Introduction to Algebraic Topology (Part 1)
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Lecture No. 13
Group Actions

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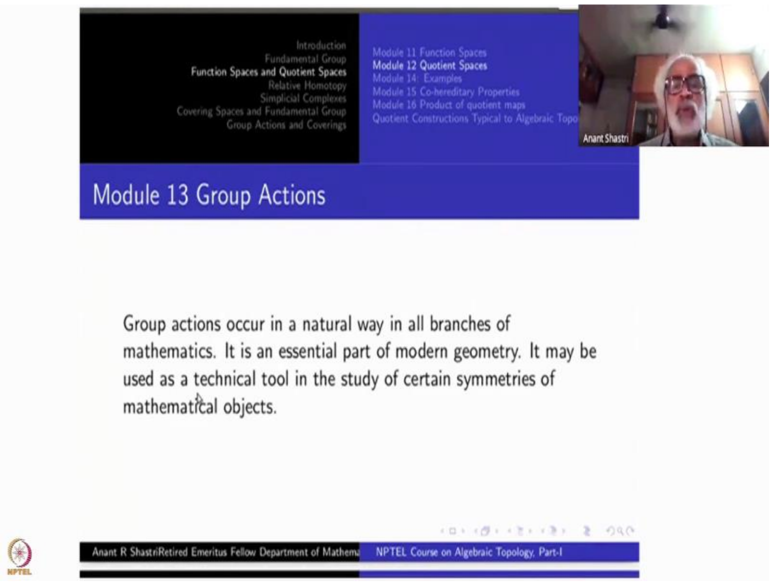

So, we have been studying quotient spaces, you may say suddenly why this Group Action? Group actions give you a big supply of quotient spaces. That is the motivation of introducing group action at this level. It will be used in many other ways, later on. Group actions occur in a natural way in all branches of mathematics, it is an essential part of modern geometry. It may be used as a technical tool in the study of certain symmetries of mathematical objects.

The symmetry of a certain structure is defined by its group actions. So, we cannot expose, go on exposing that aspect of it. But let us start with what is the definition, what are the basic concepts of the group actions. Some of it you might have seen while studying groups themselves.

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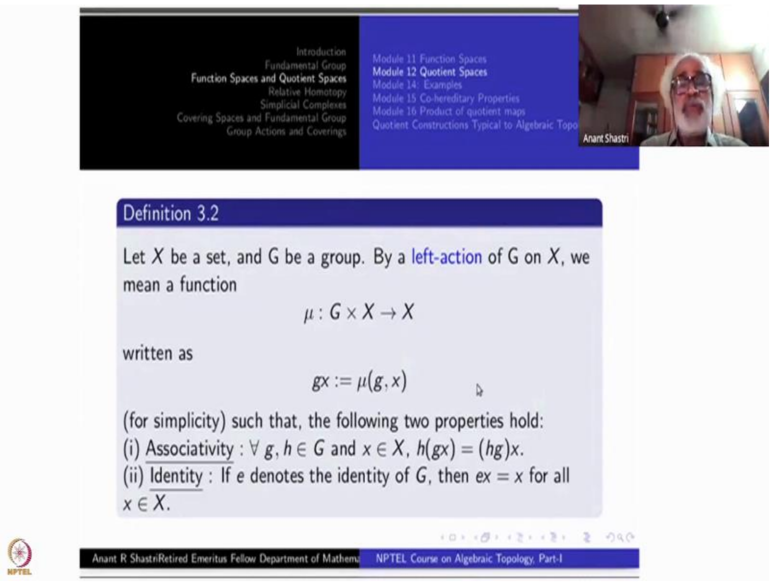

Start with a set X and a group G. By a left action of G on X, we mean some kind of a binary operation but not exactly. G × X to X, take μ, a function, $\mu\mu$(g ,x) we shall write shortly as gx, as if it is the multiplication in G. This notation follows the special case when X is also G and $\mu\mu$ is the multiplication in G, G × G to G. So, we are using the same notation here, but if you have different actions, may be, then you will have to use different notations. That is what you have to do.

So, this is the simplified notation, if there are more than one action, you cannot simplify all of them to the same thing, that is one thing you have to remember. So, this $\mu\mu$ has two important properties: One is associativity, g on x followed by action of h is equal to the action of (hg) on x. So, bracket hg (x) is (h g) bracket x. Secondly, the identity hypothesis namely, if e is the identity element of G, then the action of e on any x namely, ex, which is simply $\mu\,\mu$(e, x) , that must be always x itself.

Once a map like this satisfies these two conditions, you will call it an action. Because we are writing it on the left side, this composition etc is taken from left side, we use the term left action. Exactly same way, you could define a right action also. I will not bother to define that here, it is very straightforward.

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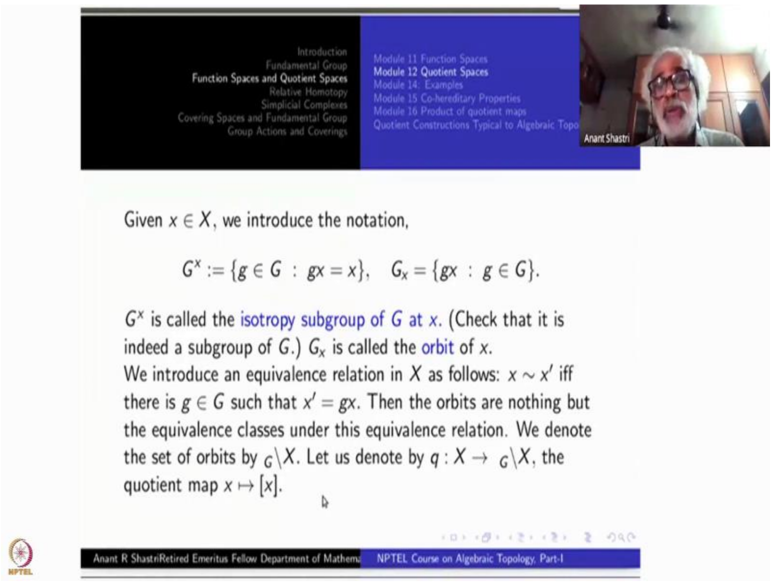

Now, there are a few inbuilt structures here. Take x belong to X. We denote G^x, G^x denotes all elements of the group G which keep the point x fixed-- g (x) is x. See you think of the action of g on X as a motion, it is moving the particles of X, elements of X are moving. That motion is given by g. That is the dynamic way of thinking about a group action.

Similarly, G_x is set of all gx, where g varies over G. So, this is called the orbit, G^x is called isotropy subgroup of G at x. G^x depends upon the x, if you take a different x, it will be different. Of course, isotropy is a subgroup of G which also depends on x. We introduce an equivalence relation in X as follows: x and x′ are equivalent, if there is a g which will bring x to x′, gx equals x′.

Obviously, g^{-1} will bring x′ to x. And identity takes x to x. And if g takes x to x′ and h takes x′ to say x″, then hg would have taken x to x″. So, this is a equivalence relation. So, just like we had three different ways of looking at the quotient map that is precisely what is happening here. So, all orbits are obviously disjoint and for each x there is an orbit, therefore X is the union of all these orbits under the G action.

When you have this equivalence relation that just means that two elements are in the same class, same orbit. If you look at all the orbits as a set Y, then you have a surjective mapping from X to Y. Here I am writing Y as X slash G on the left, because it is a left action, if it is a write action then I would write x slash G on the other side. So, this map is nothing butx going to it is orbit, it's equivalence class. So, all the three pictures are here of a quotient map. So, you have already got a quotient set here X to $G\backslash X$.

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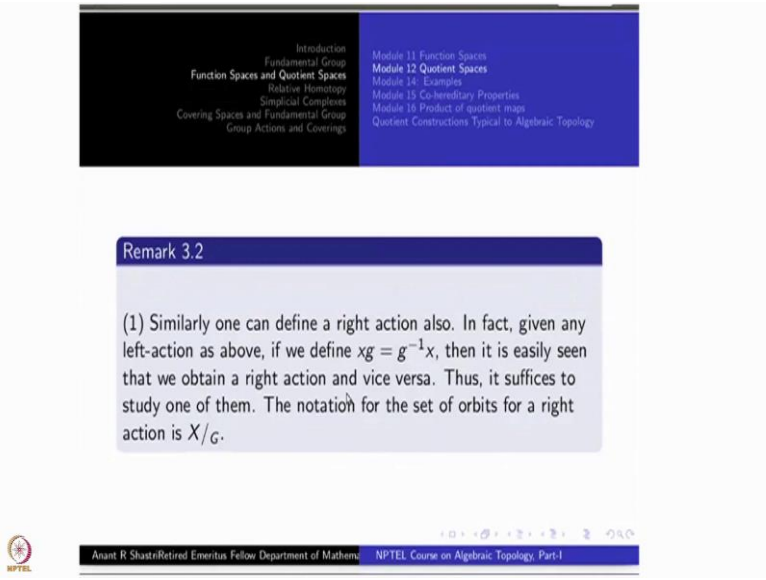

One can define, I told you, a right action also. In fact, every left action can be converted into right action by defining xg as $g^{-1}(x)$. (If xg is equal to gx will not work, the law of associativity will give trouble.) So, $g^{-1}(x)$ then it is easily checked that this becomes right action. And if you have a right action by the same formula , you can make it left action. Therefore, roughly speaking there is no need to study right actions, once you have studied left actions or vice versa.

So, depending upon whether you are lefty or a righty, you can choose which ever you like. Sometimes there are both actions, one action on left and the other on the right. That is why we need both these concepts.

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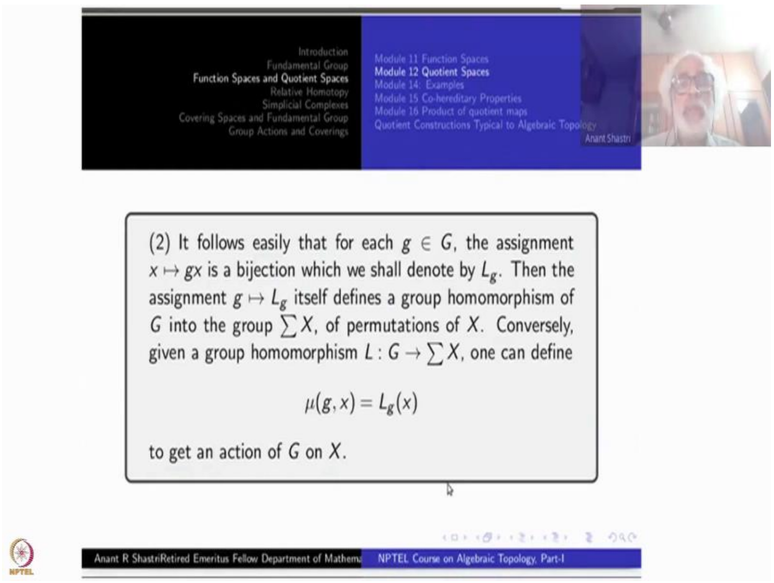

Here is another remark. For each g in G, the assignment x to gx is a bijection. because G is a group you see, so, every element in it is invertible. Therefore inverse image of x is g^{-1}(x). Let us write this left action by L_g. Now, L_g is a map from X to X and I can say that it is a permutation of X, because it is a bijection. So, what you get is: for each g, the assignment g going to L_g defines a map from G into the permutation group of the set X.

This itself is a group homomorphism, because of the associativity. L_g composite Lh is L of gh that is what associativity says. ... So, given an action, I have a group homomorphism from G into the group of all permutations of X. Conversely suppose you have such a group homomorphism, you may call it capital L. Then you can define $\mu(g, x)$ i.e., little gx, equal to L_g operating on x. Then $\mu\mu$ will become a left action on X and if you do again L_g of this action will give you back the same L. So, there is a one-one correspondence between actions and group homomorphisms from G into permutation group of X.

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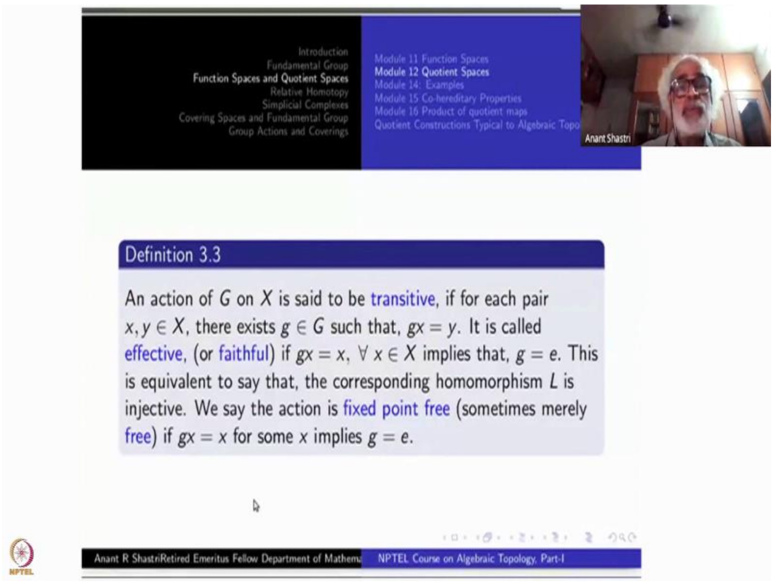

Here are few more terminology. We say an action is transitive if for each pair x, y belonging to X, there exists a g in G such that g of x is y. So, any two elements are related by an action of G. The element g may depended upon the elements x and y.... It is the same thing as saying that there is one single orbit, the entire X will be one orbit. Start with any point you can go to any other point by an action of g, so, there will be just one orbit which is same thing as saying that the action is transitive.

Now, the action is called effective, respectively, faithful, first let us look at the definition of effective: gx is x, for every x implies x must be identity. what is the meaning this? what is the meaning of this? --- that every non-trivial element defines a non-trivial permutation. So, some people call it faithful also. This just means that the corresponding homomorphism from G to ΣX is an injection, is a monomorphism. So, faithful is same as saying the corresponding homomorphism is injective.

You say the action is fixed-point free, (sometimes merely `free action') whenever gx is x for some x implies g is identity. So, this is very, very strong in the sense that if we have a non-trivial element, then it will not fix any element, all the members are moved. gx is x for some x even if it fixes one x, then it must be the identity. Identity, of course, fixes everything. So, such an action is called a free action. Some people call it fixed-point free action also. Free does not mean that it does not cost you anything, it costs very high actually.

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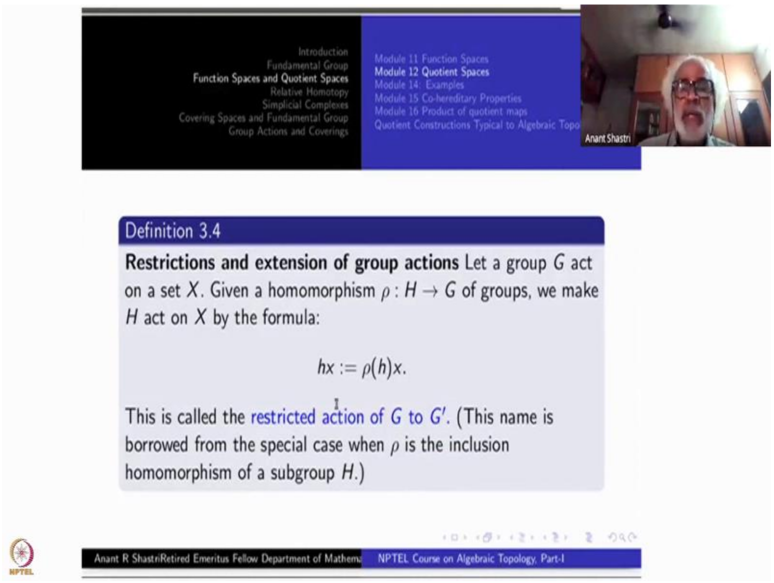

There are a few other concepts like this, which I will use much-much later, but right now I just introduced them. Take a group G which is acting on a set X. Suppose you have a homomorphism $\rho\rho$ from another group H to G. Then through this homomorphism we can make H act on X by the formula h of x is : $\rho\rho$(h) that is an element of G, take that action of ρ ρ(h) on x. A generic name for this is restricted action of G to H via ρ . (This is a typo here.)

This name is borrowed from the special case when $\rho\rho$is the inclusion homomorphism of a subgroup H. Suppose H is subgroup and $\rho\rho$ is the inclusion. Then there is this name restricted action';--- makes sense. The same thing has been generalised to any group homomorphism, whether $\rho\rho$ is injective or not.

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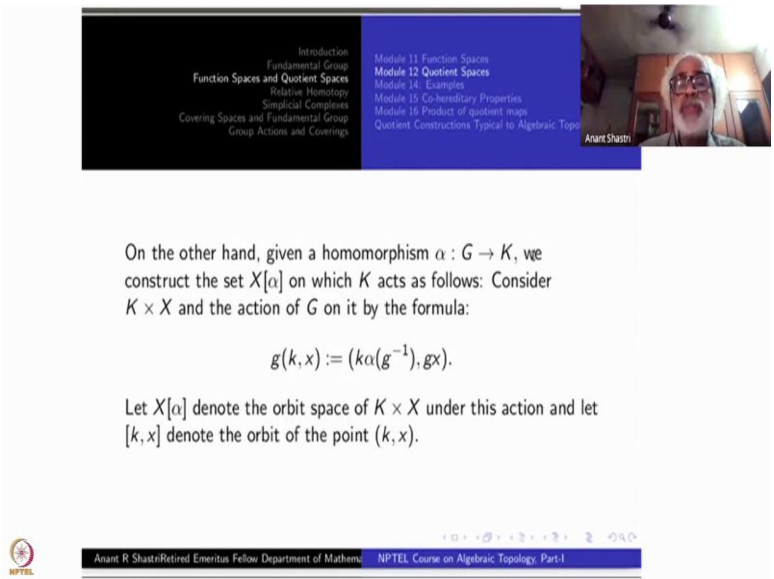

On the other end, you can have a different point of view. Suppose, you have a homomorphism α from G to another group K. Then you want to have what is called an extension here. For that, you have to enlarge the space X as follows: This $X[\alpha]$, X together with α is $X[\alpha]$ α, this is a set on which K will act. But how this $X[\alpha]$ α is defined? For that you have to wait here, let us look at this, how this is done.

Look at K × X as a set and on this set take the action of G, via the first slot K. See there is a homomorphism from G to K. So, these are group homomorphisms. You can think of G acting on K by the formula namely g k equal to... wait, I want to take some action on the right side, namely, gx equal to k times αg^{-1}, see αg is an element of K. So, I am multiplying them inside K, k into αg^{-1}. On the other slot, take just the action of G on X.

Essentially, if you do not write this α at all, suppose it is inclusion map, it is k into g^{-1}, gx like g and g^{-1} are cancelling out each other. We are introducing that g, g^{-1}, if you combine them it will be as if no action at all that is the kind of thing that we are thinking about. So, let X α denote orbit space of K × X under this action. So, this is going to be a quotient space of K× X under this action and let bracket [k, x] denote the orbit of the point (k, x).

So, these are the equivalence classes here for example, if I write αg^{-1} on this side and G on this side, it will represent the same element. So, that is my $X[\alpha]$; this will already indicate what is K, because α is a homomorphism G to K.

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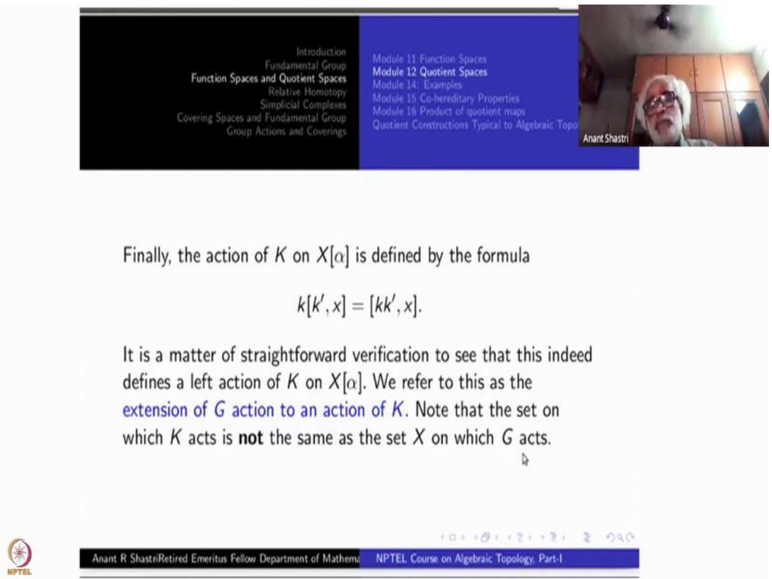

Now, you can define the action of K on X $\alpha$from the left slot. In every equivalence class, the first slot is from from K -- (k′,x). So, k times that can be taken to be bracket [kk′,x]. -- the class of that.

So, we first created room by enlarging X created some room for K to act, by just taking K× X, then you can take a left action, but we do not want the whole of K × X, we have to take the quotient of that.

So, it is a matter of verification to see that this indeed defines an action. It is all straightforward there is nothing to be verified here. We refer to this action as the extension of G action to an action of K. Note that the set on which K acts is not the same as the set on which G acts-- we have extended X. So, this is somewhat larger set and larger action.

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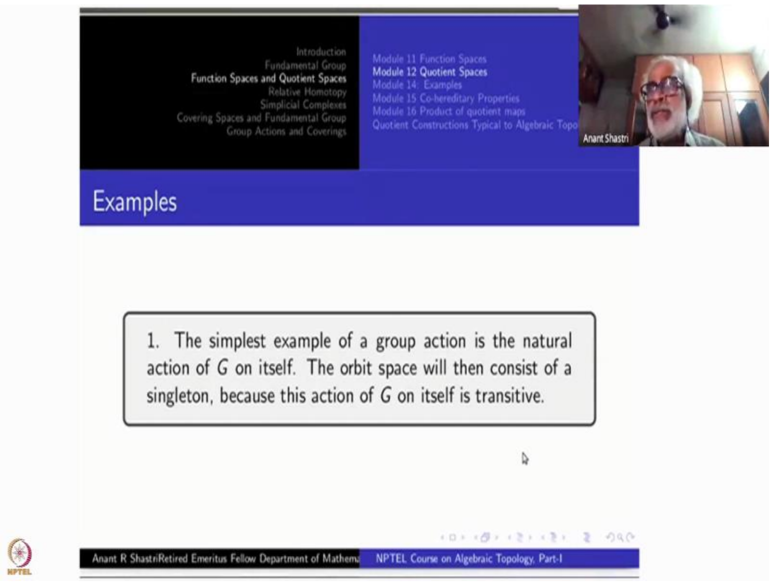

Let us take some examples now. The simplest example of a group action is the natural action of G on itself, G acting on G, where G is a group. The orbit space will then consist of just one single element because the action of G on G itself is transitive--- if we have g and h, which element will bring g to h? Namely, hg^{-1} operating upon g will be h. That is all.

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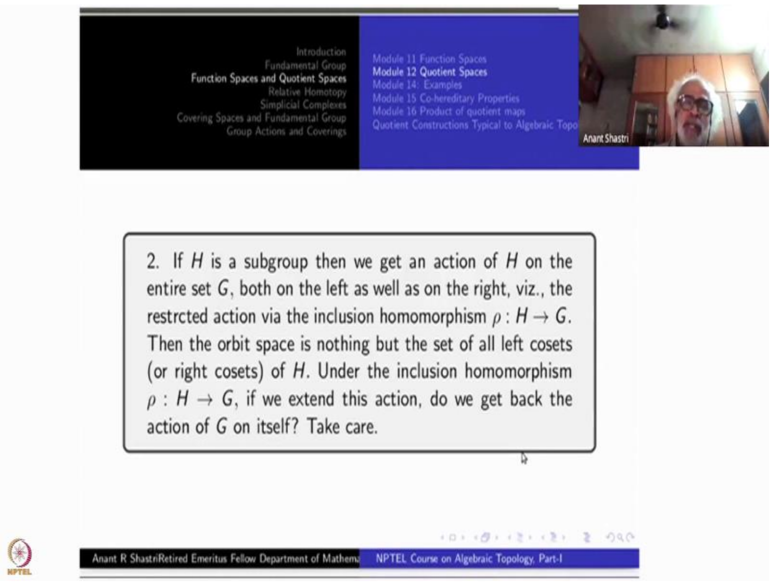

If H is subgroup then we can take the so called restricted action of H. So, h will act on the entire G, say on the left or on the right whichever way you want, it is called restricted action through the homomorphism ρ from H to G of the action number 1, the example 1. The orbits here are nothing but the left cosets or the right cosets according to which action you have taken. So, this is what you study in group theory right in the beginning. Under the inclusion of homomorphism ρ from H to G.

Now, if we extend, see first we had G × G, G acting on G itself, G × G to G you restrict it to an action of H because H is a subgroup via, some monomorphism, you will not get back G,--have to be careful, you will get something different. So, it is interesting to check what you get.

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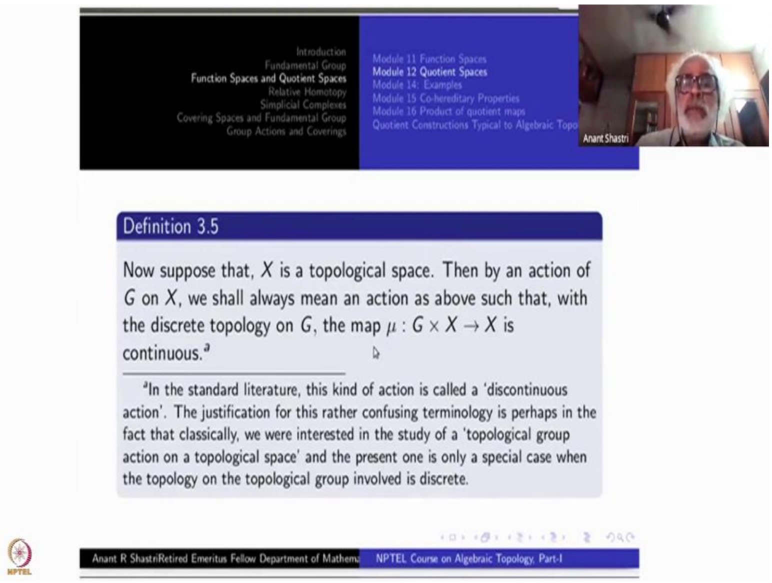

Now that is all about general group action on a set. Now in group theory you study this one very deep, all the way going to Silow's theorems and so on. So many interesting results can be obtained by just studying this one. But now we want to go the topology.

Suppose now, X is a topological space, then we are not satisfied by just permutations as actions. But the homomorphism must be into the homeomorphism group, not permutation group, the set of homeomorphism is a subgroup of the group of all permutations.

So, that is the extra condition that we need when X is a topological space, which is the same thing as saying that the action from G × X to X is now continuous. In what way ? We could take G with the discrete topology. X has its own topology. Under that $\mu\mu$ must be a continuous map G × X to X, along with the two hypotheses of associativity and identity that we have seen. That is called a topological group action.

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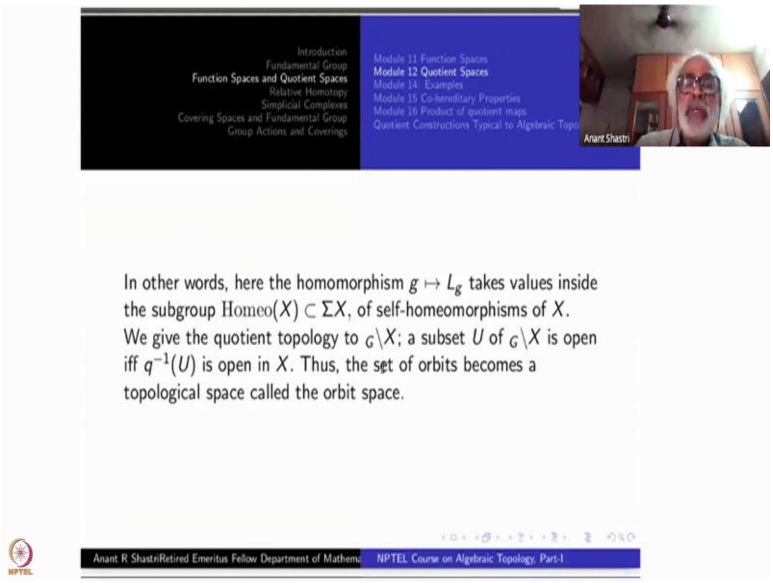

In other words as I told you already, the homomorphism on G takes values inside the homeomorphisms of X. This is a subgroup of all permutations, --self-homeomorphism of X. Again, we have the same quotient set now, you give quotient of topology to this, this becomes an orbit space not just orbit set. What is the topology? Remember, a subset U of $G\backslash X$ is open if and only if $q^{-1}(U)$ is open in X. The set of orbits becomes a topological space called orbit space.

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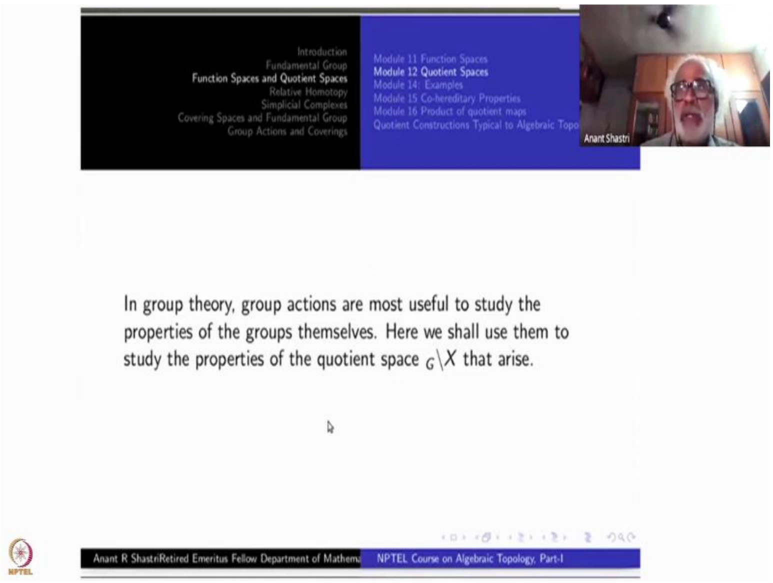

In group theory, group actions are most useful to study properties of groups themselves. Here we shall use them to study properties of the quotient spaces. Pretending as if we know everything about X, what can you say about $G\backslash X$? That will depend upon the kind of action that we have.

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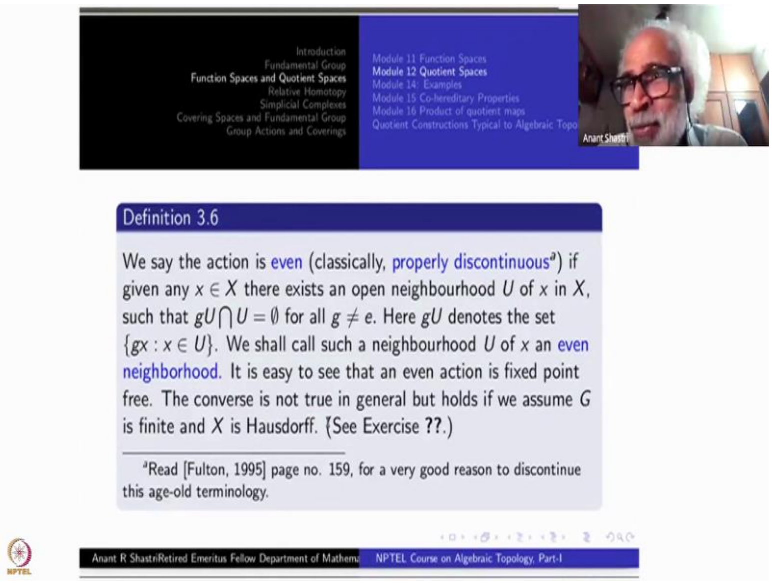

So, accordingly you introduce a few more concepts here—more definitions. An action is called even if for every point in X, you have a neighbourhood U of x in X, such that if you translate U by g, i.e., g U, it will never intersect U, g U ∩ U is empty for all g ≠ e. So, this is much stronger than the fixed-point free action. This cannot be defined for arbitrary sets because there is no neighbourhoods or no topology there and so on.

So, now, we have stronger notion of fixed point free action. gU ∩ U is empty all g ≠e, where here gU denote the set of all elements which look like gx where X runs over U, g is fixed, we shall call such a neighbour of U(x) and even neighbourhood. It is easy to see that an even action is fixed point free. That is what I told you. Converse is not true in general. But of course if we assume G is finite and X is Hausdorff, now you start doing topology see? So, this is an easy exercise, I have left it as an exercise.

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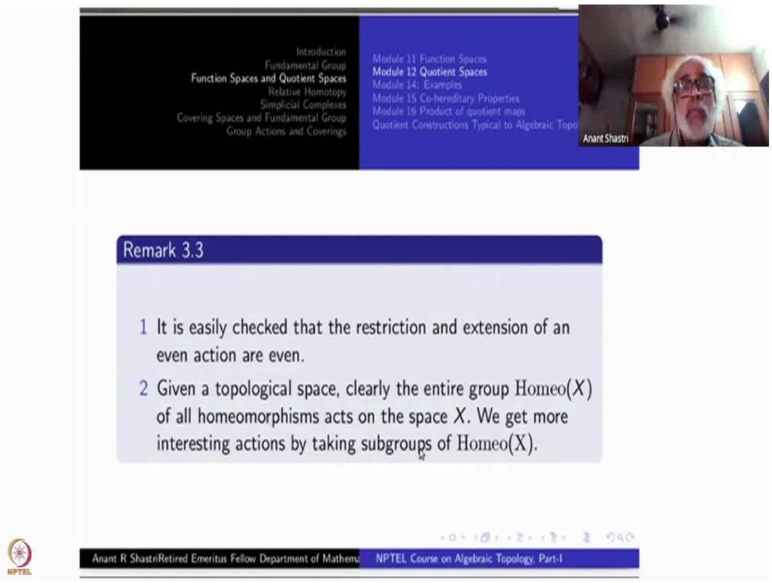

It is easily checked that restriction and extension of an even action are even. Given a topological space, clearlythe entire group of homeomorphisms acts on that space. We get more interesting actions by taking subgroups. What kind of subgroups you take? That is all. Once you know the action of all homeomorphisms on a topological space you do not have to define it for subgroups separately. Which subgroup you take will define the symmetry of X. So, let us stop here and next time we will see examples.