Introduction to Algebraic Topology (Part 1)
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Lecture No. 14
Examples of Group Actions

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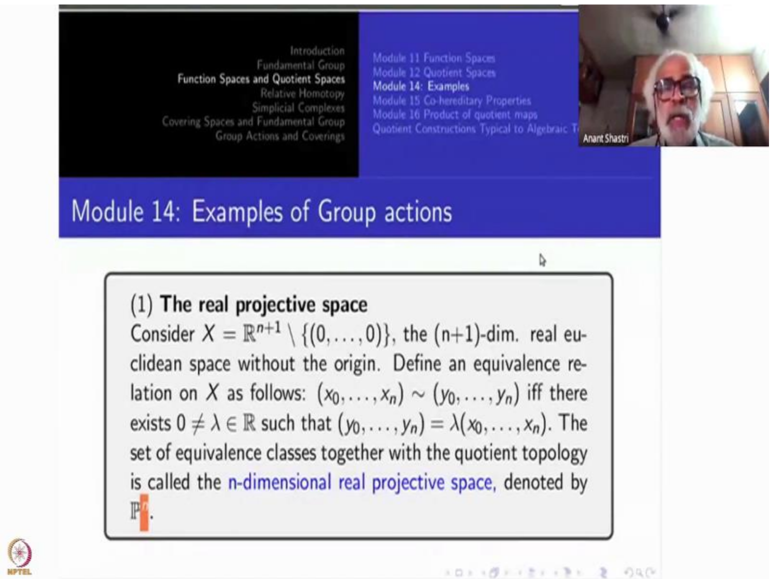

Welcome to module 14. Last time, we introduced the notion of Group Actions. Now, we will study a number of group actions and the quotient spaces arising out of that the orbit spaces. The first example is $\mathbb{R}^{n+1}$ minus the origin, n dimensional real Euclidean space minus its origin. What is the action? What is the group? Group is a non-zero real number, non-zero real numbers will act on the Euclidean spaces by multiplication, scalar multiplication.

You throw away 0, then every non-zero vector will go into non-zero vector. You the zero vector, capital 0 also no problem, but under the action of left all the elements would have fixed 0, 0 would become a singleton orbit. So, that is not an interesting element, you are throwing it away. That is all. Now, what will be the orbit of a vector $(x_0, x_1, \ldots, x_n)$ which is non-zero? It is equivalent to $(y_0, y_1, \ldots, y_{n-1}, y_n)$ if one is the scalar multiple of the other.

So, under these equivalence classes whatever you get, the collection of all equivalence classes is called n dimensional real projective space. You might have studied elsewhere that this space actually parameterizes all lines passing through the origin, each line is given by a non-zero vector, namely, whatever is spanned by the non-zero vector. Any two non-zero vectors lying on the same line are equivalent in this sense, namely, one is a multiple of the other, because they are dependent on each other. That is the real projective space with the quotient topology coming from $\mathbb{R}^{n+1} \setminus 0$.

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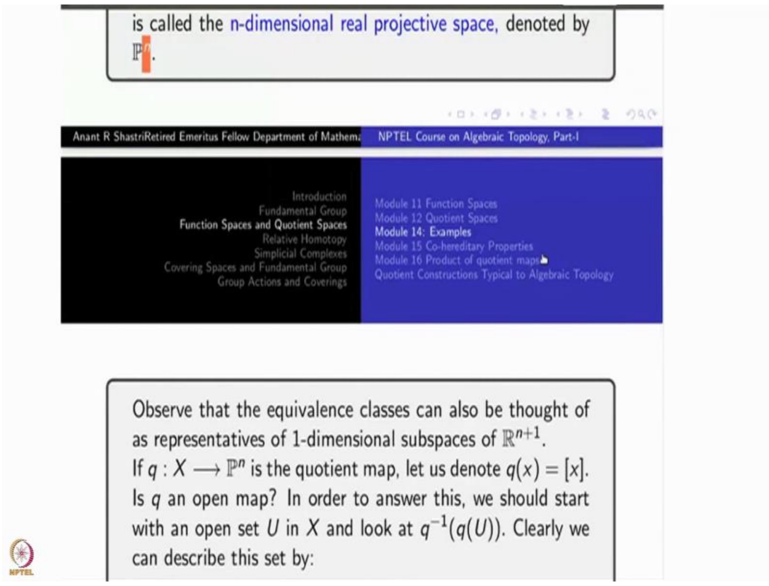

So, write q from X to $\mathbb{P}^n$ as the quotient map or you can write q of x as bracket x to denote the class of x. My first question is, is q an open mapping? So, how to answer this--- q is open map or not? Start with an open set U, q(U) must be open. But q(U) is inside the projective space, the topology there is the quotient topology. That means what? q^{-1}(q U) must be open.

Start with an open set U, q^{-1}(q U) must be open. What does that mean? All λ times v where v is a vector in U and λ varies over all the non-zero scalars that must be open which is nothing but the union of all λU's, as λ varies over $\mathbb{R}$ star. In any case, it is a union of open sets, λ times U is just a copy of U under the homeomorphism namely multiplication by λ, all these L_λ are homeomorphisms. Remember that.

So, U is open and hence λ U is open for each λ. So, union is open. So, do you see the phenomena always valid for any topological action? The quotient is always an open quotient. You see in this argument I have never used that it is actually $\mathbb{R}$ or $\mathbb{R}^{n+1}$ and so on. It is a general argument. So, all quotient maps coming from a group action-- they are open maps.

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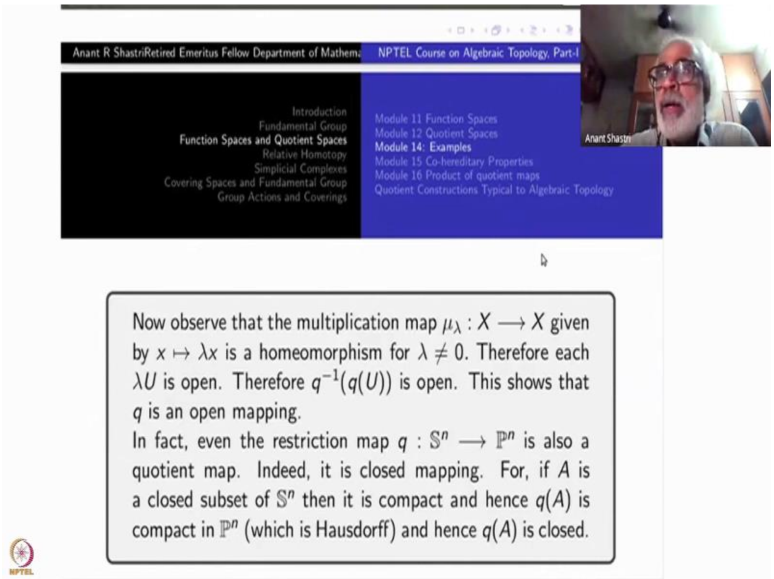

Now observe that the multiplication is a homeomorphism, therefore each λU is open all this I have told you already. The beauty here is you can restrict the whole action to unit vectors, but then the actions should be also restricted to a subgroup, namely you cannot take all scalars because if you take any arbitrary scalar and multiply by a unit vector it will not, it may not be a unit vector. Then what you have to take, you have to take unit scalars. There are only two unit scalars in $\mathbb{R}$ namely ± 1.

So, action of ± 1 on $\mathbb{S}^n$, look at the orbits, it is again the same set of orbits, every orbit of $\mathbb{P}^n$ is represented by a unit vector, in fact it is represented by two unit vectors if v is a vector $-$ v will be also a unit vector representing the same line and they are related by the action of ± 1. Therefore, q from $\mathbb{S}^n$ to $\mathbb{P}^n$ itself is a quotient map. This map is a closed mapping, the entire thing from $\mathbb{R}^{n+1} \setminus 0$ to $\mathbb{P}^n$ was an open mapping, when you restrict to $\mathbb{S}^n$, it is a closed mapping, why?

For, suppose A is a closed subset of $\mathbb{S}^n$, $\mathbb{S}^n$ is compact. Therefore, A will be compact, if A is compact, q(A) will be compact. Now q(A) is a compact subset of $\mathbb{P}^n$,-- we should verify that it is a Hausdorff space. You should know that, then q A will be closed. So, it helps to verify why this projective space is Hausdorff. Perhaps I will leave to you as an exercise. Now, the entire discussion that we have just done can be carried on with complex numbers. Wherever $\mathbb{R}$ comes here, replace it by $\mathbb{C}$: $\mathbb{C}^{n+1}$ and $\mathbb{C}$ star, i.e.,$\mathbb{C} - 0$. So, complex numbers non-zero complex numbers are scalars. What do you get is the complex vector space. So, that is what I am doing here.

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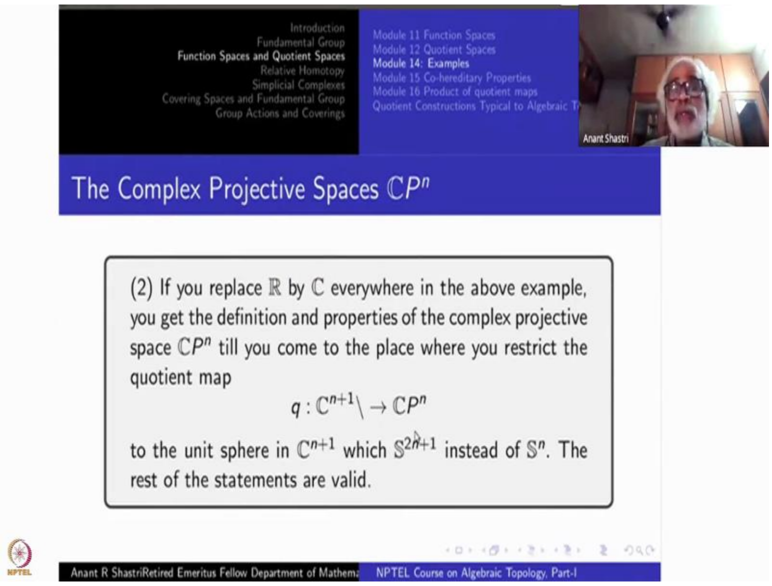

Namely to replace $\mathbb{R}$ by $\mathbb{C}$, everywhere in the above example, you get the definition and properties of the complex projective space which is denote by $\mathbb{C}P^n$ Did you come to the place where you restrict the quotient map $\mathbb{C}^{n+1}$ − 0 it should be to $\mathbb{C}P^n$, to the unit sphere in $\mathbb{C}^{n+1}$, only thing is unit sphere in $\mathbb{C}^{n+1}$ is not $\mathbb{S}^n$, but it is $\mathbb{S}^{2n+1}$ because this is 2n+2- dimensional real vector space.

So, the unit sphere will be $\mathbb{S}^{2n+1}$, instead of $\mathbb{S}^n$ that is all. And $\mathbb{C}P^n$ will be a quotient of this by unit scalars in complex numbers. That is not just $\pm$ 1 but it is the entire circle. So, the circle acts on $\mathbb{S}^{2n+1}$. So, you should write elements $\mathbb{S}^{2n+1}$ as, say, z naught z_1, dot dot z_n, all complex numbers. Then you use the complex multiplication by complex numbers that is the action here. the quotient will the same. Once again $\mathbb{S}^{2n+1}$ is compact, therefore this is compact. But these things are actually Hausdorff. Can be verified. It is not all that easy. It is easy in the real case, here it is slightly more complicated.

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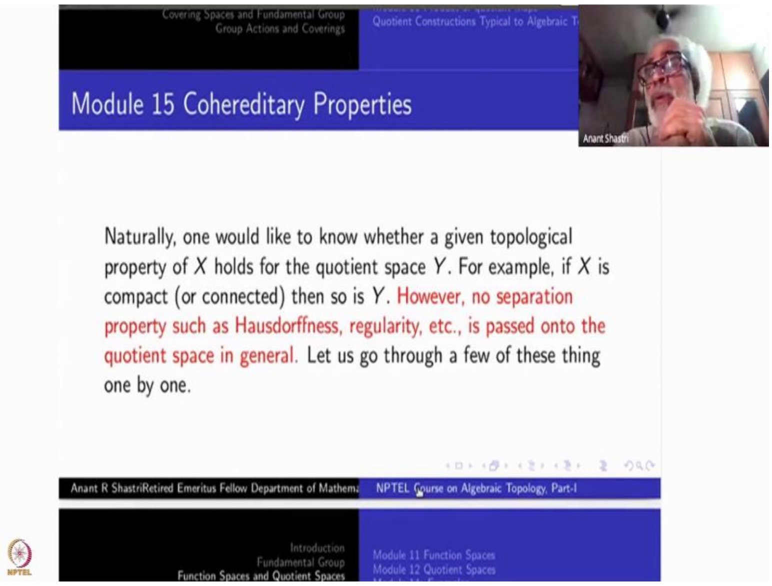

Let us carry out the most important aspect of quotient spaces, what are the properties of the original space which will reflect, which will remain intact when you go to the quotient space, such things are called co-hereditary properties. So, naturally one would like to know, whether a given topological property of X holds for the quotient space. For example, if X is compact, the quotient map being a surjective continuous, the quotient space is compact. This is what we have used already. Similarly, if X is connected, the quotient space is also connected, these are the two easy things.

On the other hand some very elementary properties like T_1, T_2 regular, normal etc., none of these largeness properties is passed on to the quotients easily. That is why in the previous examples, you have to verify separately that $\mathbb{P}^n$ and $\mathbb{C}P^n$ are actually Hausdorff spaces. So, let us carefully go through these exercise what are the kinds of things that will give you the properties below I mean, for the quotient space.

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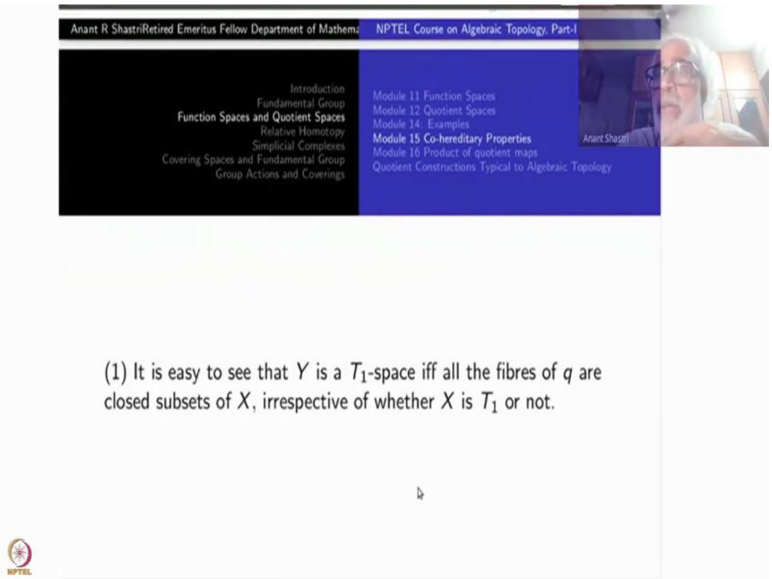

The first thing is, if Y itself is a T_1 space. When it is a T_1 space this means that each point is closed. By the very definition, inverse image of each point must be closed, because q is a continuous function. And conversely also, because this is the quotient topology. Whether X is Hausdorff or not, if each orbit is closed, then Y, which is the orbit space, will be automatically a T_1 space.

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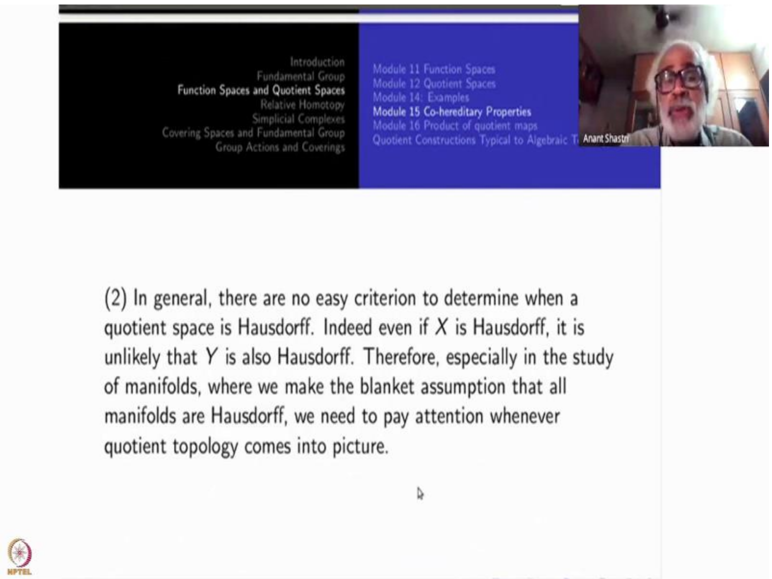

Second thing is, I told you that Hausdorffness is a very tricky thing. There is no easy criteria to determine Hausdorffness like this, even if X is Hausdorff, Y may not be Hausdorff. Therefore, especially in the study of manifolds, wherein you want to have the Hausdorffness, (the same thing in algebraic topology) all the time we HAVE to assume the spaces are

Hausdorff. So far we have not come across such a situation. But that is what essentially essentially algebraic topology is built on, assuming Hausdorffness.

A blanket assumption that all manifolds are Hausdorff that is used. So, when you construct something as a quotient space you will have to verify not only it is locally Euclidean-- you have to verify Hausdorffness also.

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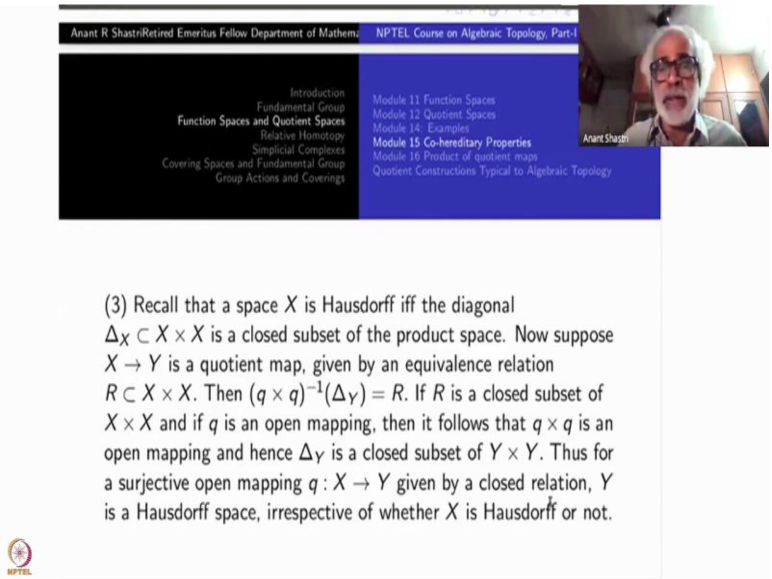

So, let us spend some time on Hausdorffness. So, recall that something is Hausdorff space if the diagonal in X × X in the product of X with itself, the diagonal Δ, namely elements of the form x , x, they form a closer subspace. So, that is an equivalent condition. Now suppose X to Y is a quotient map, given by an equivalence relation $\mathbb{R}$, an equivalence relation is actually a subset of X × X. Whenever x is related to y that means what? You put them in a subset $\mathbb{R}$. So, if a pair belongs to that subset, then they are related. So, that is the meaning of this $\mathbb{R}$. $\mathbb{R}$ is a subset of X × X which gives you the relation.

Now, look at $(q \times q)^{-1}\Delta_Y$, Δ_Y is now what? Diagonal in Y × Y or the set of all elements (y ,y), y belonging to Y. That will automatically equal to $\mathbb{R}$, because if x_1, x_2 is inside $\mathbb{R}^2$, they will come to same point in Y. So, this is the meaning of that Y is the quotient of X by the relation $\mathbb{R}$, that is all set theoretically I have written. So far it just theory. If $\mathbb{R}$ is a closer subset of X × X, then by the very definition, you would like to have Δ_Y as a close subset of Y × Y.

Unfortunately, Y × Y is given the product topology not the quotient from X × X, if you had given the quotient space topology on Y × Y through X × X, then your problem was over. All that you wanted is that $\mathbb{R}$ is a closed subset of X × X. So, that may not be the case. If $\mathbb{R}$ is a closed subset of $X \times X$ and if Q is an open mapping, then we are through. Because if q is an open mapping, then $q \times q$ is also an open mapping. Obviously, it is surjective also. Therefore, it is a quotient map.So, the product topology on $Y \times Y$ is a quotient topology from $X \times X$.

In general, first you take the product then take the quotient or the other way around they do not commute with each other. These two operations do not commute with each other. That is the problem. So, for open mappings, we have solved this problem. More generally, start with a quotient space Y such that $Y \times Y$ is also a quotient of $X \times X$, then only this will work.

Namely, all that you need is that the relation as a subspace of $X \times X$ must be a closed set. Then whatever happens to X, X may not be Hausdorff, but Y will be Hausdorff. So, this is similar to our earlier result when the quotient space is at T_1 space, remember. It is similar to that but you have to work harder over here.

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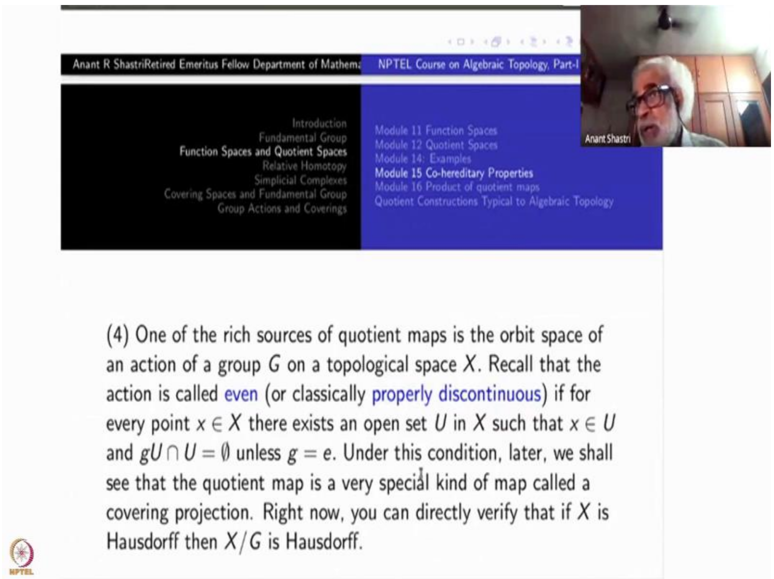

As I told you why we studied the group actions. They give you a large number of quotient spaces. So, now you recall that you had an even action, what is an even action? It is stronger than fixed-point free action. Remember, each point has a neighbourhood U, such that all the translates of this neighbourhood, actual translates other than identity, they will never intersect U. It is the same thing as saying that two distinct translates will not intersect at all. That is the evenness condition. It is also called properly discontinuous action.

Under this condition, we shall see that the quotient map is actually a very special kind of map called covering projection. Similar to the exponential function from $\mathbb{R}$ to $\mathbb{S}_1$. That we will see later. Right now, what you can verify is that if X is Hausdorff, then the quotient X / G is

Hausdroff, under an even action. All these things are very straightforward computations. Unless you sit down and do it by yourself--just me rolling out the solution --- will not go into your head.

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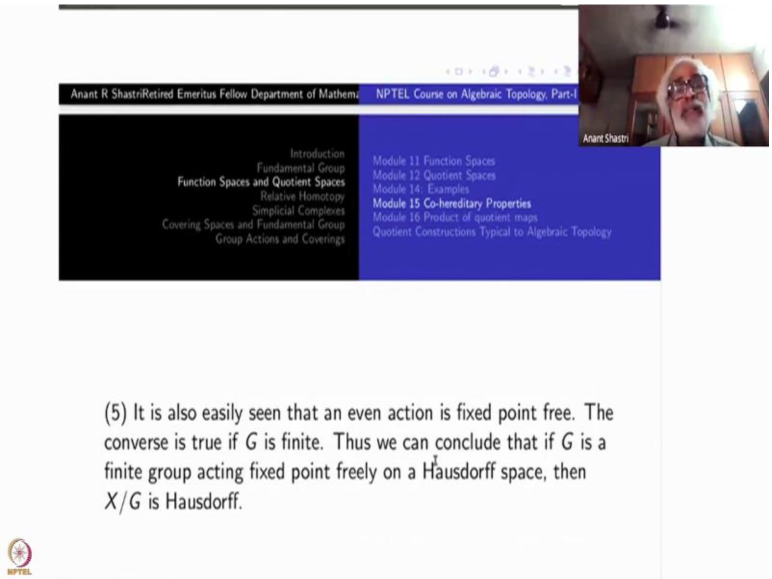

Next thing is, I have already told you that even actions are always fixed point free. Converse will be also true, if G is a finite group. Thus, we can conclude that if G is a finite group, acting fixed point freely on a Hausdroff space, then X / G is Hausdroff. That is because we can go back to the example, to the previous example.

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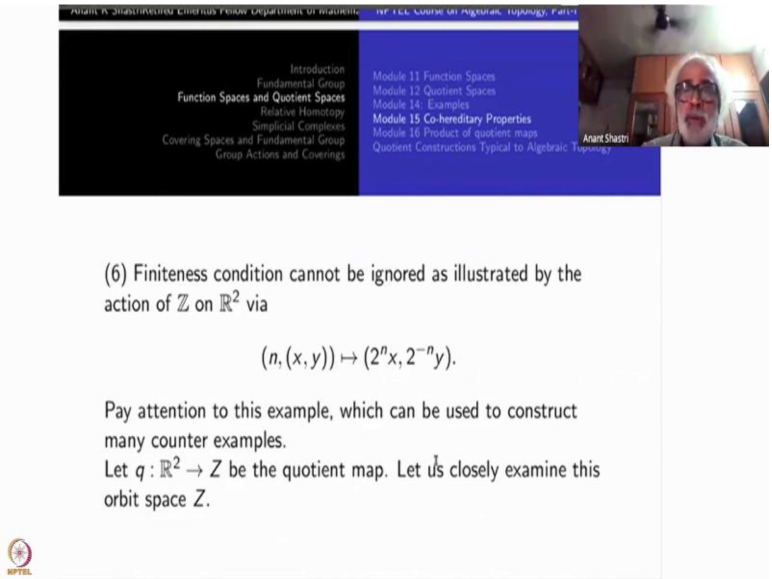

Now I look at this finiteness condition here a nd delete that. What can happen? Maybe it still works! This example says no. What is it? I will give you a fixed point free action yet the quotient is not a Hausdroff space. The new thing here is that the group is infinite. To be precise, we just take the infinite cyclic group written multiplicatively. Instead of that I have represented it by integers here.

Actually, if you see, it is better to write multiplicative notation, take a generator t and then $t^2, t^3, ...t^n, t^{-n}$ and so on, that forms the group. So, never mind, you take the action of $\mathbb{Z}$ on $\mathbb{R}^2$ as follows. So, elements of Z are written as integers n. (x, y) will go to $(2^n x, 2^{-n} y)$. The first coordinate is multiply by 2^n, the second coordinate is divided by 2^n.

So, you can see immediately that this is a fixed point free action, if you throw away 0. You have to throw away 0. That will give automatically a fixed point free action. Keep the 0 for a while, because you would like to study the entire thing, how things look like. So, pay attention to all the details here. Let us write down the quotient map from $\mathbb{R}^2$ / Z is a quotient space here, and you have the quotient map. You can write q (x,y) to denote the orbit space also, whichever one you like, or just bracket (x,y). So, I will use both the notations here.

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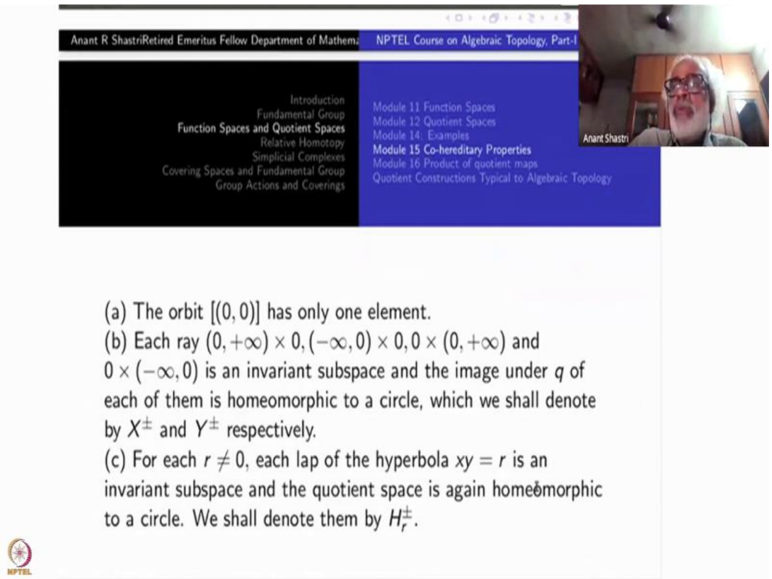

Orbit of 0, (0 remains 0 always 0,) it consists of only one element, multiplication, or division, you only get 0. So, singleton 0 is one orbit. Now, take the ray from 0 to ∞ on the x- axis, i.e, 0 to ∞ cross 0. What are the elements? x,0. So, when you multiply by 2^n or divide by 2^n, it will still remain on the x- axis.

Not only that, it will remain in the positive x- axis. So, positive x-axis, negative x-axis, positive y- axis, and negative y-axis, there are four of these subspaces which are invariant under the action. So, these are invariant subspaces, what are the quotients of this one? What is the image of this one? I claim that image of each of them is a circle. Just examine one of them. The argument is similar.

What happens to 1, 1 will go to 2, 4, 8 on this side it will be 1 / 2, 1 / 4, 1 / 8 and so on, what happens to in between, 1 to 2 what happens? That interval gets just mapped homeomorphically to 2 to 4 expanded. Again it will be mapped to 4 to 8 and then 8 to 16 and so on. On this side it will be contracting. So, half to 1, it will be halved, then one fourth to 1 / 2, it will be one fourth and so on.

So, each part is mapped homeomorphically to the other, and endpoints are identified. So, when you identify all these, what you get is-- just like exponential function wraps the infinite, line $\mathbb{R}$, from $-\infty$ to $+\infty$, onto $\mathbb{S}_1$, this will also give you the quotient space as $\mathbb{S}_1$.

So, in the quotient space, there is one single point which is the orbit of 0, 0 and then there are these four circles.

Now, let us look at beyond this one. Let us look at the first quadrant, second quadrant, third quadrant and fourth quadrant. What is happening there, the first quadrant you can look at this one. Suppose, you take a point x, y, you are going to multiply the first coordinate x / 2 and dividing y / 2. So, each time inverse you will multiply this 1 by 1 by 2 and divide that 1 / 2. So, in effect, the product remains the constant, x y is a constant. That means what? The hyperbola x y = r, one of the branches of x y = r, where r ≠ 0. (r equal to 0 will give the union of x axis and y -axis). So x y = r, when r is not 0, r +, has two branches one in the first quadrant, other one in the third quadrant.

Similarly, if r is - these two hyperbola, the two laps of the hyperbola will be in the second quadrant and fourth quadrant, they are themselves invariant under the action. Just like the positive x-axis is invariant, these lines are also invariant and what happens to them exactly is what happened to the real line and so on ,they really get wrapped up into a circle.

So, for each hyperbola x y = r there are two circles. So, let us denote them by $H_r^{\pm}$. See $X^{\pm}1$, $Y^{\pm}$ 1, are the images of the positive real axis, negative real axis, positive imaginary axis, negative imaginary axis. Now $q(H_r^{\pm})$, these are all images of hyperbolas. So, the quotient space is actually a union of a lot of circles along with the [0], 0 a single point orbit.

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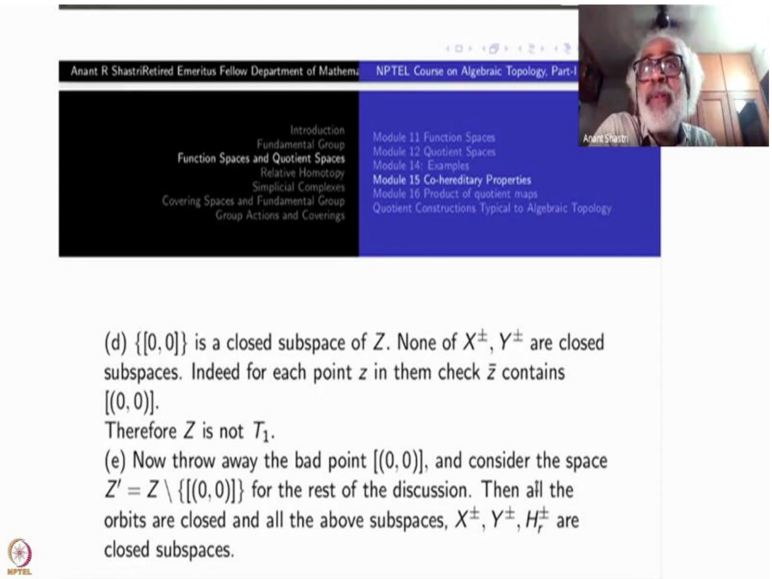

Now [0,0], singleton \{(0,0)\}, inverse image is (0,0), that is a closed. Therefore, in the quotient space [0, 0] is a closed orbit. But look at $X^{\pm}$ and $Y^{\pm}$. I started with an open ray. So, open ray is not a closed subset of $\mathbb{R}^2$ So, what happens is these things are not closed. In fact, you can see for each point z in them, if you take the closure of that, that will contain (0, 0), no matter where you take z. It is the class that will come very close to, as close to (0, 0) as possible, under multiplication (or division), by going on multiplying (or dividing) by it by powers of 2. It will come as close as you please. So, each set [z] will have (0, 0) in its closure. Each [z] bar will contain [0, 0], [z] bar is the closure of [z]. So, no point of this is closed. what I have shown is that none of these points is a closed point in the space Z. So, what does it mean? Z not T_1. All this looks like because we have put this bad point [0, 0]. So, let us throw away this point, let us look at the quotient space Z− the orbit [0, 0]. Then, we are left with all the circles, they are packed up in a very strange way.

So, let us look at Z′, which is Z− singlenton [0, 0]. Now, all these ± X axis, Y axis parts, they are all closed subspaces. All the hyperbola laps are closed subspaces. Therefore, each orbit, each of these spaces, they are close sub spaces. They are themselves close subspaces. Not only that, each orbit is also a closed space now.

On the X axis and Y axis, 0 was the only limit point. On the hyperbola there are no limit points, they will keep going to infinity, so there is no problem. So, the orbits are discrete spaces. So, each orbit is now closed. So, it is definitely T_1. But we will show that this cannot be Hausdorff.

So, what we can show is that after throwing away [0, 0] this becomes a T_1 space, every point is closed, and that it cannot be Hausdorff is what we want to show.

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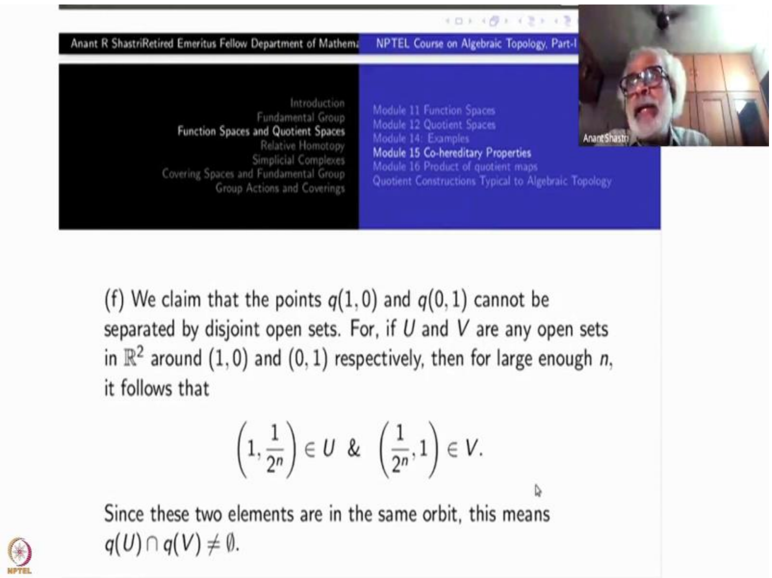

So this is the claim. Look at the image of (1,0) and (0,1). Namely the class of 1,0 and the class of 0,1. I want to say that these two points cannot be separated by disjoint open sets in Z'. Take an open set U and another open set V, which contains classes [(0, 1)] and [(1,0)] in $\mathbb{R}^2$, respectively. Start with $\mathbb{R}^2$, take open sets there. Then, if n is very large, look at the point (1,1) under the action of t^n, y-coordinate will be very close to 0 as close as you want by choosing n large enough. That means it will be inside the open set V.

Similarly, under the action of t^{-n}, (1,1) will be inside V. We can choose a common n such that both of them will happen, no matter how small U and V, we can find a common n, large enough n that these things happen. Namely, (1,1) by 1 by t power is inside U and under t^{-n} it will be inside V. If I multiply the second 1 / 2^n, I get the first 1. I mean action of 2^n . See the first coordinate is 2^n, the second coordinate is 2^{-n}. So, that is precisely this point. It means these two are the same orbit. So, U and V when we go down ot Z, will contain image of these two elements, the same element that is there. That is why the intersection is non-empty, what I mean, q U ∩ q V is non-empty.

If you started with two opens of sets containing q (0 1), and   q (0 1), there inverse images would have been like these neighbourhoods U and V. Then q of that would have been this. Therefore, for every open subset containing q (1, 0) and q (0,1), intersection is non-empty. So, that shows that this is non Hausdorff. The same argument can be done with many other points on the X axis and Y axis.

Since this cannot be Hausdorff, but T_1, the conclusion is much stronger, namely,

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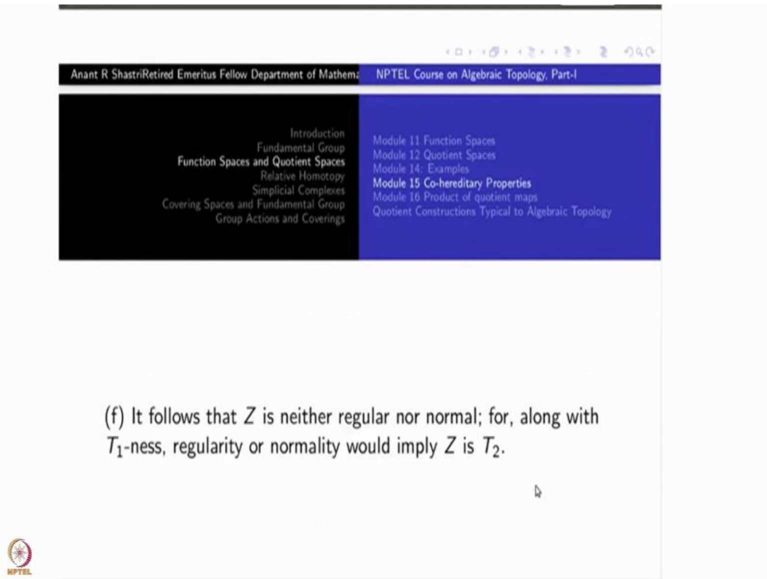

this space cannot be regular or normal. Why? Under T_1-ness, regularity will imply T_2-ness. Similarly, under T_1-ness normality will imply Hausdorff. But we have shown that is not Hausdorff. So, this is neither regular nor normal. Whereas our original space $\mathbb{R}^2 \setminus (0,0)$ is a metric space. It satisfies all this properties. The quotient does not satisfy anything other than T_1. Convincing? We stop here now. We will study some more examples next time. Thank you.