**Introduction to Algebraic Topology (Part - 1)**
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**Lecture 12**
**Quotient Maps**

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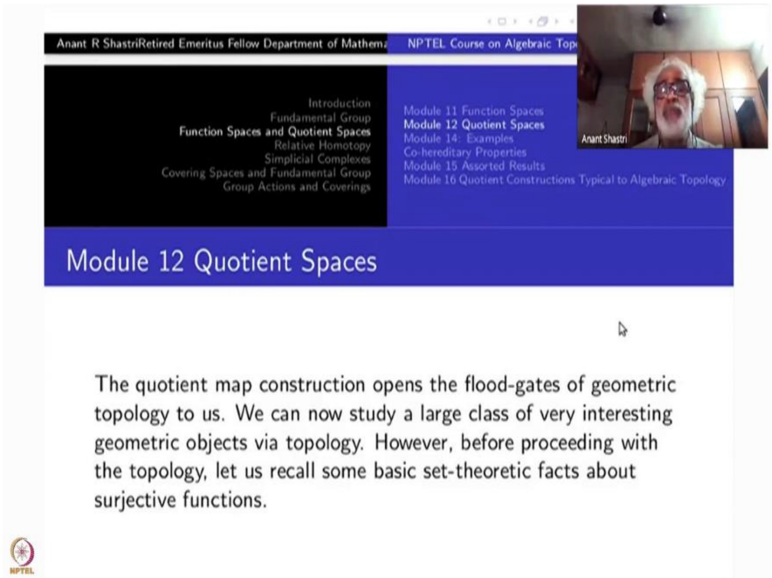


Welcome to Module 12-- Quotient Spaces. The idea of a quotient map and the construction opens the flood gates of geometric topology to us. We can now study a large class of very interesting geometric objects via topology. Before proceeding with the topology, let me make it clear that all of you understand the meaning of a surjective function from one set to another set, and what it means in terms of decomposition of X or equivalence classes. There are three different pictures of this which imply the same thing, the same concept.

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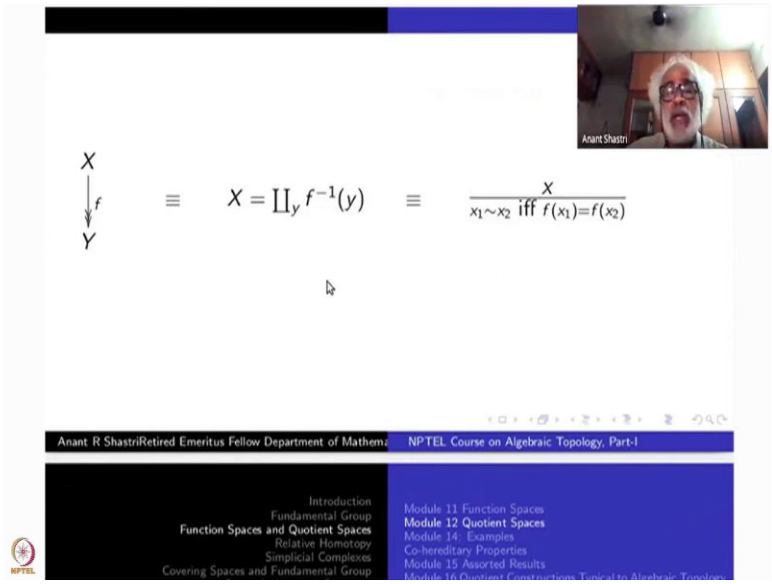


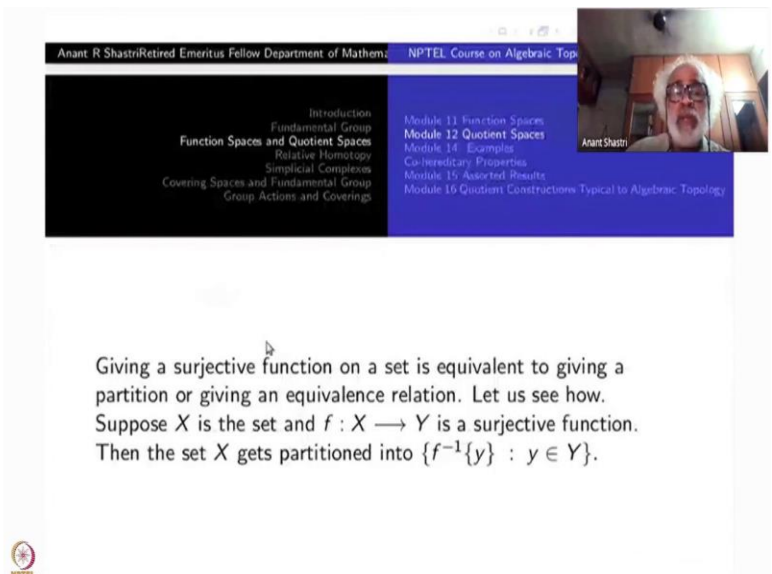


Look at this diagram, the first portion here represents a surjective function, an onto-function from one set X to another set Y. When you have this, you can take $f^{-1}(y)$ as y ranges over capital Y. That will give you a decomposition of X, as I have written down here. $f^{-1}(y)$ means all points of X which come to the point y here. So by the very definition, they are disjoint and the entire X is a union of these. Because f is surjective, each $f^{-1}(y)$ will be non-empty. Therefore, what we get is a non-empty disjoint decomposition of X.

So, such a thing is called a decomposition or a partition. So, what we have got here is a surjective function of X into another set, and this implies a partition of X or a decomposition of X.

Now suppose you have a partition of X, then you can define an equivalence relation on the set X namely, in which the partition members will become equivalence classes, namely, x is equivalent to y if and only if both are in the same subset.

In particular, here what will be the relation? $x_1$ will be in equal to $x_2$ if and only if f $(x_1)$ is equal to $f(x_2)$. Your partition giving rise to a relation or a function directly giving a relation. This picture you can get from here directly or from directly from here.

Finally, suppose you have an equivalence relation like this on X. Look at the collection of all equivalence classes. That is a set. Call that set Y. Then what should be the function from X to Y? Take any x and take its equivalence class as f(x).

So, from an equivalence relation you can come to a surjective function. If you follow this procedure again, this cycle, what you get is whatever you have started with, namely suppose you have an equivalence relation and then you have defined f to be the function x going to the equivalent class of x. Then what will be the disjoint union here? They will be the equivalence classes. And what is the equivalence relation induced by that? The same equivalence relation because these are the equivalence classes to begin with. So this is what I am saying, giving a surjective function on a set is equivalent to giving a partition.

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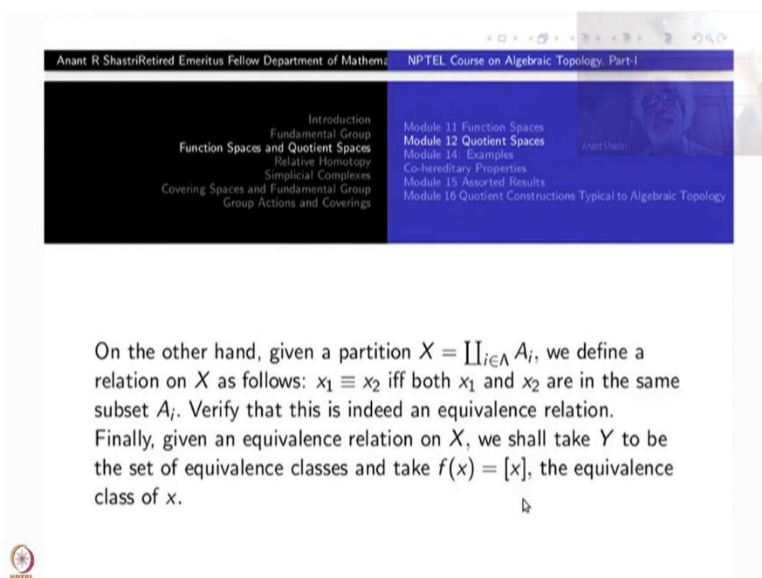

And then, giving a partition say X is disjoint union of $A_i$, we can define x_1 is equivalent to $x_2$ if and only if both $x_1$ and $x_2$ are in the same subset $A_i$. This is equivalence relation. Finally, given an equivalence relation we can take Y to be the set of equivalence classes and define f by saying that f( x) equal to equivalence class to which x belongs.

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So, verify that if you run this cycle of arguments again, what you get is wherever you started from. You come back to the same point, same concept. So, when you have an equivalence relation or a surjective function or a decomposition, even if one of them is given you should have all the three in your mind. So that you can use any one of them,  whichever description you want you can use.

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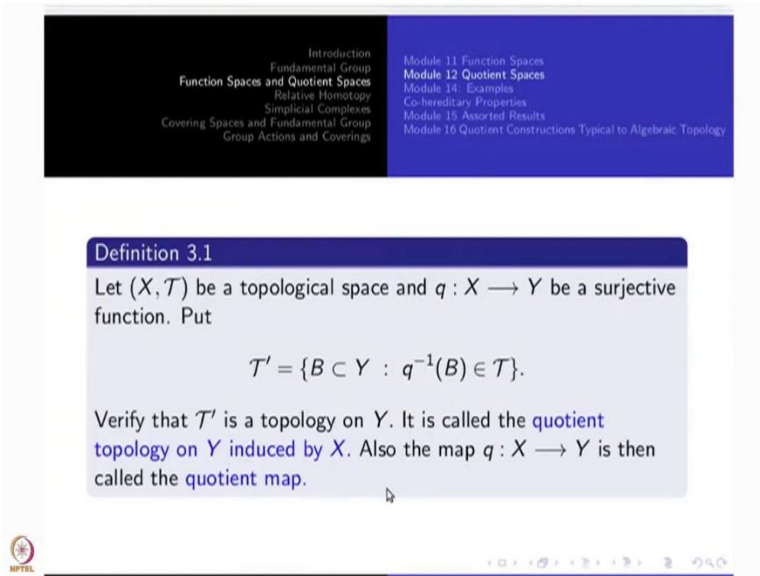


Now let us come to the topology. Suppose (X ,$\tau\mathcal{T}$) is a topological space. X is a set and $\mathcal{T}\tau$ is a topology and q is a surjective function from X to Y where Y is any set. We want to make Y into a topological space such that q becomes continuous and this we want to do in a optimal way. How many open sets can we put in Y? As much as possible, that is the whole idea. So, I define $\mathcal{T}'\tau$ which is going to be a topology on Y as follows.

Take a subset of Y if $q^{-1}$ of that subset is inside $\mathcal{T}$, then put it in $\mathcal{T}'$. That means a subset B belongs to $\mathcal{T}'$, if and only if $q^{-1}$(B) belongs to $\tau\mathcal{T}$. This $\tau$ $\mathcal{T}'$ will automatically be a topology on Y. Very easy to check. Namely, $q^{-1}$ of intersection of two sets $B_1$ and $B_2$ is nothing but $q^{-1}$( $B_1$) intersection $q^{-1}$($B_2$). This $\mathcal{T}$ is a topology. Therefore, if $B_1, B_2$ are in $\mathcal{T}'$, their intersection belongs to $\mathcal{T}'$.

Similarly, $q^{-1}$ of the union of a collection Bi, union of all Bi (say they are arbitrary union) is nothing but $q^{-1}$(Bi) and take the union. That will be inside $\mathcal{T}$,$\tau$ because, each $q^{-1}(B_i)$ is there. Therefore, union of $B_i$'s will be inside $\mathcal{T}'.\tau$ So, $\mathcal{T}'$ is a topology. Automatically q is continuous. Why? Because take a set here in $\tau$ $\mathcal{T}'$, its inverse image by the very definition is in $\mathcal{T}$. So, this topology thus satisfies the condition that q is continuous. This topology is called the Quotient Topology and the space Y with this topology is a quotient space, and the map q will be called quotient map.

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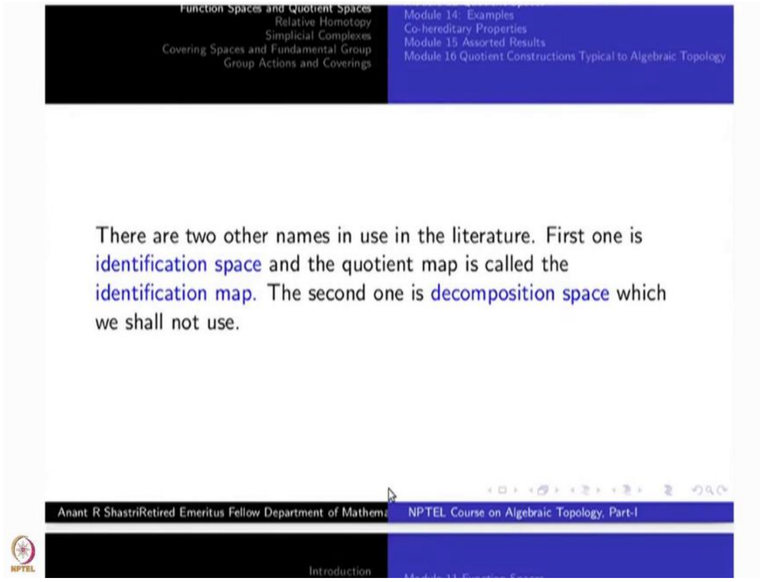


Some people call it the Identification Space because a surjective function (may not be injective) identifies a certain number of points which are taken to same point in Y. It is also called decomposition space using the second description that we have there. See every surjective function gives you a decomposition of the set or partition of the set. So those names are also used by some authors.

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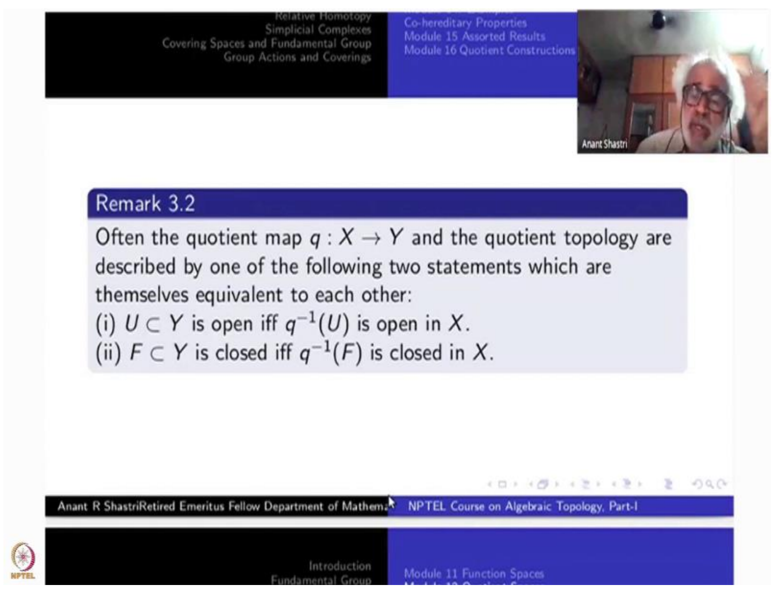


Often a quotient map from X to Y and the quotient topology are described by one of the following two statements. So you should know various different definitions here, they are all, they could all be taken as definitions of a quotient space. The first one is U contained inside Y is open if and only if $q^{-1}(U)$ is open in X. The second condition is, by taking a De Morgan's law, F contained inside Y is closed if and only if $q^{-1}$ (F) is closed in X. The `if and only if' is important. `Only if' gives continuity of q. We are telling that everything that whatever satisfies this condition has been put in there.

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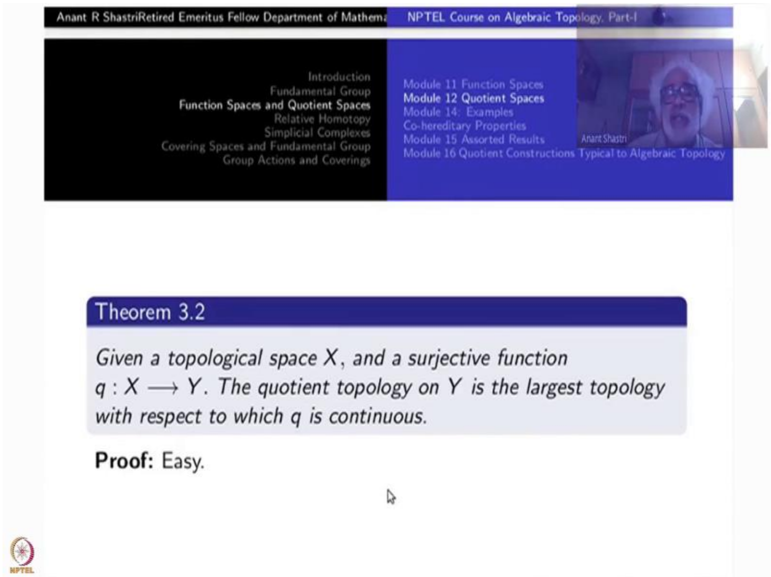


Therefore, I get a third one which describes the quotient topology, namely, this theorem which says that quotient topology on Y is the largest topology with respect to which q is continuous. If some $\Lambda$ is a topology on Y such that q is continuous with respect to that topology, (X has not changed, topology on X has not changed) , then I want to say that every element of $\Lambda$, every open sub set in this topology is in the quotient topology which we have to denoted by $\mathcal{T}'.\tau$ Take an element of $\Lambda$, so that its inverse image is open in X -- that is enough to put this one inside $\tau\,\mathcal{T}'$. Therefore, $\Lambda$ is contained inside $\mathcal{T}'$. Therefore, $\mathcal{T}'$ is the largest topology.

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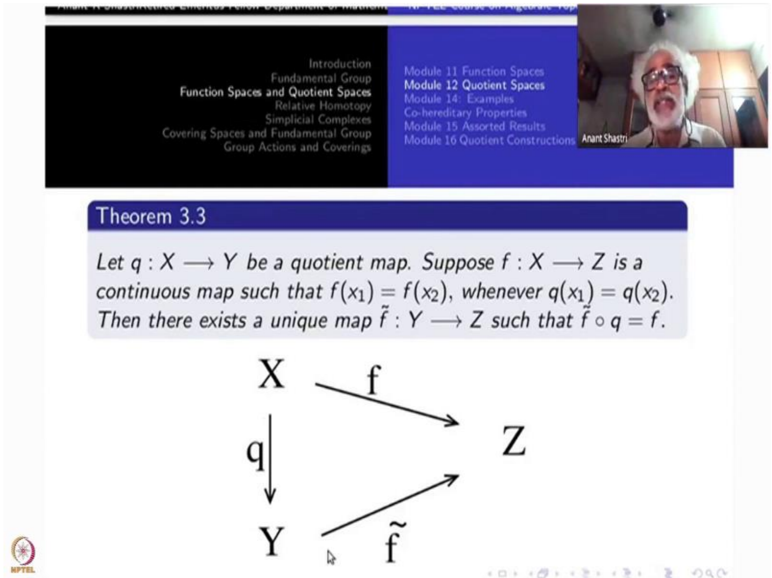


One more criteria. So this will be the fourth one, it is called in some parlor as the universal property of the quotient map. So let us go through this carefully. We start with the quotient map okay? Now take any function, any continuous function from $f : X \to Z$, which, as a function factors down from Y to Z. Factoring down from Y to Z just means that there is a function $\tilde{f}$ here such that if we start with q and then follow it with $\tilde{f}$ i.e., composite with q, we get f. This is the same thing as saying that whenever f takes two points to the same point here, sorry, whenever q takes two points of X to same point here, f must take those two points to same point here. Then $\tilde{f}$ of q (x) is defined to be f(x). You get function here. This is a set theoretic fact.

But what this theorem says is that this $\tilde{f}$ is automatically continuous. There is a unique continuous map.

No matter what f is, what Z is, once they satisfy the set theoretic property correctly to get a function $\tilde{f}$ like this, continuity of f will imply continuity of $\tilde{f}$. You see continuity of $\tilde{f}$ automatically implies continuity of f because f is $\tilde{f} \circ q$. Composite of two continuous function is continuous. Here what this theorem says is that if f is continuous then $\tilde{f}$ is continuous.

So this is the so called universal property of q because it is true for all f and all Z. And there is no other function which is continuous like this which satisfy this one. Such maps are called quotient maps. You can take this as definition. Let us go through this one, see why this is so.

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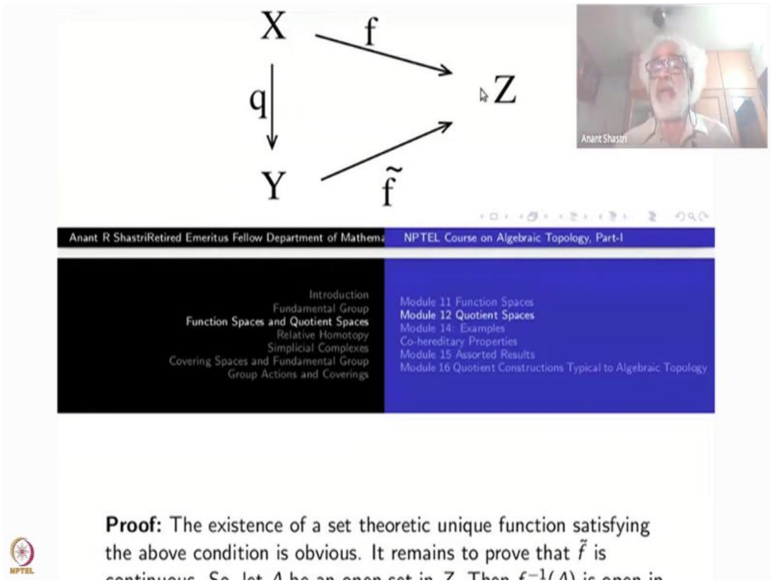


Set theoretically that condition will tell you the existence of $\tilde{f}$. But why $\tilde{f}$ is continuous is what you have to show. Start with an open subset in Z. Its inverse image must be open here, but what is the condition for a set to be open here? Its inverse image under q must be open in X. So I have to take the open subset U here, $\tilde{f}$ inverse then $q^{-1}$. That must be open in X. But f ~ inverse and then $q^{-1}$ is nothing but $f^{-1}$ because f is nothing but $\tilde{f} \circ q$. So if f is continuous, $f^{-1}(U)$ is open here, so we are done.

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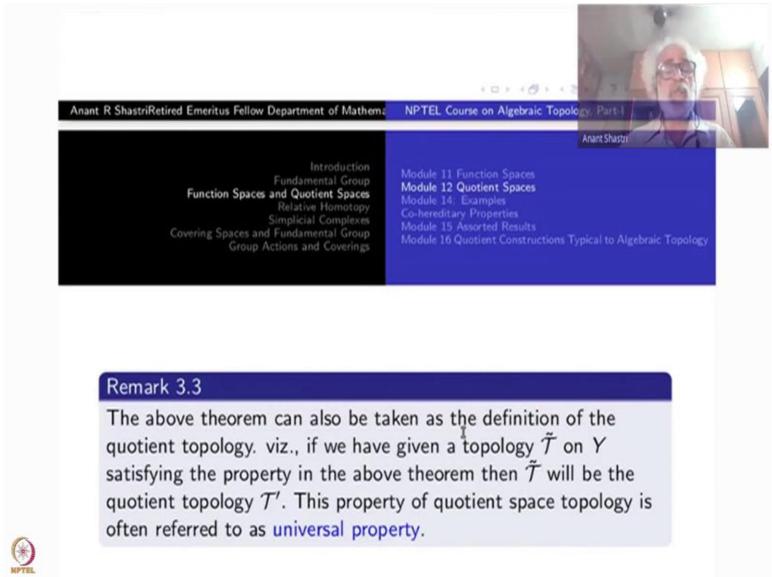


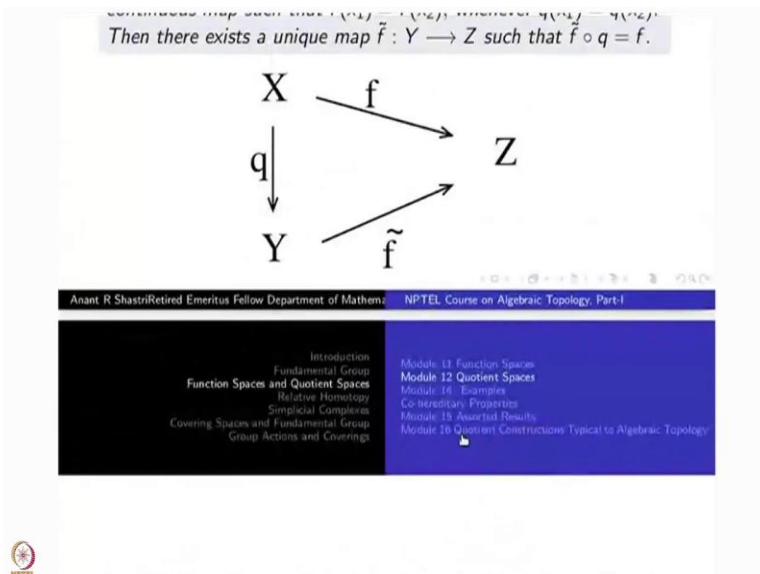


The above theorem can also be taken as a definition of the quotient topology, if we have given a topology $\tilde{\mathcal{T}}$ on Y and it satisfies this property, then $\tilde{\mathcal{T}} = \mathcal{T}'$. There is no other choice. That is the meaning of this, so I once again repeat: such a property is called universal property.

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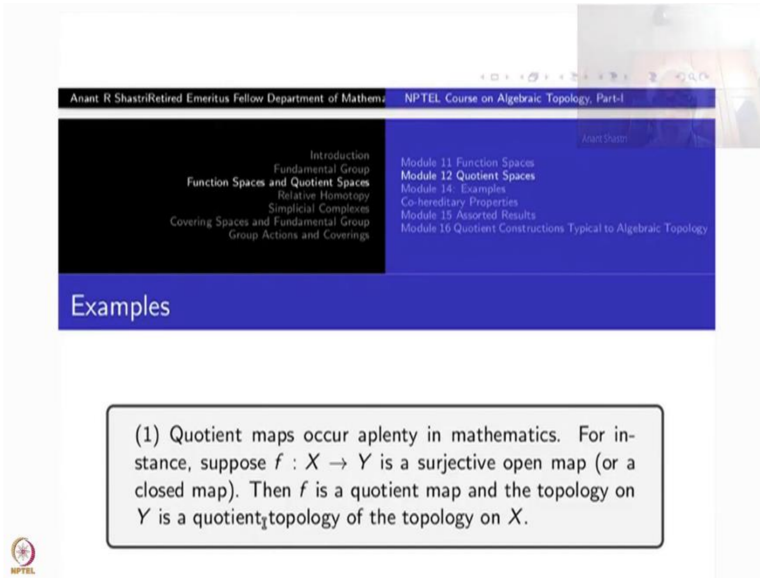


Here is an example. Quotient maps occur a plenty in mathematics. For instance, take a surjective open map. It will be automatically a quotient map, surjective, open and continuous, automatically it is quotient map. Why? Because take any subset U in Y. Suppose that the inverse image is open then I want show that U is open. But that follows because f of the inverse image of U under f is U itself; $f(f^{-1}(U)) = U$. $f^{-1}(U)$ is open and f is open therefore U is open. So this is the `only if' part.

Now, f is continuous so if U is open then $f^{-1}(U)$ is open. That is easy because f is continuous. So openness gives you that it is a quotient map. Also, exactly the same way, by using De Morgan law, if we have a closed map, closed, continuous and surjection, that is also a quotient map. You may be under the impression that these are the only examples. No. Of course, quotient maps are much more general than open maps or closed maps.

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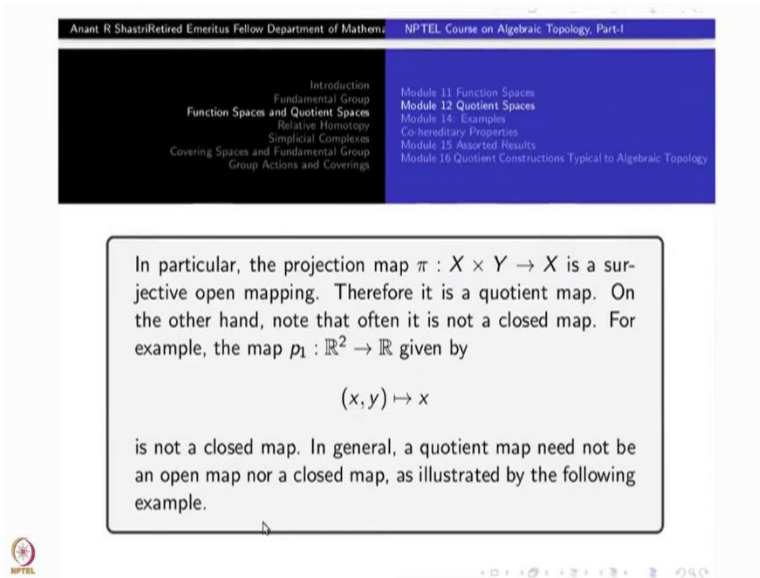


But open maps are plenty again namely all coordinate projections from any product space into any factor, X × Y to X, X × Y to Y, etc. From $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$ etc., to any of these coordinate spaces they are all open maps-- the projection maps. But be careful-- projections maps are not closed, you may be knowing that the standard projection X × Y to X , or, X × Y to Y for instance, $\mathbb{R}^2$ to $\mathbb{R}$ is not a closed map. Just look at the image of the hyperbola xy equal to 1, by the very definition the set of all xy equal to 1 is a close set inside $\mathbb{R}^2$, its image inside the x- axis is all the real numbers minus the 0. That is not a close set.

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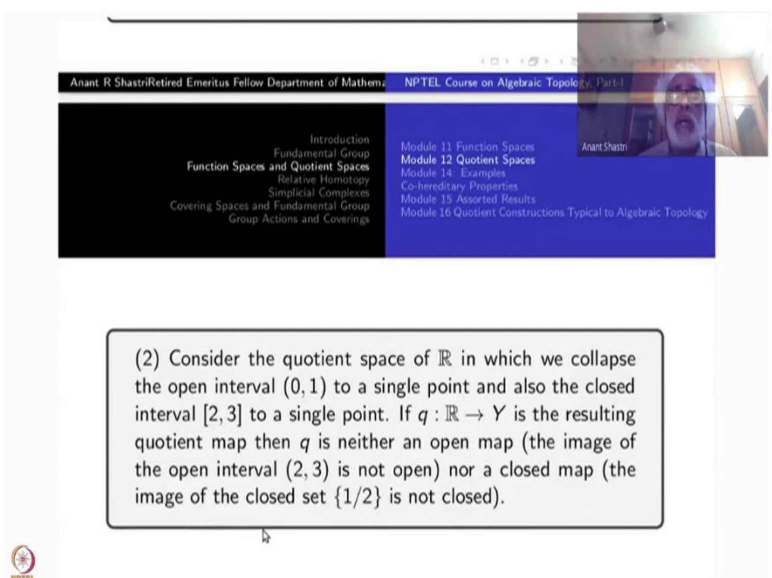

I will give you some examples of maps which are quotients maps but they are neither open nor closed. So I have just cooked up this example, this is not a standard one, it is not occurring naturally, there are such natural things also. Here just for simplicity, in one single example, I will show you that a quotient map need be neither open nor closed. So what I do? I take the whole of real line or some large interval. So let me take the whole of real line, then look at the open interval 0 to 1, all the points in (0,1), I identify them to a single point. 0 less than t less than 1, all of them will be identified to single point.

Similarly, look at the closed interval [2,3]. All the points in between, 2 and 3 included, identify them to another single point, to a different point, another single point. So these two are different points not the same points. So this is all the relation; rest of the points are not identify with any other point. Look at the quotient space Y obtained and let q from $\mathbb{R}^2$ to Y be the resulting quotient map, so Y is the quotient space.

Then I want to say that this map q is neither open nor closed. Why it is not open? The open interval 2 to 3, the image of that under q, is equal to one single point namely q of the closed interval 2 to 3. The closed interval 2 to 3 and open interval 2 to 3 have a same image point. That point is not an open set in Y. Why? Because its inverse image is closed interval 2 to 3 which is not open inside $\mathbb{R}$. Therefore q of the open interval 2 3 is not open-- that means q is not an open mapping.

It is not a closed mapping either. Here, you come to the first part, take any point in 0 to 1. Any single point is close set in $\mathbb{R}$, but the image which is the same thing as the image of the open interval 0 to 1, one single point, open interval 0 1 goes to one single point. That point is not a closed point because its inverse image is the open interval 0 1 which is not a closed subset of $\mathbb{R}$. So this map is neither close nor open.

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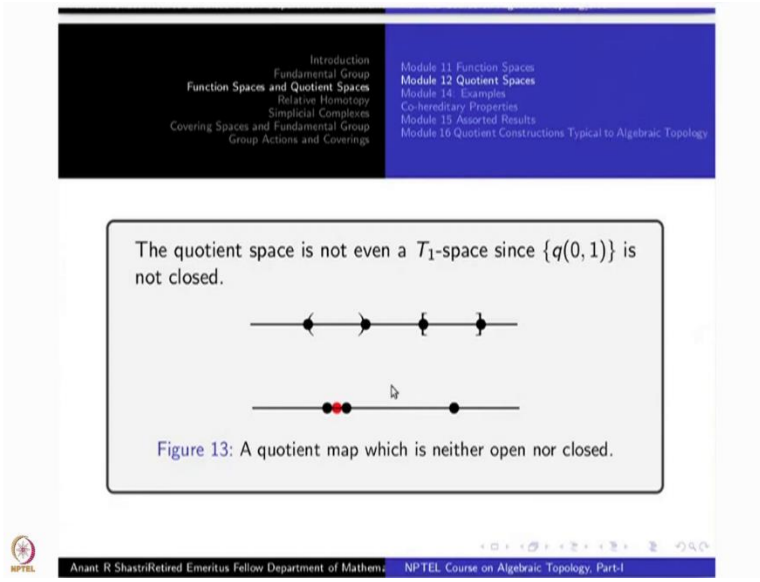


So here is a picturesque representation of what I have done. This is the open interval 0 1 and this is 2 to 3 a close interval. This entire close interval is going to a single point. Here the open interval is coming to this red point, but the end points of this interval, they have come just near to that point. What is the meaning of this? If you take the closure of this point in the quotient topology that will contain both the end points. So in particular the red point, the singleton red point is not a closed subset. …..

In particular, the quotient space Y is not a $T_1$ space, because this image of 0 here and the image of 1 here cannot be separated by open sets. If you take any neighborhood of the red point, it will contain both of them. So Y is not a $T_1$ space. So singleton 0 is not closed-- this is enough to conclude that it is not $T_1$ space.

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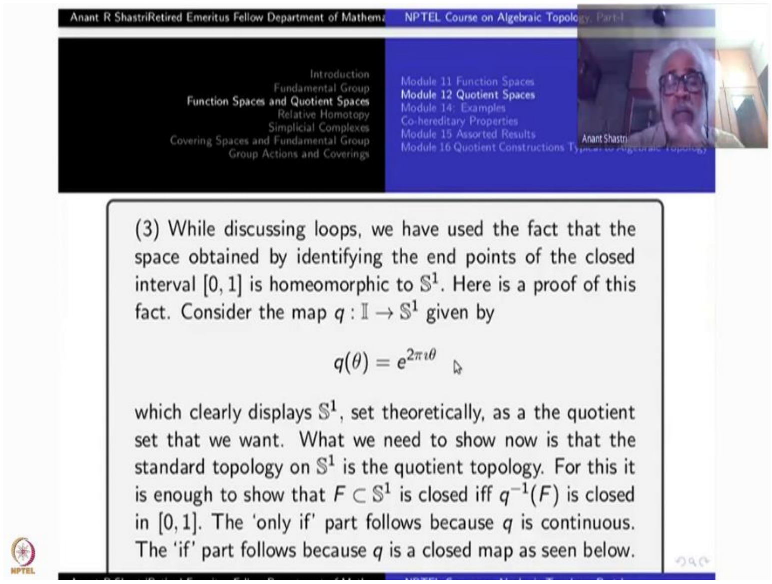


Finally, here is an another example. While discussing loops we have used the fact that the space obtained by identifying the end points of the closed interval 0 1 is homeomorphic to $\mathbb{S}^1$ . So let us write down a neat proof of this one, by taking the exponential function on the interval [0 1]: $q(t) = e^{2\pi\imath t}\theta$. Then what happens to q(0) and q(1)? They go to the same point and no other point is identified with any different point.

In other words, q is injective on the open interval (0 ,1). Therefore, we look at the inverse image of a point in $\mathbb{S}^1$, only the inverse image on 1 contains two distinct points, all others are singletons. So this is the same thing as saying that the identification set, as a set, it is $\mathbb{S}^1$. 0 and 1 are in the same equivalence class and rest of them are in different class, that is the meaning of that. What we have done? by identifying 0 and 1 in the interval close interval 0 1., we got the set $\mathbb{S}^1$.

So, set theoretically we are fine. So what we have to verify is that the subspace topology of $\mathbb{S}^1$ coming from $\mathbb{R}^2$ is the quotient topology on $\mathbb{S}^1$ coming from $\mathbb{I}$ via this map q. This is what we have to verify. In other words, q is continuous function is fine, because we know that the exponential function is continuous, it is surjective is also fine. What we want to say is, q itself is the quotient map.

If you know q is the quotient map then our justification, our claim that identifying 0 and 1 inside the closed interval 0 1 to a single point ,whatever space you get, is homeomorphic to $\mathbb{S}^1$. So what we have to observe is, we have to verify that this exponential function is a quotient map. So we know that this can be done if we verify that this is a close map because we have observed that close maps are quotient maps. So that can be done very easily.

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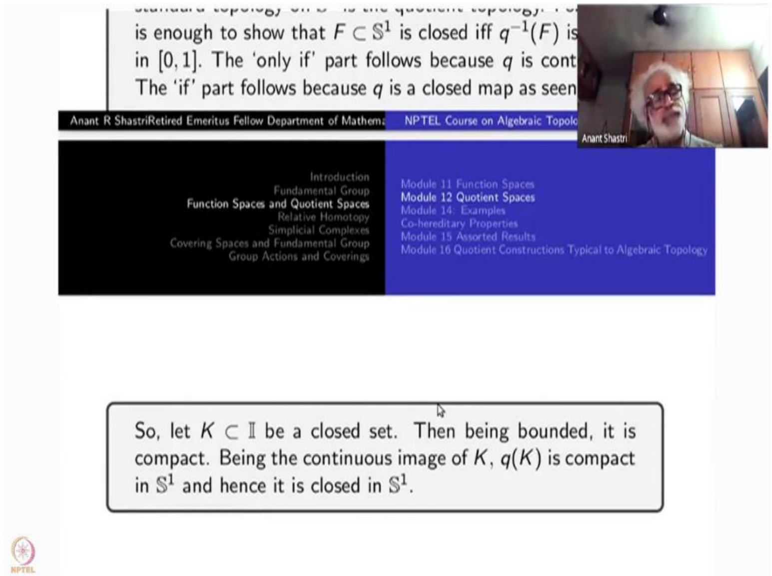


Take any subset K of $\mathbb{I}$ which is closed. K is bounded set because $\mathbb{I}$ is bounded, bounded and closed subset of the Euclidean space $\mathbb{R}$, is a compact space. K being compact the image of K under continuous function q, q of K, must be compact. A compact subset of a metric space, $\mathbb{S}^1$ with the standard topology, so it must be closed subset. So, q(K) is closed. So closed set goes to closed set means the function q is closed. And that is the end of this argument here. So we will study more of quotient spaces next time. Thank you.