition on z; is a vertical cylinder of radius r, centre is at the origin and axis will be the z axis. So, this is a model for what we call infinite cylinder. You can restrict the coordinate z say - 5 to + 5, then it will be a cylinder of height 10. So, z axis can be restricted to some extent that is also called a cylinder.

So, topologically what is this? nothing but $\mathbb{S}^1$ cross an interval. That interval could be of infinite length, or finite length, closed or open interval; all these things are called cylinders. $\mathbb{S}^1\times J$; so these are actually called circular cylinders in physics.

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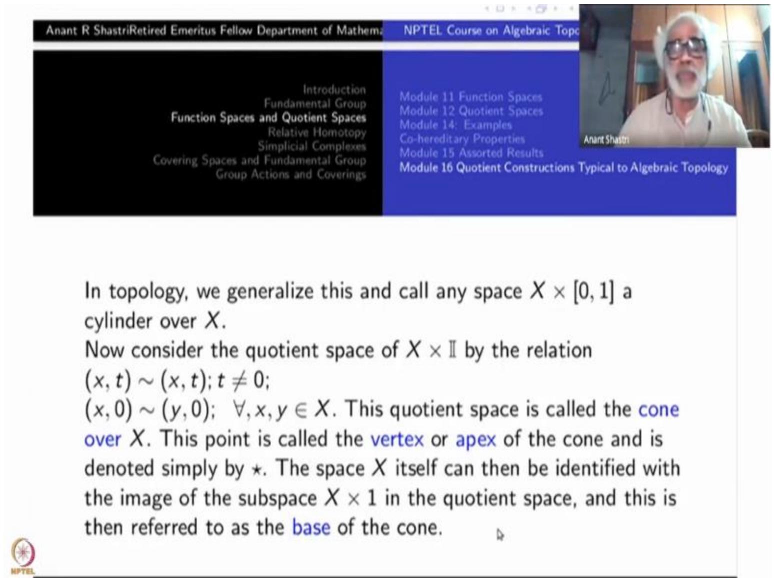

So, we will generalize this concept. You can call any space cross an interval as a cylinder; and this cylinder has X as the base instead of the circle. X a topological space cross interval 0, 1; you can think of this as a cylinder. Now, this kind of generalization that we are going to do for what is known in layman's language or in physics, the usual definition of a cone. So, in algebraic topology we define the cone based on any topological space X. What we do? Take X $\times\mathbb{I}$--- you can take any interval instead of I, but let us standardized to unit interval. Take X $\times\mathbb{I}$ and then identify (x,0) with (y,0), (x,t) with (x,t) for all t $\neq$ 0.

What is happening here? $X \times \{0\}$ all become single point. Other (x,t) remain (x,t), no identifications, where t ≠ 0. So, when t = 0, all (x,0) is identified with (y,0), for every x, y in X. So, this is the space. For t ≠ 0, the only identification is with itself, there is no identification; (x,t) is (x,t). But, on the base namely (x,0) in X $\times$ $\mathbb{I}$, every point you identify with every other point; so, the entire $X \times \{0\}$ is identified to a single point.

So, this quotient space is called the cone over X; the point namely, the entire X $\times$ 0 which identified to a single point, that point is called the vertex of the cone or apex of the cone. So, you can just denote it simply by star. The space X itself can then be identified with the image of X $\times$ 1 on the top in the quotient space. And this is then referred to as the base of the cone. To get an idea of a cone, you may always think of X as $\mathbb{S}^1$, the circle. Instead of circle, it could be an ellipse, it could be just be a curve and then you can talk about the cone over the curve. So, this is what this generalization is about.

So, the base of the cone is: starting with topological space X, that is the base; but then that will a subspace of the cone via several ways. But, I will be taking it in one particular way, namely, x goes to (x ,1).

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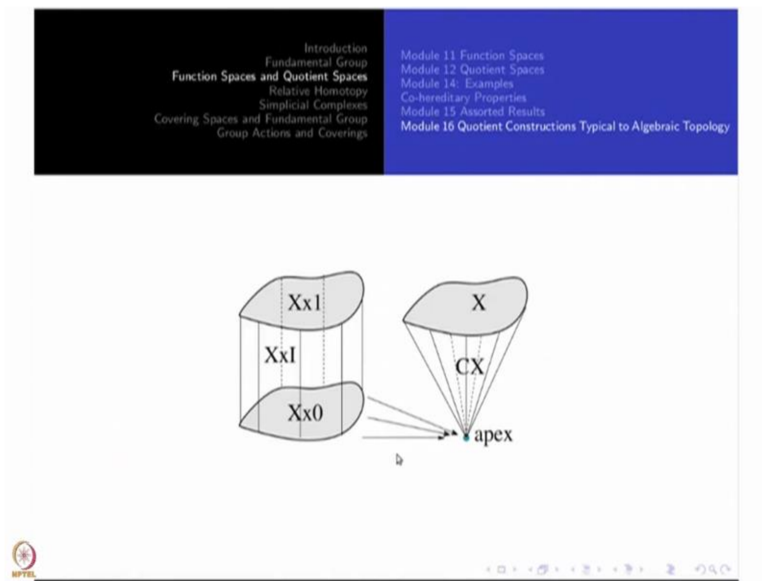

So, here is a picture which starts with $X \times 0$, and then the bottom $X \times 0$ is completely identified to a single point; whereas, $X \times 1$ remains as a copy of the original X here. In fact, you could have put X cross half here that will be also a copy of X here. In the picture it will be of different size, but it is homeomorphic to X-- all $X \times t$, $t \neq 0$, will be homeomorphic to X itself, X going to (x,t). This cone construction is very fundamental in homotopy theory; we will see why is it so important.

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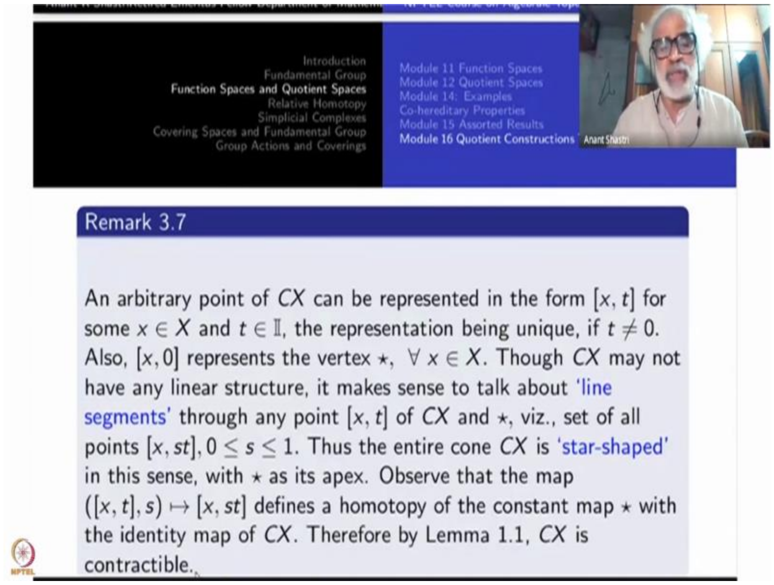

So, for an arbitrary point of this cone over X see this is the notation. A point in the product is represented in the form (x,t), with curly bracket usual---usual parenthesis. Now, we have put square brackets on (x,t), for some x belong to X and t belongs to $\mathbb{I}$. Normally, square brackets denote the equivalence classes. Here the representation being unique if t ≠ 0. But, for t = 0, any (x,0) will represent the same point. So, (x,0) will represent the vertex star for all x.

Though CX may not have any linear structure as such, though it is not a vector space, but something nice happens, namely, it makes sense to talk about line segments through any point (x ,t) of CX and star. Passing through star, there are lines; what are they? Namely the image of little $\{x\} \times \mathbb{I}$.

So, that whole thing image of X × $\mathbb{I}$ is homeomorphic to $\mathbb{I}$ itself for each x; t going to (x ,t), t give you an embedding of the interval $\mathbb{I}$, inside $X \times \mathbb{I}$ and then to again into CX. So, from star, you can go to any other point of x, t by a line segment; all the points between star and (x,t) are there; you take s going to (x,st). When s is 0, it will be the star; when s is 1, it will be your point (x,t). Thus the entire cone is star-shaped in this sense-- the star as its apex. Recall that we have defined apex for any star-shaped set earlier. Here also we call it an apex.

We call it a vertex also;--- vertex of the cone. that terminology is also used. Consider the map (x,t), s going to x(st);-- that map here defines a homotopy of the constant map star. When s is 0, it is the constant map; and for s is 1, it is the identity map. So, in particular this shows that CX is contractible; of course you do not need this proof. Because once something is star-shaped, we know that it is contractible. But, that was in a vector space, so you better see what is the meaning of this here. Here too it is similar. Thus cones are always contractible irrespective of what the original base space is.

Suppose you take X as just two point space, a discrete space; what is the cone over X? It will be just union of two lines joined at one single point. Because, two points space cross $\mathbb{I}$ is the disjoint union of two lines; two copies of $\mathbb{I}$, but one at of the point namely at 0, both the points will be identified; so it will be again a line, so it is contractible. X here, which has two points is not even connected.

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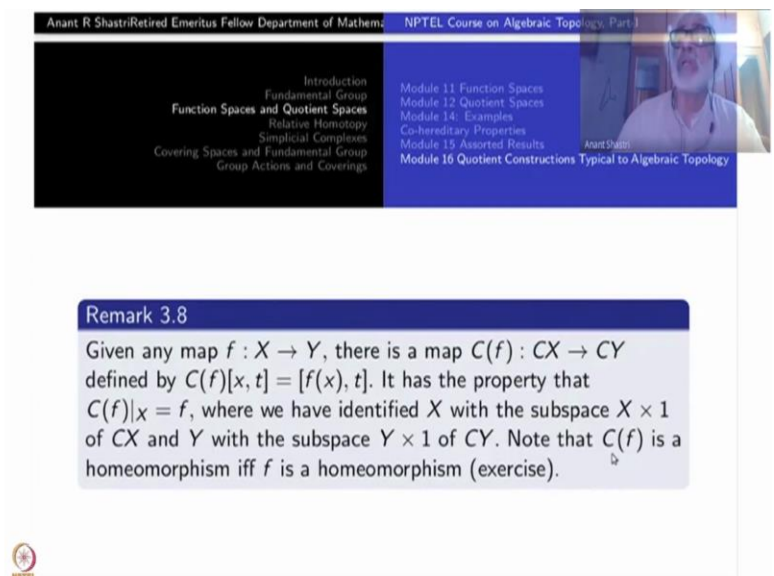

So, given any map f from X to Y, we can talk about the cone of f; namely the map corresponding to, the map induced from CX to CY. It is a natural way to get a map on $X \times \mathbb{I}$ to $Y \times \mathbb{I}$. Namely, f of (x,t) goes to (f x, t), i.e., f × identity. But, then you can pass down to the quotients; so that is C f: the class of (x,t) goes to the class of (f x, t). The second coordinate t is not affected; it has the property that if you restrict it to X × 1, then it can be identified with f; when you identify x with (x,1). You can ignore that one and then it is just x, to fx. That is f.

So, X to Y you have a map, CX and CY are larger spaces containing X and Y; then we can think of C f as an extension of f. Extension of f from the cone over X to cone over Y. So, this is what --- will have this picture; X is sitting inside CX and Y is sitting inside CY; as (X,1) here and (Y,1) here.

So, one of the nice property of this construction is that C f will be a homeomorphism, if and only if f is a homeomorphism; very easy to check. The more deeper thing here is that a cone construction actually is a functor. So, if f from X to X is the identity map; C f is the identity map. If X to Y, and Y to Z you have maps f and g, respectively, then g composite f will be defined, it will be from X to Z. Then C g composite C f is C (g) composite f; so that is the meaning of that cone construction is a functor. Just like our π_1, which was from topological spaces to group. It was a functor from topological spaces to the groups. Here, C is from space to space itself, topological spaces to topological spaces.

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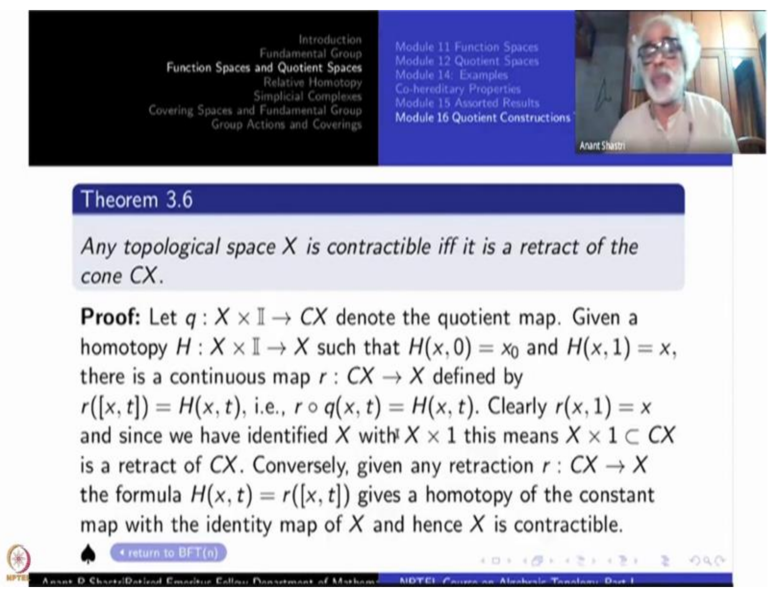

Now, the usefulness of the cone construction will come into picture in the study of homotopy theory. Any topological space X is contractible if and only if the space is retract of the cone; cone itself is contractible always. But, X is itself is contractible, if X is a retract of this one and conversely; so, this is the statement. Let us go through the proof of this. Take q from $X \times \mathbb{I}$ to CX the quotient map; $X \times 0$ being identified to a single point. So, this is just the notation of the quotient map. Suppose, now you have a homotopy H from $X \times \mathbb{I}$ to X, such it is the constant map at zeroth level; and identity at the level 1. So, between a constant map and the identity map, H is a homotopy.

Corresponding to this, there is a continuous map r from CX to X, defined by r(x, t) equal to H of (x,t). r of the class x, t =H (x, t). You see this should be independent of the class; so let us verify. If t is not 0, then x, t is a single point so it is H (x, t)no problem. If t is 0, all this is independent of x, one single point; but H (x, t) depends upon x. But, if t is 0 H is the constant map; therefore this is well defined. Not only it is well defined, automatically it is continuous; because H is continuous on $X \times \mathbb{I}$ and it goes down to CX. Remember CX is a quotient map, this is unique; q followed by this, i.e., r composite q is H.

So, that is why it is well defined. So it goes down to the quotient space CX. Also, this r on (x,1) namely on the subspace X it is identity; because H(x,1) is x. So, since we have identified X with $X \times 1$, this is what we are using here; that is how we can talk about X as retract of CX. This is the meaning of retract, you may recall it; namely, a function r from the whole space to a subspace which is identity on elements of the subspace, and r must be continuous. We have used the retracts earlier, we have shown that in the disc $\mathbb{D}^2$, the boundary namely $\mathbb{S}^1$ is not retract a of $\mathbb{D}^2$. We have shown that one in proving Brouwer's fixed point theorem.

What we have proved so far? If the constant map is homotopic to the identity map of X; namely, if X is contractible, then X is a retract of CX. Now, let us look at the converse. Suppose you have a retraction CX to X, then you just define H (x, t) by r; H x, t by same formula. Earlier I used H to define r; now this r is given here, I am defining H. Automatically, H has the property that when t is 0, it is a single point, and when t is 1-- means r (x,1) by definition is r (x) which is (x,1). So it is the identity map. So, H gives you a homotopy of the constant map with the identity map of X.

Just to remind you, I have to use the fact that CX is the quotient of $X \times \mathbb{I}$, where all of $X \times 0$ is identify a single point. And homotopy of the identity map to a single point has also this property; therefore everything works fine.

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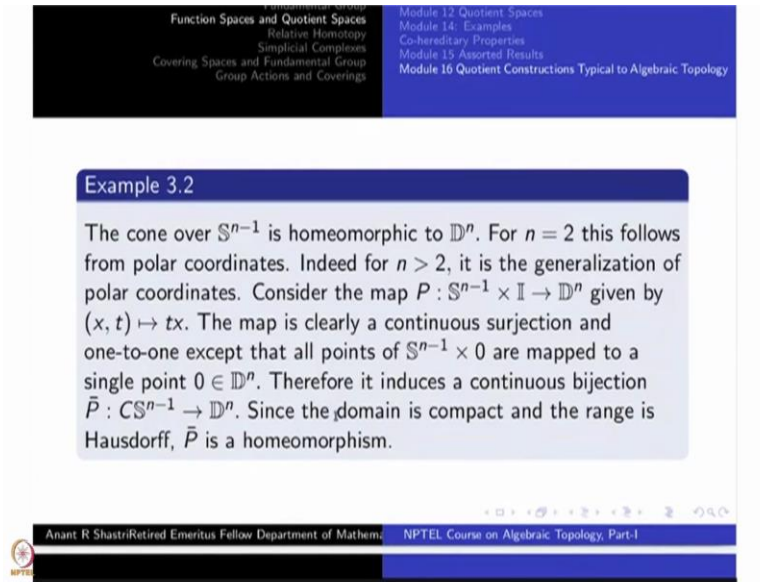

Now, here is an example. Let us take the circle, more generally take any sphere; you can even take $\mathbb{S}^0$ also. So, take the sphere $\mathbb{S}^{n-1}$. Take the cone over it. That will be homeomorphic to the disc of 1-dimension higher. So, start this with $\mathbb{S}^0$, a cone over that will be homeomorphic to the interval - 1 to + 1. If you take $\mathbb{S}^1$, this will be the disc $\mathbb{D}^2$ and so on. For n = 2, this is the familiar polar coordinates, polar coordinates of complex numbers, r times cos θ, sin θ. The cos θ, sin θ is a point of $\mathbb{S}^1$; r ranges over 0, 1; so the domain of polar coordinates is $\mathbb{S}^1 \times \mathbb{I}$ you can say.

But, when r is 0, it gives you one single point; so the entire $\mathbb{S}^1 \times 0$ is going to a single point; so, that is a quotient map. So, that is why the cone over $\mathbb{S}^1$ is $\mathbb{D}^2$; for $n \geq 2$, it is a generalized polar coordinate. When you have a vector, unique vector in $\mathbb{S}^{n-1}$; namely, in $\mathbb{R}^n$ a unit vector. The rest of the $\mathbb{R}^n$ can be thrown away. r times a unit vector. When r is 0, it will give me just a single point; but if r not equal to 0, there is one-oneness. So, what we do? $\mathbb{S}^{n-1} \times \mathbb{I}$ to $\mathbb{D}^n$; just write (x, t) going to t times x. This map is clearly continuous surjection, and one-to-one except for points in $\mathbb{S}^{n-1} \times 0$; the entire thing is mapped to a single point.

Therefore, this map p factors down to the cone over $\mathbb{S}^{n-1}$ to $\mathbb{D}^n$; this is a quotient map. So, it gives you a map p bar on this quotient space $C\mathbb{S}^{n-1}$; so p bar is the unique map defined by p. So, p bar q composite p bar is p; since the domain is compact, that is C of ($\mathbb{S}^{n-1}$)is compact, for $\mathbb{S}^{n-1}$ is compact; product with $\mathbb{I}$ is compact; therefore quotient is compact. So, the domain is compact.

And this is Hausdorff. We have a continuous bijection; so it is a homeomorphism. This argument will keep occuring all the time.

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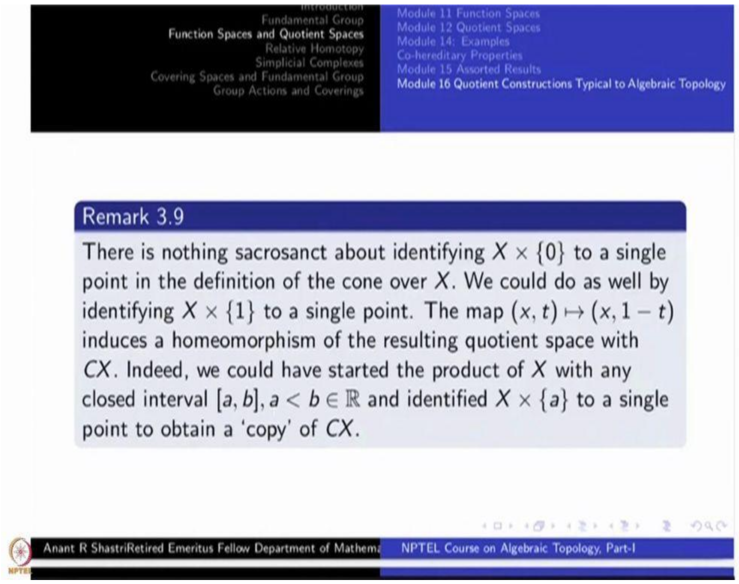

So, we identified X × 0 to a single point. So we got an ice-cream cone like that. Instead, you could identify to X × 1 also to a point; then you would have got a tent; both of them are cones. Homeomorphically. all that you have to do is change (x,t) to (x,1) - t; then you get the other one, so there is no problem. In fact, this can be done on any closed interval also, I have told you. But we have just standardized these things so that again and again we do not have to be warned---changing coordinates and all that.

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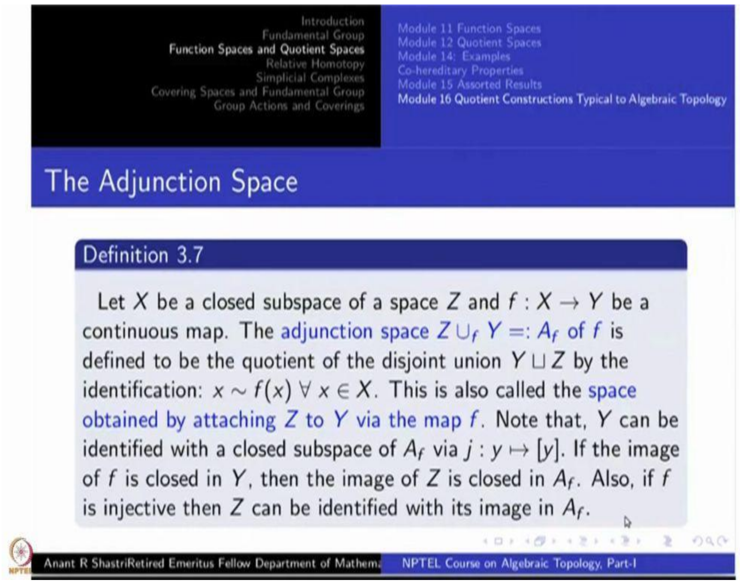

Now, let me tell you about another important construction namely, adjunction space. So, pay attention to the definitions, so you should not have any confusion in the definition. You start with a space X and a closed subspace Z; Z is a closed subspace of X. Sorry the other way round . Let be X is a closed subspace of Z; let us do stick to the notation in the slide that I have already fixed and let us not change it here. Let X be a closed subspace of Z, and take a continuous function from X to Y, where Y is another space. On the subspace you have a continuous function; on this closed subspace X, you have a continuous function.

Now, the adjunction space is defined as the quotient of the disjoint union of Z and Y modulo some relation. So, it is a quotient of Z disjoint union Y modulo some relation; and I am going to denote it by A_f or Z union over f Y. So, these are the notations for the final quotient space. You start with the disjoint union of Y and Z, then you have to make an identification. What is the relation? Every point x in X will be identify with f x in Y for every x in X; so this is the identification. No other identification there. If z is a point not in X then it is not identified with any other point. Each such z= z itself. That is all. Similarly, each y in Y is equivalent to y itself. There is no identification inside Y. A point y in Y and the point x in X, they are identified, when? y must be equal to f(x); then only those two points are identified. This space is also called the space obtained by attaching the whole space Z to the space Y via the map f. This is the description of attaching. This one will come very much into i operation, maybe, a later, in algebraic topology, when you study cell complexes and so on.

So, Y can be identified with a closed subspace of A_f; Here A_f is the adjunction space, each singleton y being its class. There is no identification; each Y itself is a class there.

So, that inclusion map from the disjoint union to the quotient space; that will be a closed subspace. Why? Because inverse image is just Y in the disjoint union Y and Z; that is closed here. In the disjoint union both Y and Z are closed subspaces. Only thing is we will not get a copy of Z in A_f . Because there is some identifications inside X. Suppose, f x_1 = f x_2; then x_1 and x_2 will be identified with the f x_1 = f x_2. So, it depends upon whether f is injective or not; so, Z may not be a subspace of A_f, but Y is subspace.

If the image of f is closed in Y,--- this is an extra hypothesis-- then Z will be closed in A_f, sorry I mean the image of Z will be closed in A_f. Also, if f is injective, then Z can be identified with its image in A_f. If f is injective, then there will not be any identification within points of Z. Then z going to its class, it is a single point; that each class a single point. So, then Z can be identified with its image, that is how it will be also a subspace.

The adjunction space is such a wide, generic definition, it gives you a lot of examples. In some special cases they have special names now.

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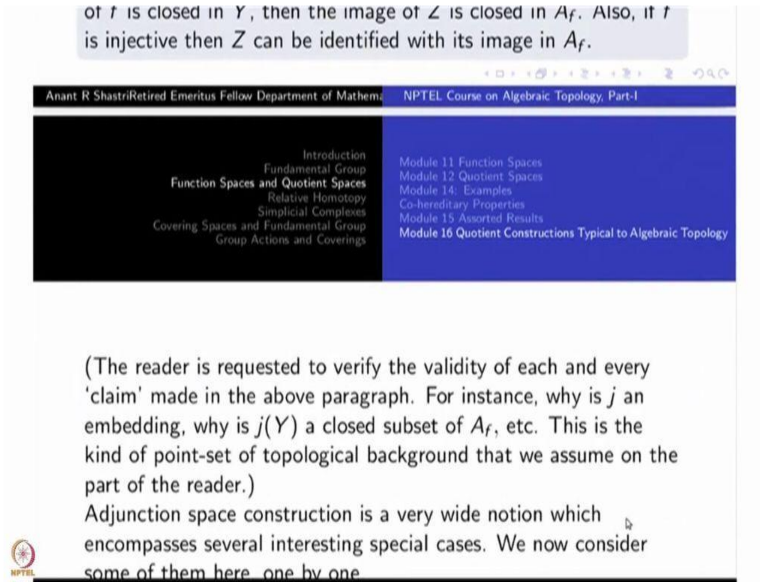

Now, I have made several claims here, which are all topological claims; which are all easy. But, unless you verify them each, step by step, you will not get the full picture of these new concepts. So, you have to spend that much of time. So you are is requested to verify the validity of each and every claim made in the above paragraph. We have to see every thing, because this is that blah blah blah. So you have to verify. For instance, why is j an embedding? Why Y is closed subspace of A_f etc., please make sure that you have verified them yourself.

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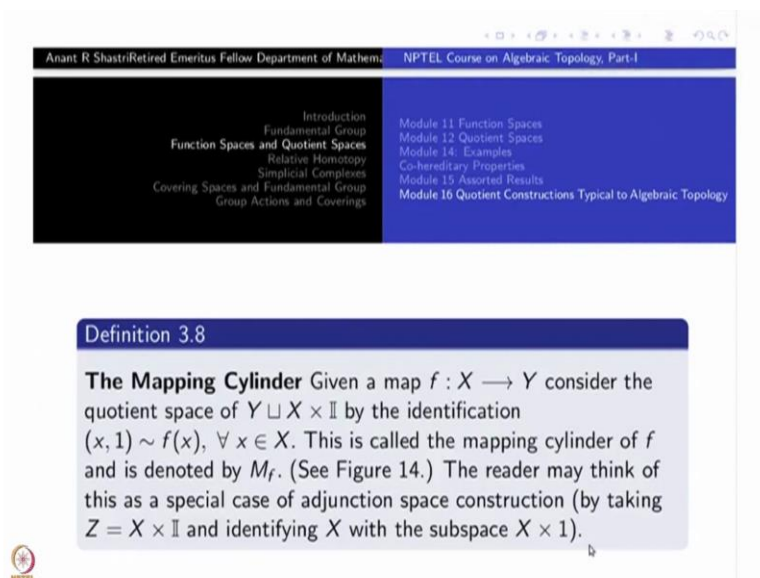

So, we will take some special cases now. But, mapping cylinder, remember just now we have defined cylinder; now I am going to define a mapping cylinder. f from X to Y is a map, it is a just continuous function; then mapping cylinder is got all follows: First of all, on X, I have a cylinder X × 𝕀; then I take the disjoint union with Y, just like I took Z cross disjoint union with Y, in the adjunction space construction. Here it is Y disjoint union X × 𝕀. Then make the identification, namely, (x ,1) with f(x), for every x in X.

So, this is a special case of the adjunction space namely Z is X × 𝕀; X × 𝕀 here and X is X × 1 On X × 1 think of f being defined there. If you put Z = X ×𝕀 and X= X × 1 , in the earlier example adjunction space, you get the mapping cylinder of f; so this a special case. This is called the mapping cylinder of f, and is denoted by M_f. The reader may think of this as a special case of the adjunction space construction; that is what I have told. So, here is a picture of that. Let me show you a picture of mapping cylinder.

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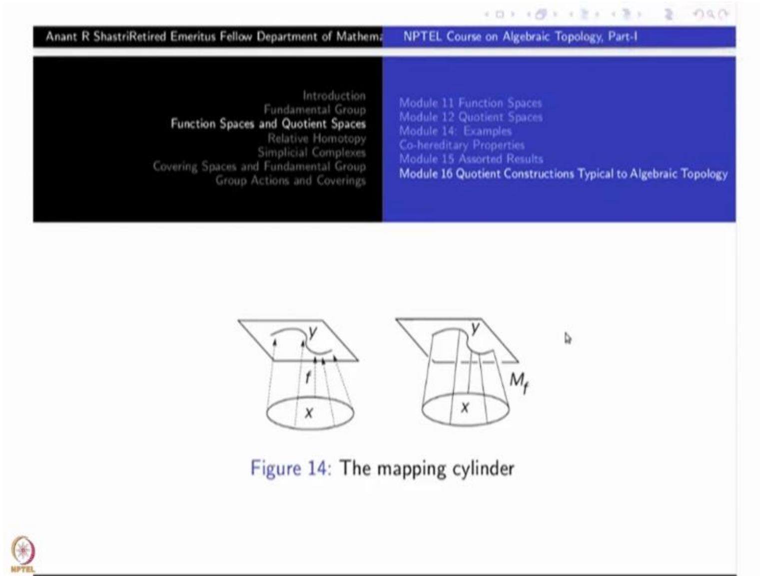

So, X × 𝕀. To begin with I a map X to Y. In this picture I am showing X as a disc, and Y as just a curve here; sorry that is the image of f. So, image of f is a curve in Y which is a rectangle. Y is a rectangle, X is a disc; but the image is only just a curve here. For the mapping cylinder, I have taken X × 𝕀, but the point x cross 1 do not remain as it is x × 1. Because each point x × 1 has to be identified with its image under f. So, this point goes to this point, then x × 1 will be identified with this point and so on.

So, this is the picture for mapping cylinder M_f; M for mapping cylinder of f. So, there are various inclusions and so on; because of the importance of this construction, let us pay some attention to this one.

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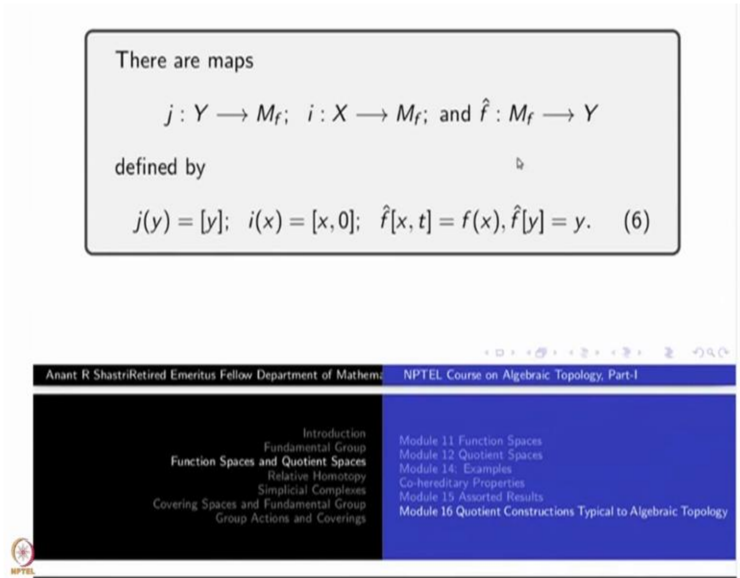

Just like in the case of adjunction space, there is a map j from Y to M_f; this is always the inclusion because the space Y is not disturbed. There is a also an inclusion map i from X to M_f; this is at X × 0, or any other X × t not equal to n $_1$; On X × 1, there are identifications. So, this inclusion is X × 0; so i of x goes to x × 0. And then there the map f hat from M_f to Y; f itself is extended; f hat (x, t) goes to f x. If you take f hat (x,0), it is just fx , each (x,1) will get identified with f x for all x. f hat of y I have defined, it is just y. When y is fx for some x, it is f hat (x $_1$ will be y of course. So, this map f hat is continuous. That is what we have to verify.

I have defined, on X cross I, I have defined f hat like this; and on Y, it is defined this way. Whenever, we make identification, these two coincide. Because f hat of x comma 1 is fx and we identify it with y=f x . So, f hat is an extension of f, for X is contained in M_f as X cross 0.

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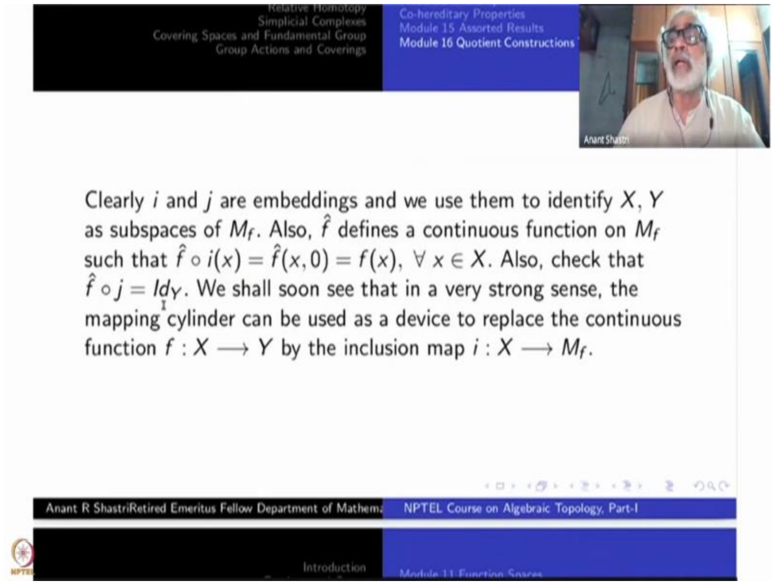

So, here are the claims, i and j are embedding; i is an embedding of X and j is embedding of Y. We use them to identify X and Y as subspaces of M_f. To begin with X and Y are arbitrary spaces; there is a map between them. Now, M_f will include both domain and co-domain of f as subspace. It brings them together, this is the whole idea. Now, $\hat{f}$ which defines a continuous function on M_f, such that $\hat{f}$ composed i is f; $\hat{f}$ (x,0) is equal f (x) for every x. Check that f composite j is identity of Y; so that is how we define it here.

We shall soon see that in a very strong sense, the mapping cylinder can be used as a device to replace the continuous function f from X to Y by the inclusion map from X to M_f. What is the meaning of this replacement? We will let you know later. The mapping cylinder, is a special case of the adjunction space. So, there are some more things we shall consider them next time. Thank you.