Introduction to Algebraic Topology (Part 1)
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Lecture No. 15
Assorted Results on Quotient Spaces

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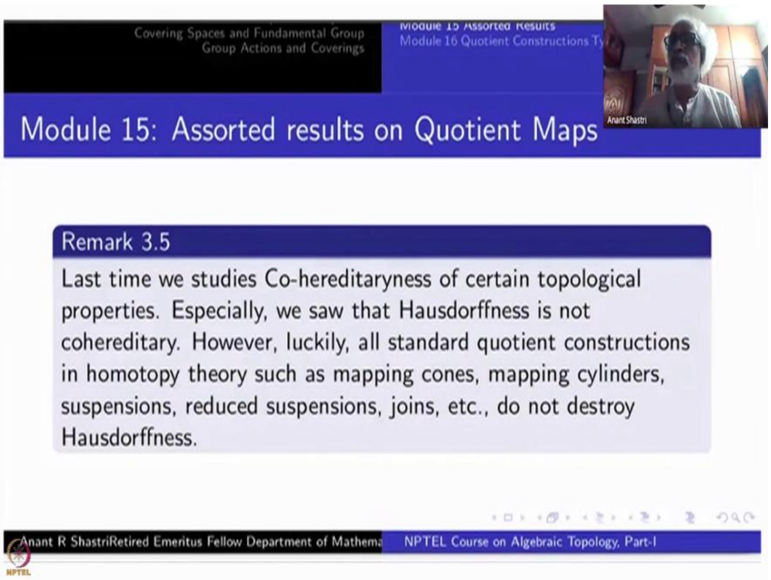

Welcome to Module 15. Last time we studied co-hereditaryness of certain properties, topological properties. Especially, our concern was about Hausdorffness which is a part and part and parcel of the assumptions in algebraic topology and we saw that Hausdorffness is not co-hereditary and it is not all that easy hypothesis all we need to have so that the quotient is Hausdorff, therefore quite often, case by case, you will have to check whether something is Hausdorff or not.

Luckily, what happens is the kind of quotient constructions we do in algebraic topology, such as cones, mapping cylinders and so on, as we will see later, they all have Hausdorffness going into the quotient spaces.

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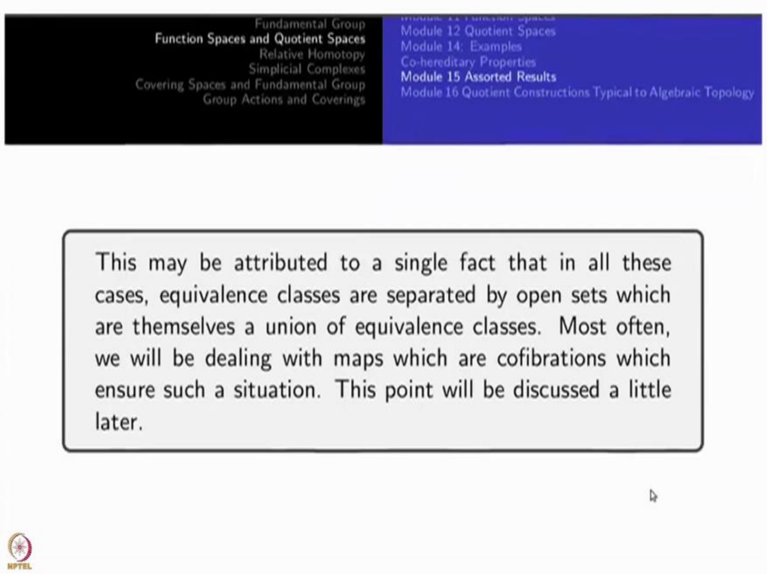

The one single fact which can be attributed to this phenomenon is that equivalence classes are separated by open sets, which are themselves union or equivalence classes. Most often we will be dealing with maps which are co-fibrations, which will ensure the situation. So, when the topic comes up, we will discuss this point in a little more detail.

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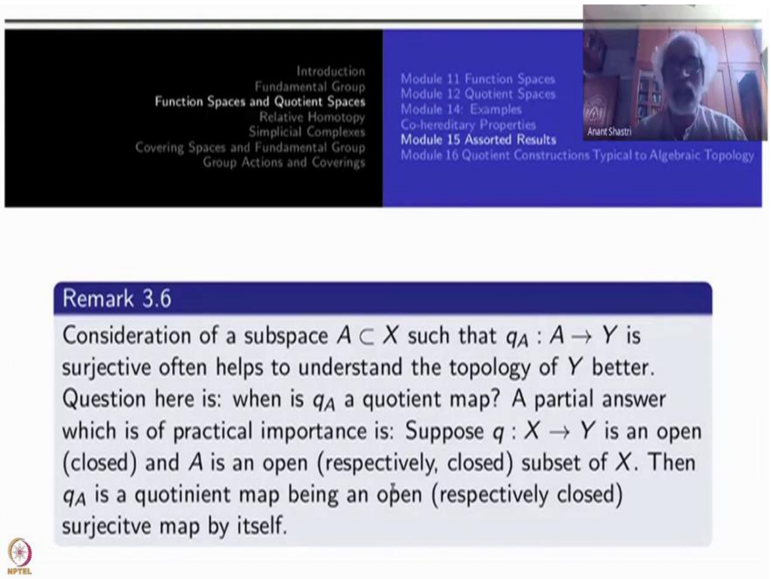

So, let us start a few assorted things about quotient spaces like we observed that if you restrict the quotient to some smaller subspace in the domain in the mother space, such that the map is still surjection and if that turns out to be a quotient map again, then that will be of quite help for us. But you may ask when such a thing is possible.

Once again there is no easy criteria for that, we can say if this one, if that one, if that one and so on. So, you have to verify it case by case. For example of practical importance, suppose you have an open quotient map or a closed quotient map, then restricting it to an open sub space or a close subspace, (respectively), it will continue to be an open map or close map, whichever the case may be. So, all that you have to do ensure is that the subspace is large enough namely suppose we have taken open subset of X, restricted to that open set the function must be surjective, that is it. If you would like to know why this is good, it is very easy. Because if q is an open map and A is an open subset, then q restricted to U will also be an open map. Similarly, if A is close set and q is closed map, the restricted map will be also closed map. That is all. Open surjections or closed surjections are automatically quotient maps.

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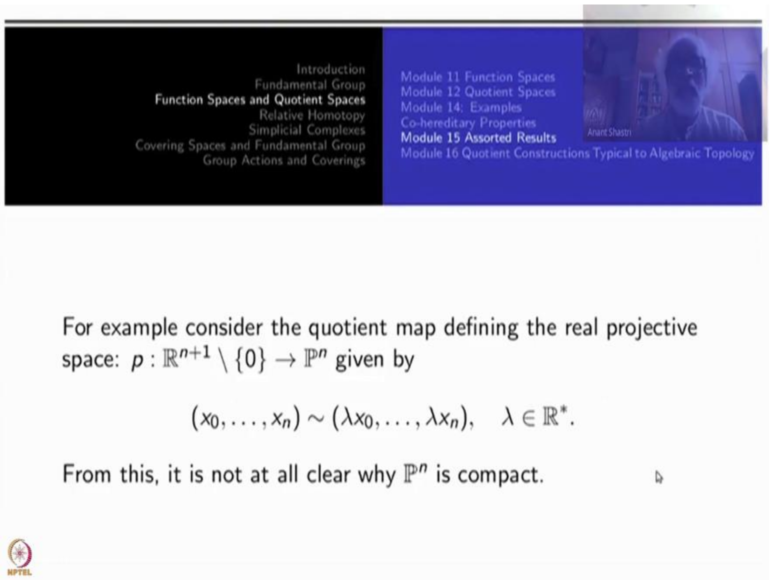

We have already noted this one,-- I will now recall that, namely, if you take the projective space, which is defined as a quotient of $\mathbb{R}^{n+1} \setminus \{0\}$; you can restrict it to just the unit sphere there. The original map p from $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is actually both open as well as a closed map and the sphere is a closed subset. So, a closed subset of the sphere is closed in the whole space, that is our idea. So, restricted to $\mathbb{S}^n$ also, p will be a quotient map.

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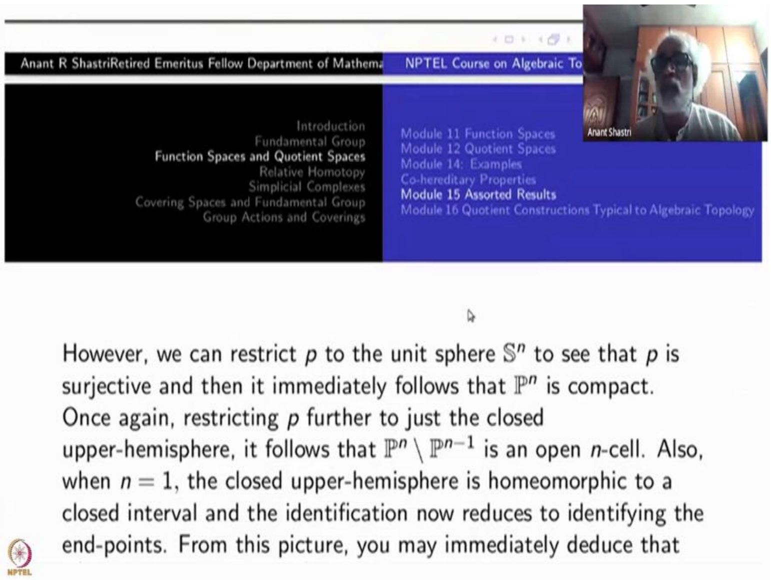

Why this is important? The first thing is that you immediately know that being a quotient of $\mathbb{S}^n$, $\mathbb{P}^n$ is compact. You can take further restriction namely, only on the upper hemisphere. That will give you a better structure of the projective space itself. If you take the upper hemisphere, elements in the lower hemisphere are all represented by elements in upper hemisphere. Therefore, the identifications are only on the equator.

But the equator is one dimension lower sphere and the identification is again antipodal. Therefore, what you get from the equator is Pn minus 1. The rest of $\mathbb{P}^n$ is one open cell, which is the upper hemisphere, strictly upper hemisphere, actually remains as it is. It is attached to $\mathbb{P}^{n-1}$. So, that is an open cell because there, the map is injective. So, this description will be very very helpful in understanding the projective space inductively. In the case when n is equal to 1, it already tells you that $\mathbb{P}^1$ is nothing but $\mathbb{S}^1$ again.

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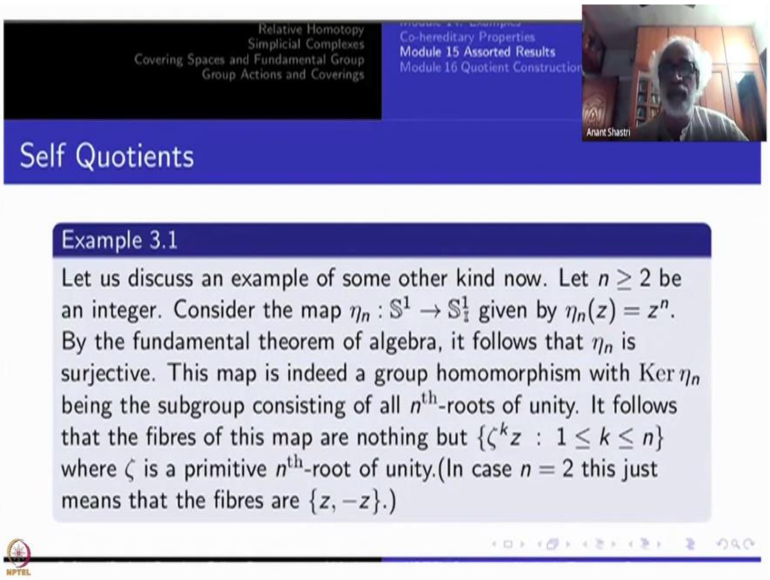

Now, let us consider a slightly different kind of quotients. They could be quite weird that is what I wanted to tell you, namely, there are spaces which are self-quotients. Here is a simple example, but there are many such examples, cannot go on discussing all the examples. For n greater than or equal to 2, look at the map from $\mathbb{S}^1 \to \mathbb{S}^1$, namely, $\eta_n(z) = z^n$. If n is 1, this is the identity map. It is not of much interest. But if it is a square, or a cube etcetera, n greater than 1, then you know that the kernel of this map is precisely the nth roots of unity.

So, it follows that the fibres over w are nothing but nth roots of unity multiplied by some single element, translated by an nth root of w. This will be all the fibres over w. If you take ζ such that $\zeta^n = 1$, i.e., ζ is an nth root of unity, then $(\zeta z)^n = z^n = w$, i So, all of them go to the same point z raise to n. But now, what I need is what fundamental theorem of algebra says, all these maps are surjective.

So, being a map from a compact space to a Hausdorff space they will be automatically quotient maps-- η_n is automatically a quotient map. Only thing is it is not injective. So, in an infinitely many different ways, $\mathbb{S}^1$ is a quotient of itself.

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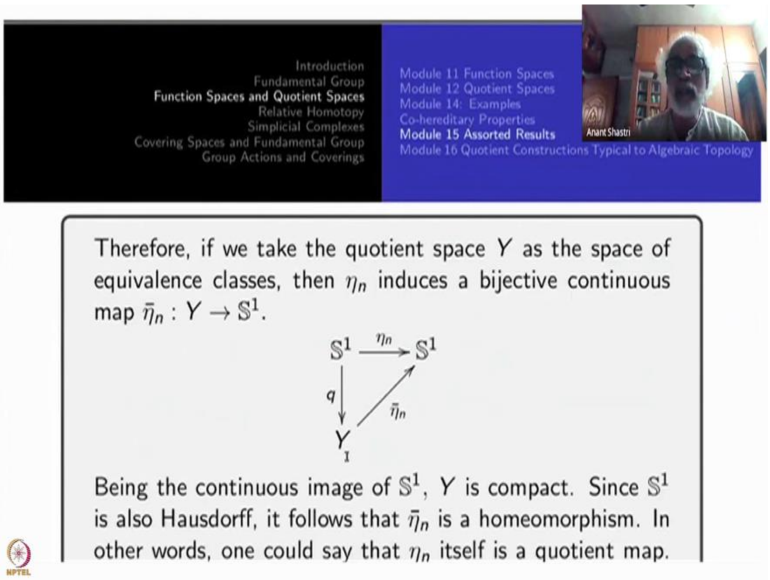

This is a picture that we have, $\mathbb{S}^1$ to $\mathbb{S}^1$ to take the quotient by action of say nth roots of unity here, so this the action of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{S}^1$. So, again what do you get is the orbit space is $\mathbb{S}^1$ homeomorphic to $\mathbb{S}^1$.

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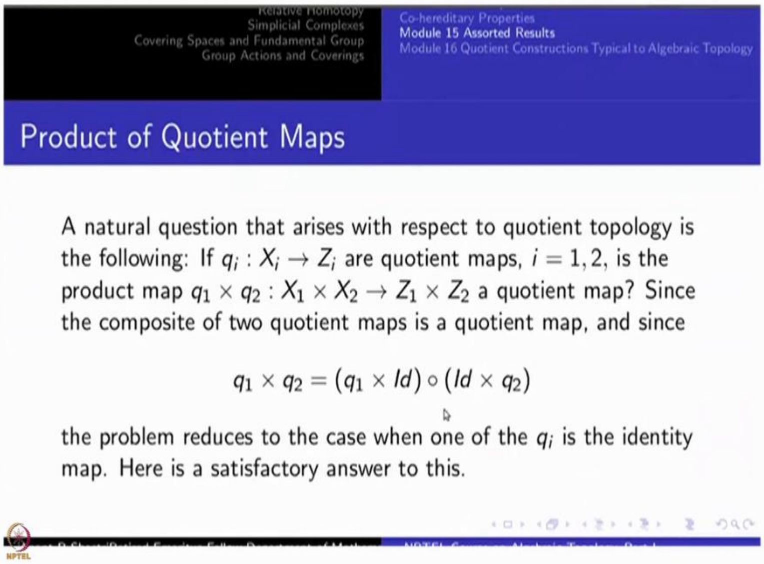

Now, let us come to another important question namely, if you take two quotient maps, will the product be a quotient map? It looks such a nice things to have. But this is something which is not true in general. You should not be surprised because product is quite misbehaved with many many topological properties. The cartesian product that we take in point- set -topology as well as in algebraic topology has this problem.

So, you have to be very careful in extending results to products spaces. The product of two maps say $q_1 : X_1 \to Z_1, q_2 : X_2 \to Z_2,$ they are quotient maps, look at their product $q_1 \times q_2.$ Is this a quotient map? The general answer is no. However we will not leave it like that. So, we want to understand why this is so and what best we can do.

We can decompose $q_1 \times q_2$ as q1 cross identity composed with identity cross q2 --this identity, first one is the identity of X2, other one is the identity of Z1, identity maps of different spaces. The problem reduces to--- you know that the composite of two quotient maps is quotient map, therefore, we want to have a positive answer in the special case, namely, when either of them, (condition is symmetric), say q1 is a quotient map and second one is identity map. If I can show that this product is a quotient map, then by symmetry, this will also be a quotient map. So, the composite too will be a quotient map. So, we ask when is q cross identity is a quotient map? So, here is a satisfactory answer.

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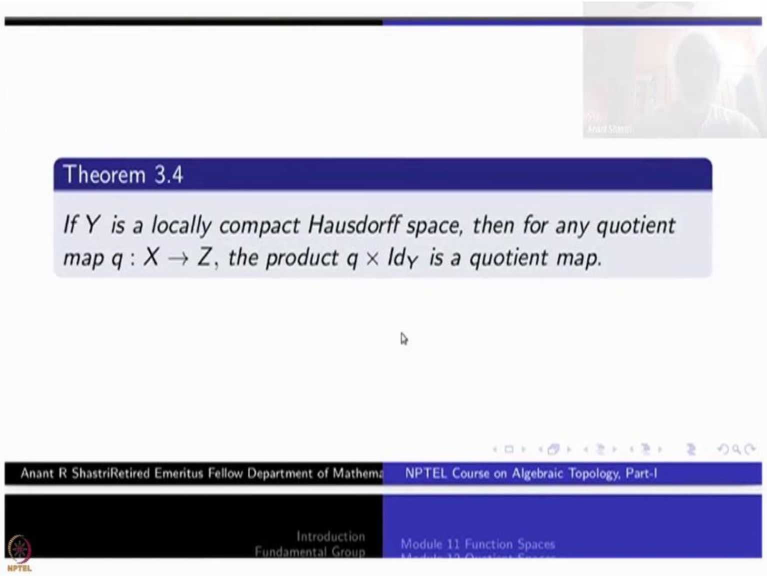

Suppose Y is locally compact Hausdorff and q X to Z is any quotient map. What you are doing is taking product with a locally compact Hausdorff space, then q cross identity of Y is a quotient map. So, the new creature comes here, q could be any quotient map. This one should be locally compact, then no matter what q is, q is some quotient map implies q cross identities is also quotient map. This itselt is not so difficult.

The point is why put such locally compact Hausdorffness condition? To make the problem easier? The stranger thing is nothing else will work, as soon as Y is not locally compact Hausdorff space, there will be some quotient here, such that the product is not a quotient, product with identity is not a quotient. So, that is the beauty of this hypothesis. So, in some sense it is a full answer also.

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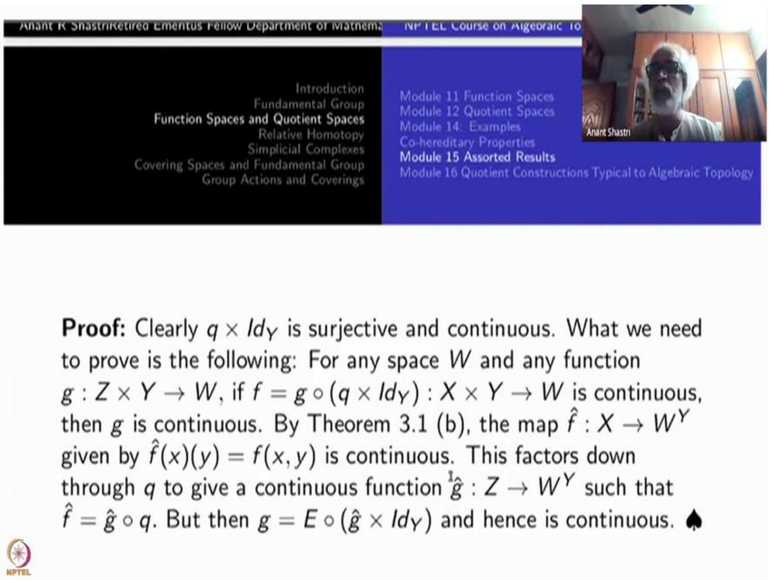

Let us go through this one. Now you see the function space theory that we have studied comes to help, because we have locally compact Hausdorff space. q cross identity, because q is a surjective map, is also surjective and continuous, there is no problem about that, what we need to prove is the following: The universal property,-- we have to prove that for any space W and any function Z cross Y to W, if g composite q cross identity, say f equals to this one, is continuous, then g must be continuous.

So, this is the property that says q cross Id is a quotient map. Any quotient map has this property: any map from the quotient space to any other space, is continuous if and only if composing with the quotient map it should be continuous. Here we are talking about q cross identity, q cross identity will be quotient map, if it has this property. You see the universal property of the quotient map will define the quotient map: if this is true for all g, as soon as f the composite is continuous, g is continuous.

Now go back to the exponential correspondence that we have established earlier. Saying that a function f like this is continuous is the same thing as $\hat{f} : X \to W^Y$, X, you get a map hat f that is continuous. Namely, the map f hat from X to W raise to Y, given by $\hat{f}(x)(y) = f(x, y)$.

Remember f is a function from X cross Y to W. So, if this is continuous, then f hat will be continuous and conversely. So, we have passed the problem to this one. And here we have used the fact that Y is locally compact.

Now, this map f hat factors down through q to give a continuous function $\hat{g}: Z \to W^Y$ because q is a quotient map. And in the beginning, f itself is g composite q, for that reason, it factors down to a map $\hat{g}: Z \to W^Y$ i.e., such that $\hat{f} = \hat{g} \circ q$. So that is the meaning of `factors down'. But then $g = E \circ (\hat{g} \times Id_Y)$, therefore, if this g hat is continuous, g hat cross Id is continuous, composite with E is continuous. That is g is continuous. So, that is what we wanted to prove.

So, you pass to the exponential via exponential correspondence, use the function space argument. So, this becomes such an easy thing. Alternatively, you can try directly to write down the proof. But then you will have to repeat the proofs of the exponential correspondence etcetera, repeat it out more or less exactly the same way you have to worked there. Instead of doing that exponential theorem is a readymade result for you.

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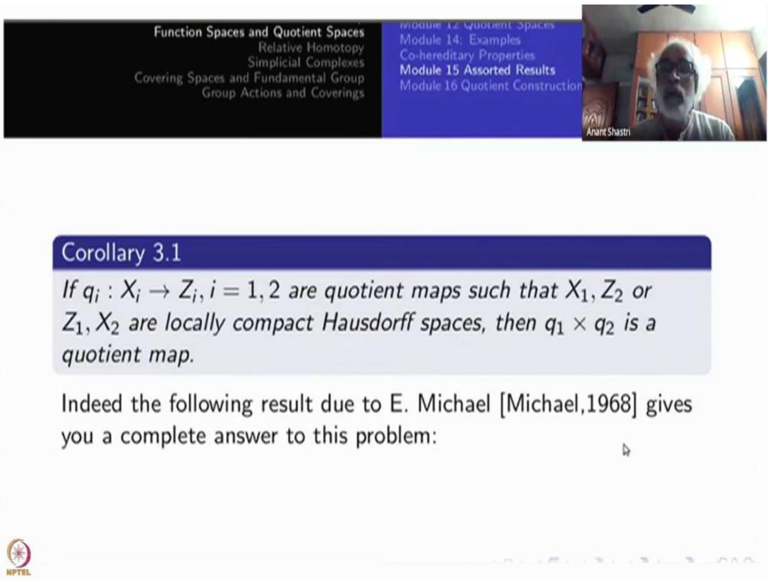

So, once we have this, we have observed a corollary. That is, suppose you have two quotient maps qi from Xi to Zi, such that X1 and Z2 are locally compact or Z1 and X2 are locally compact as the case may be. Depending upon that q1 cross q2 can be written as q1 cross identity composite identity cross q2 or the other way around. Therefore, it will be a quotient map. Each time, you have to have both the cases, you have to have that the identity factor that you are taking must be locally compact. Quotient map could be anything, crossing with a space, then you are taking identity, that space must be locally compact.

Now, I repeat this one, namely, though I started saying that it is a partial answer, this is a full answer. There is no other way, namely, this result is due to Micheal which is very recent.

actually it appeared 1968. In topology, by the way, 1960s it is quite recent, I would think. Either everything, by 60s is completely proved or left as too hard topology-- all major problems were solved and the rest of them were very hard, this is what happened in around 60s.

So, Michael gives you a complete answer to this problem. Namely, I have stated it here. It may not be exactly as it is there. Because that is a research paper –that may contain many other things, as you may expect.

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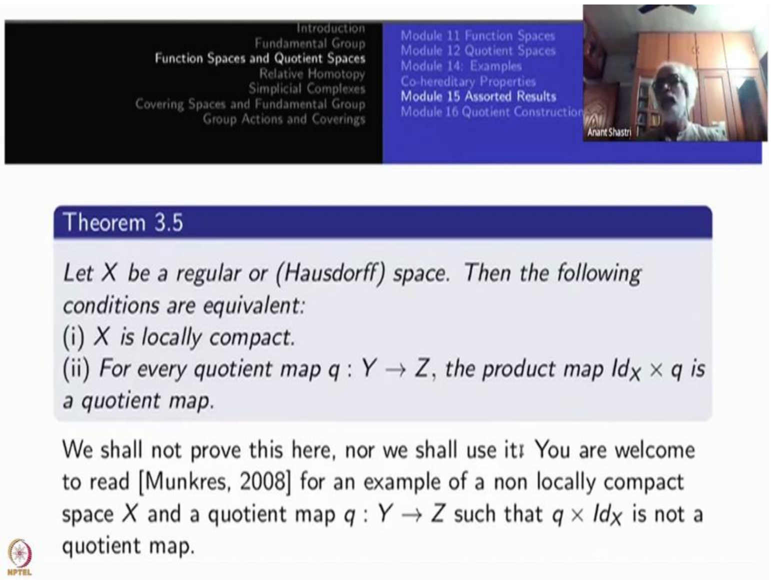

So, what is it? That X be a regular or a Hausdorff space. Then the following two conditions are equivalent, namely, (i) X is locally compact (ii) For every quotient map q Y to Z, the product with Id_X (Y and X are interchanged changed here), identity cross q is a quotient map.

 Whatever identity you take, this X must locally compact. Then this is true. Conversely, if this is true for all q, then X must be locally compact. That is the theorem of Micheal.

So, I declare that I am not going to give the details here. We will not prove it here. Nor we have any use for this theorem. As such   one way is all that we need, but other way will not be needed. You are welcome to read in Munkres book, an example of a non-locally compact space, and a quotient map q from Y to Z such that the product is not a quotient map. But if you know this result, then reading Munkres example is redundant.

The proof of this is interesting in his some sense. Namely for each non-locally compact space X, canonically, it cooks up a quotient map, the quotient map is such that product with identity of X will not be a quotient map. It is not just one example. For each locally, non-locally compact space there is one, there is a nice canonical example. So, I would like to now discuss a few exercises here, I am not going to give you the solutions as such.

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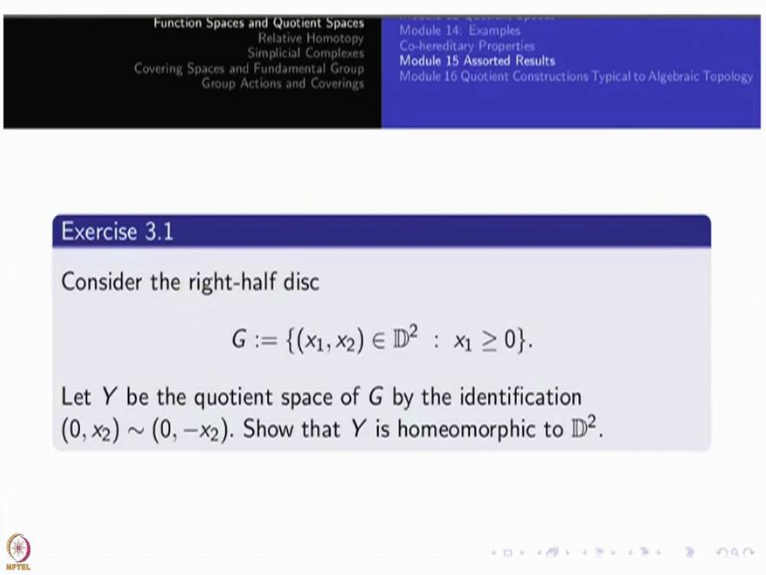

But let me go through some of this, one by one, they are illustrations of quotient spaces and they will be helpful in understanding various constructions in algebraic topology. We start with the unit disk in $\mathbb{R}^2$, this is a closed unit disk in $\mathbb{R}^2$, all (x_1, x_2) such that $x_1^2 + x_2^2 \leq 1$. Take only the subspace in which the first coordinate is greater than or equal to 0. So, it is the half the disk.

Now make the identification namely, $(0, x_2) \sim (0, -x_2)$, 0 comma x2 will be what x coordinate is 0 only y coordinate. There, x2 is identified with minus x2, only on the this line x_1 equal to 0, the second coordinate x2 (or y whatever you take), x2 is identified minus x2. If this is the case you have to show that the quotient spaces again homeomorphic to the closed disc. The proof is not hard. So, this is the beginning of the kind of things I want to look at.

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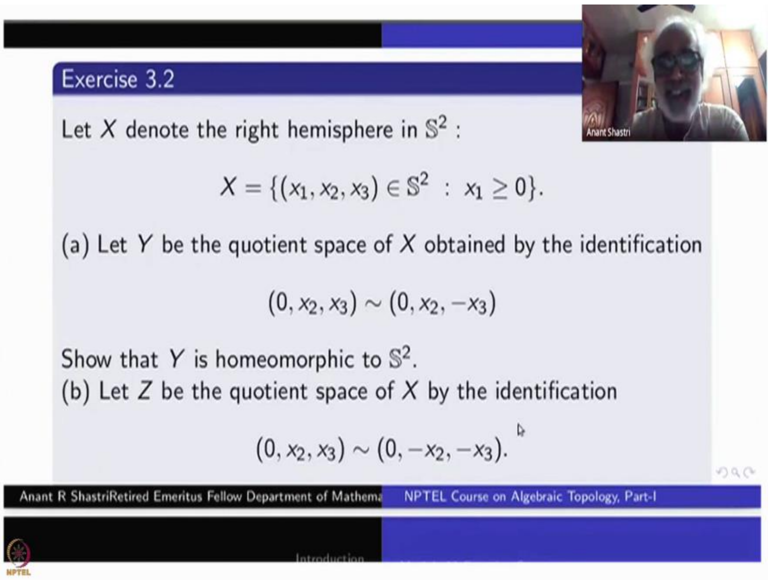

Next one is: take the right hemisphere this time, instead of this disk. I am taking the sphere, then again I am taking only half of it, namely, all points with the first coordinate is greater than equal to 0. So, it is a cup like this. Now it is in three dimension, but the sphere itself is two dimensional, this is like a cup here. On this I want to make two different operations here.

The first one is the quotient space obtained by the identification similar to the first exercise. This time I will not touch the x2 axis, x2 coordinate, 0 comma x2 is left as it is. But x3 corresponding x3 will be identified to minus x3. So, this is the boundary which is actually circle, on the circle I am identifying x3 with minus x3. So, note that this is not the anti-podal action. You have to be careful. Show that the quotient is homeomorphic to the sphere now, full S2. This was a half sphere, but the quotient is a full sphere, up to a homeomorphism. You do not see the quotient space in R3. They are not embedded subspaces.

In the second example, different operation, same space. The quotient space is obtained by taking anti-podal action. So, anti-podal action gives the projective space $\mathbb{P}^2$ which what we have already discussed, while discussing the projectional space. So, I have already shown you the solution here why it is the projective sapce. So, I think this slide is cut off a little bit, but that is all here namely, the quotient space you have to show is $\mathbb{P}^2$. (That line here is cut off, I cannot help it now.)

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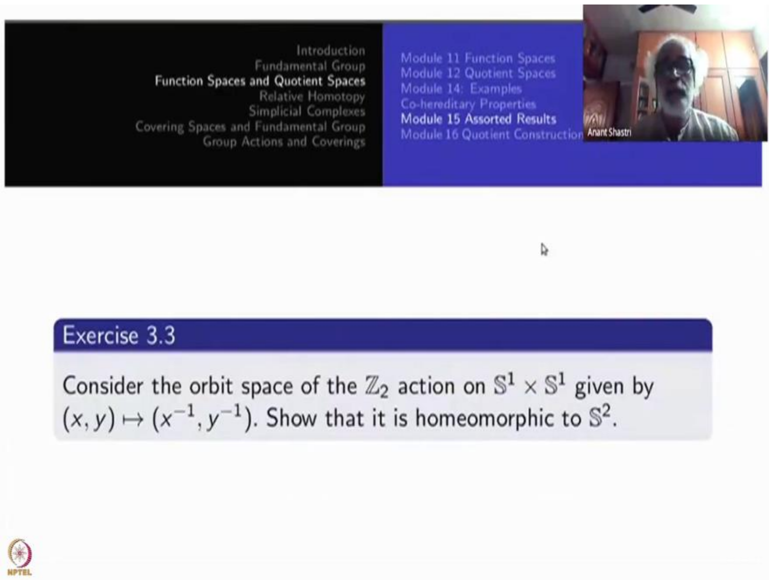

The third example is slightly more complicated, but if you have worked out the first two, they will let you know how things have to be worked out here. So, there is a hint for exercise 3, what is this? on $\mathbb{S}^1 \times \mathbb{S}^1$ you take x going to x inverse diagonal action. So, x y going to x inverse y inverse. Going to means what? this is the action of $\mathbb{Z}/2\mathbb{Z}$. So go down modulo this action. That means you have to identify x comma y for each x y, with corresponding x inverse comma y inverse. The quotient is homeomorphic to $\mathbb{S}^2$, that is what we have to show. So, the 2- sphere can be thought of as a quotient of $\mathbb{S}^1 \times \mathbb{S}^1$.

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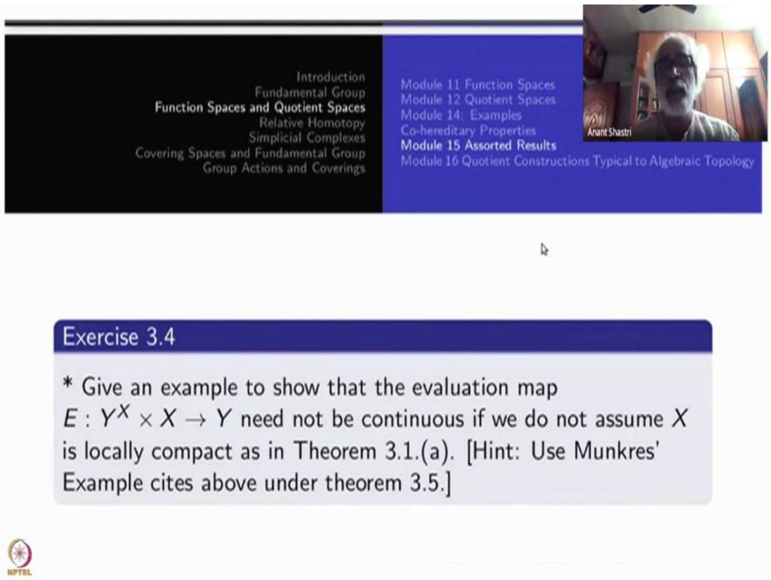

This is the fourth exercise, this is only for people who are advanced, quite sufficiently advanced with point set topology, they will go through Munkres' paper or maybe Michael's paper and so on. So, then they will be able to prove this, solve this problem. So, let us stop here. Thank you.