Introduction to Algebraic Topology (Part 1)
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Lecture 11
Function Spaces

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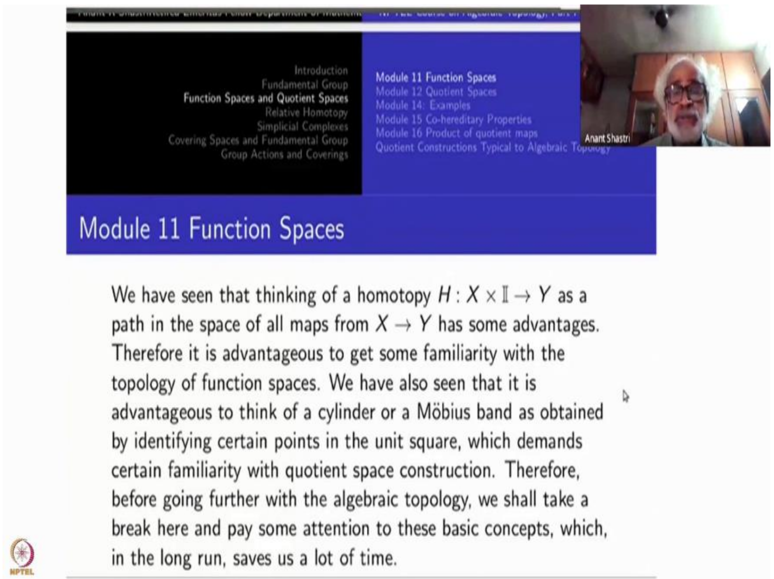

So welcome to all of you. As promised, today we will take up one topic in point-set-topology and then we will take one more topic which we will discuss very thoroughly in point-set-topology because these are very much used in algebraic topology. The first one is our function space. Recall that when you have a homotopy from some X cross to Y then you can think of this as a family of functions from X to Y indexed by the interval $\mathbb{I}$. Each function is definitely continuous therefore it belongs to the set of continuous function from X to Y.

That can be thought of as the space of the product of Y taken X times and then you have the product topology. But the product topology is not very good because it corresponds to point-wise -convergence. What we need is some kind of uniform convergence and the correct thing one has to look up for is…. different topologies on these function spaces are needed depending upon your requirement; that is why the study of function spaces, ---- the one which is more useful in algebraic topology is the so called compact-open-topology. Today, I would like to introduce you to that aspect. Later on we will talk about quotient spaces also.

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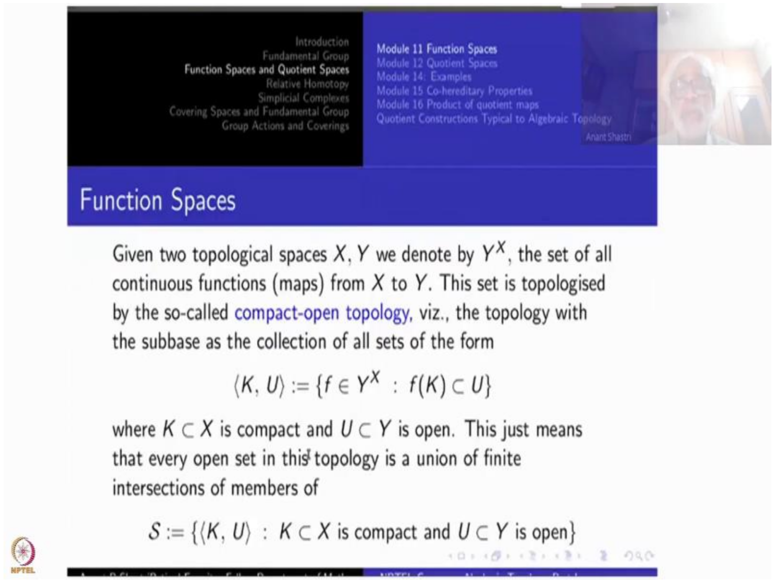

Given two topological spaces, let us denote by Y power X the set of all continuous functions from X to Y. In point set topology this notation is used to denote the set of all functions from X to Y, namely the product of Y taken X times, X many copies of Y. Since, we are not interested in all functions we are only interested in continuous functions, because X is not an arbitrary indexing set here but a topological space itself, and we are interested only in continuous function we will use this notation, the simplest notation for the most-used concept.

So there should not be any confusion with this one. So Y^X will denotes all continuous functions from X to Y for us. So now we put a topology on this o Y^X, called compact open topology. This topology is described by describing a sub base. And the sub base consists of these sets like inner-product notation bracket $\langle K, U\rangle$-- set of all those functions which are continuous from X to Y such that they take the compact set K inside U.

So this should remind you that we are more or less using some kind of continuity --epsilon delta condition, U playing the role of epsilon and K playing the role of delta. But this is not point -wise, this should be for all of K, so that is why this will correspond to uniform continuity--- uniform continuity on compact sets.

So this is the catch here: the condition $f(K) \subset U$ captures the uniform continuity aspect. So these $\langle K, U\rangle$ will form sub base just means that I have to take finitely many such sets, intersect them and look at all such sets, they will form a base. Thus, the topology on this Y power X is now such that an open set is an arbitrary union of finite intersections of members $\langle K_1, U_1\rangle, \langle K_2, U_2\rangle, \ldots, \langle K_n, U_n\rangle$, Ki are compact and Ui are open. So this the topology.

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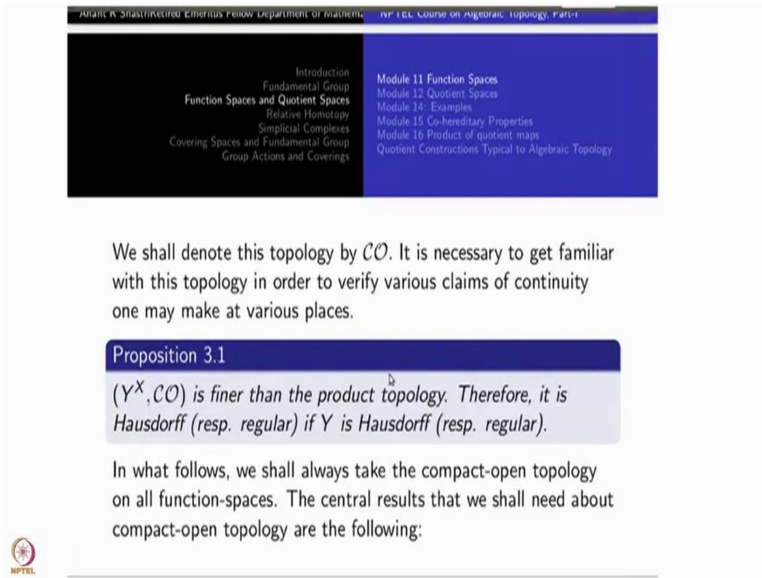

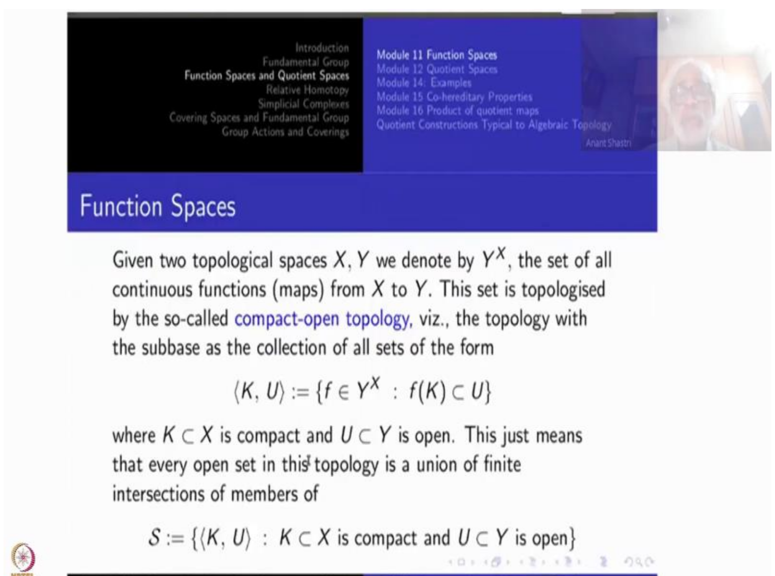

So for a while, and whenever it is unclear what topology we are using, we will use this notation CO or compact open topology. Once we are familiar with this we will not even mention it. Just like the way we are all the time writing just $\mathbb{R}$ with its usual topology. So let us look at the first statement here. Y^X with compact open topology is finer than the product topology. Therefore, it is Hausdorff (respectively, regular), if Y is Hausdorff (respectively, regular).

If Y is Hausdorff Y power X is Hausdorff. Any topology which is finer than the product topology then that will be also Hausdorff. When I say just Y power X it is the product topology. Y power X with CO is finer. I am claiming that will be Hausdorff. (Similarly, with some more effort, we can show that if Y is regular then the CO topoology on Y^X is regular.) So I have to see why this is finer than the other. To see that, you see the single point is always compact, therefore in this collection here, single point comma U, these are already there and what are those? They are the sub basic open sets for the product topology.

Therefore, this collection has already more elements in it than the standard sub base topology for the product topology, over. Singleton x is compact that is all I am using. So first proposition is very nice. You also know that there are more open sets in C O than in product topology. Therefore, convergence here will be stronger. Point-wise convergence may not be good enough. A sequence of functions which is point-wise convergent may not be uniformly convergent, you know that. Whereas if you have uniform convergence, then there will point-wise convergence also.

So if some sequence of functions is convergent in CO topology, in the ordinary product topology also it will be convergent because this is finer than the product topology. The central result that we want is summed up in one single theorem three parts of that. The second part is the most important one for us which is of immediate use. So let us be done with it and that is all about function spaces that we are going to discuss.

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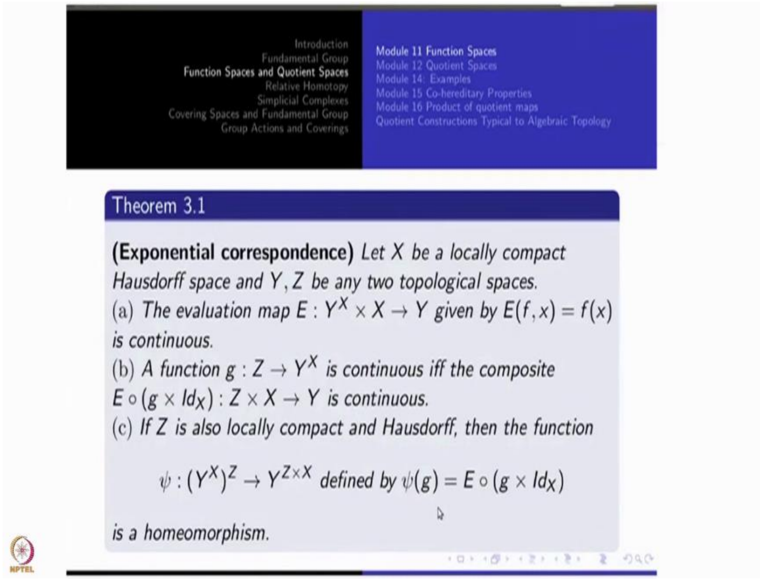

So this is called the exponential correspondence. So here is an important extra hypothesis: namely, X is locally compact Hausdorff. Y and Z could be any topological spaces. For simplicity you can assume Y and Z are also locally compact if you do not remember where to put the condition of local compactness, but assuming Y and Z are locally compact does not help you at all. X must be locally compact and Hausdorff.

So what is involved here is that evaluation map E, Y power X cross X to Y. What is this evaluation? Evaluation of f at x in X. f is a function from X to Y. The second slot is X -- E of f comma x is f of x. So this is what evaluation map is. It will be automatically a continuous function. In fact, this is continuous even if you take, even if you take here the product topology. This will be nothing but the projection map to one of the factors. This x is an element of X, so it is the projection map to x-coordinate. Since I have put a stronger topology here, automatically this will be also continuous. So this is the way you have to see but you can directly prove that also, so evaluation map E is continuous.

Now look at the function g from Z to Y power X. A function g is continuous if and only if ---you take g cross identity of X from Z cross X to Y power X cross X is continuous. So, from Z cross X you land here with g cross Identity of X and then follow it by the evaluation map. If that composite is continuous g will be continuous. If g is continuous the product is continuous composite is continuous is easy.

What is important here is if the composite is continuous then g is continuous. So this is the key thing which justifies the classical way saying `parameterized family' and so on-- it is formalized here. So let us see what is the proof of this one. Before that let us look at the third statement here. This actually tells you the law of exponentials, valid for space of continuous functions, just like it is valid for sets. You must be knowing that the cardinality of Y power X power Z is the same thing as cardinality of Y power cardinality of Z into cardinality of X.

This is same thing as A raise to B raise to C is A raise to B C for numbers. Same thing is true for cardinality, when you take these things for all functions. But now, this is also true for continuous functions as well- is the beauty here. There is the condition, namely, Z is also locally compact Hausdorff, the extra hypothesis there, then the function space Y power X raise to Z ---what is this one? This is the set of all continuous functions from Z to Y power X, which occurs in, statement (b) here. That corresponds to under (b), corresponds to a function from Z cross X to Y. If that is continuous, this will be continuous and vice versa under the exponential correspondence, Psi of g is E composite g cross identity--- which is just like here. So (c) is actually a consequence of (b) but we will have to work out. What happens is, as a bijection it is fine but this is actually homeomorphism is what you have to see. So let us go through these proofs one by one.

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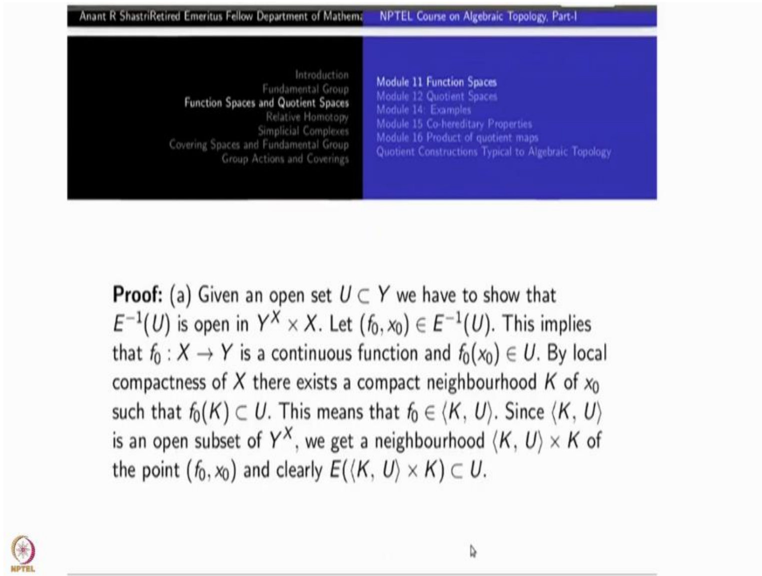

So I am writing a full detail of why the function evaluation map is continuous. I have given you the idea why it is continuous. Let us write down the full detail here. Given an open set U inside Y, we have to show that E inverse of U is open in Y power X cross X. So take a point $(f_0, x_0) \in E^{-1}(U)$. I must produce a basic open set which contains the point f naught, comma x naught and contained inside E inverse U.

What is the meaning of this belongs to E inverse of U? Notice that f naught is first of all a continuous function because it is in Y^X. And f naught of x naught is inside U, because E of that is inside U. Now you see, because we have assumed that X is locally compact , there exist a compact neighborhood K of x_0 such that $f_0(K) \subset U$.

This means that $f_0 \in \langle K, U\rangle$. This $\langle K, U\rangle$ is an open subset of Y^X. We get a neighborhood $\langle K, U\rangle \times K$ of the point $(f_0, x_0) \in Y^X \times X$. And clearly E of K comma U cross K is contained inside U. So K U cross K, K is a neighborhood, compact neighborhood, I have found a neighborhood here and a neighborhood here for x naught such that E of that is contained inside U. See, you had to find a neighborhood cross a neighborhood here. So that is the product topology from Y power X cross X, that is why.

So this is a clear cut proof ---no hand waving.

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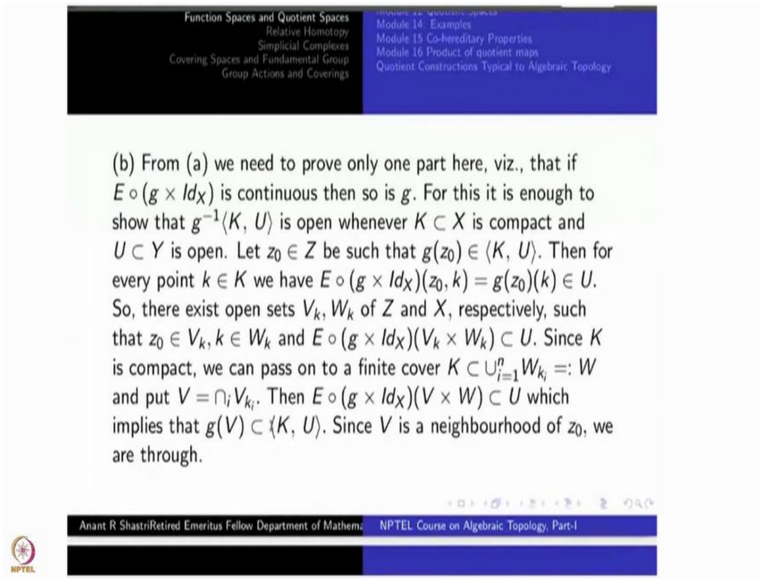

Now let us go to (b). We need to prove only one part because we have already proved E is continuous, so, if g is continuous then g cross identity is continuous, composing with E will be continuous. So only the converse part here remains. Suppose this is continuous then we have to show that g is continuous. It is enough to show that g inverse of a sub basic some basic open set is open. What is a sub basic open set? K is compact and U is open, take $\langle K, U\rangle$. Inverse image of every sub basic open set is open then the function is continuous.

So how to show that g inverse of $\langle K, U\rangle$is open? So take a point here say z naught belonging to Z. Let us say g of z naught is inside $\langle K, U\rangle$. That is the meaning of z naught is in g inverse of $\langle K, U\rangle$. Then for every point k inside K what we get is: E composite g cross identity of z naught comma k is by definition, g of z naught operating upon k. Remember g of z naught is a map from where X to Y.

So g is map from Z to Y^X so g(z_0) itself is a map from X to Y. g(z_0) (k) belongs to this open set U. So there exists open sets V_k, W_k of Z and X respectively, ---because it is the product topology, --- so a neighborhood here and a neighborhood here respectively, -- such that z naught is in V_k, k is in W_k, and $E \circ (g \times Id)(V_k \times W_k) \subset U$. If a point goes inside this U, then since the composite is continuous, so the entire function operating on the product neighborhood is contained inside U. --- follows from continuity of E composite g cross identity.

Now use the fact that is K is compact. We can pass onto a finite cover. K is contained in the union $\cup_{k\in K} W_k$ For each point k inside K we have found a W suffix k. So all this W suffix k's cover K . Therefore, I say there is a finite cover $K \subset \cup_{i=1}^{n} W_{k_i}$=: W. So let us call this union W. On the other hand, let us take the neighborhood for this z naught to be intersection $\cap_{i=1}^{n} V_{k_i}$. Then E of g cross identity of this intersection here cross union there, that will be contained inside U, because once z is in the intersection, this will be true for every element no matter where you take the second coordinate x in the union, it will be in one of W suffix k_i. So it is contained in here.

This implies g of V is contained inside K comma U, because V is inside W, the whole W goes inside U, so g of V will be contained inside K comma U. What is the meaning of K comma U? Every element takes K inside U. But g(v) is taking the whole W inside U. Therefore they are taking K also inside U. Since V is a neighborhood of z naught, we are done because we have found a neighborhood here, g of that is contained inside K U. g is titled with Z naught contained inside KU and found the neighborhood now. This shows that g is continuous. Is that clear?

Student: Yes, sir.

Professor: The compactness, the local compactness of X is used here.

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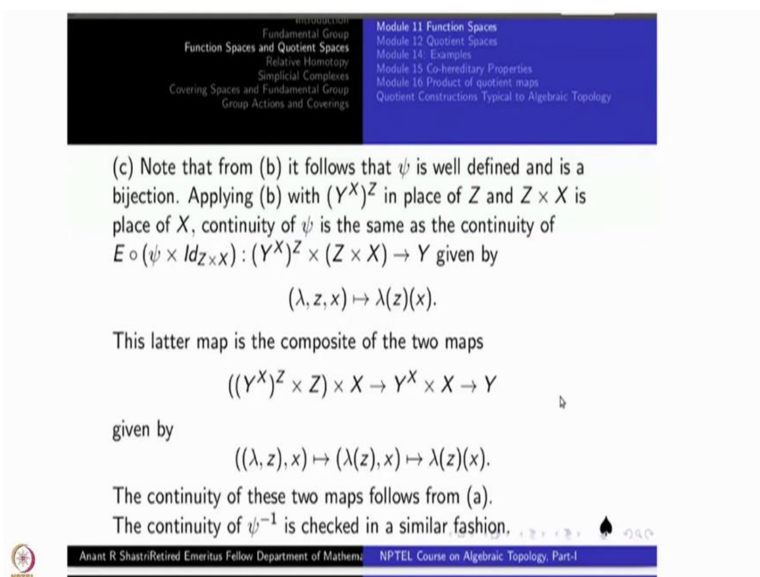

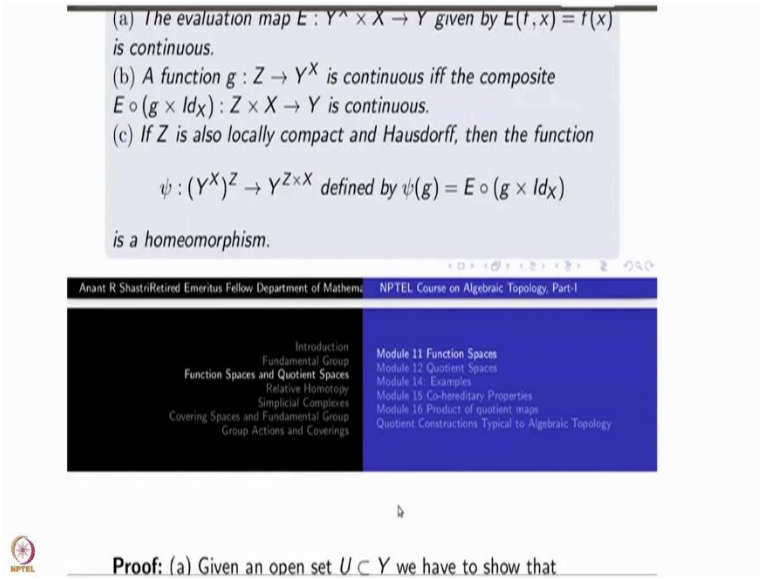

Now you see in (c), in statement (c) we have local compactness assumption on Z also, you know why? because in this statement Z is also going up exponentially, it is also a super script. So it is easy to remember--- whenever the space goes up here on the super script, those spaces must be locally compact. The space at the bottom could be anything, This is Y power X-- so if something should work properly X must be locally compact, this whole thing is raised to Z so Z must be locally compact. So Z , X both of them are locally compact so product is also locally compact.

So that is why it is necessary that Z is also locally compact. Y could still be anything, so I am telling you the trick to remember things--- whichever go up must be locally compact. So let us prove (c). The idea of giving all these proofs is so that you will get familiar with compact open topology. Otherwise one can just take the statement and go away. Finally, we are going to use only the statements.

From (b) it follows that the map, whatever you are writing here, psi, this map, how it is defined? psi of g is E composite g cross identity. Whenever g is continuous, you have defined psi of g as a composite of continuous functions, so you do not need (b) here. But to show the inverse is well defined, that will need the part (b). It is well defined and its inverse is also well defined that is important. It is well defined and is a bijection, so for that bijectivity part, you need part (b) there. Applying (b) with Y power X raise to Z in place of Z, and Z cross X in in place of X, continuity of psi is the same as the continuity of, (see I am applying (b) here), E composite psi cross Id of

Z cross X. All the way from Y power X power Z cross Z cross X to Y power X cross Z cross Z cross X to Y.

If I show that this is continuous then it will imply that psi is continuous. So this is where I am using (b) very strongly. What is this map? This is some lambda here from Z to Y power X, z is a point of Z, x is a point of X. So lambda of z of x-- we take lambda z, it is a function from X to Y we can evaluate it on X. This is lambda of z of x . By (b) it will follow that you know that if this whole thing is continuous then this is continuous. But why is this continuous? Continuity of this one is obvious because it is again from (b).

So this latter map is nothing but we can think of this as Y power X power Z cross Z bracket, (I am putting the bracket other way around) cross X. So from here where are you going? Y power X cross X so this is continuous from here to here. So this is given by lambda of z of x, lambda of z we are evaluating on X, so it is like evaluation carried twice. That is why again from (b), and from (a), it is continuous.

So that proves that the evaluation map is, the evaluation map is continuous, therefore psi is continuous. The proof of psi inverse is continuous is exactly similar. Actually, all steps are reversible. Everything is if and only if you can go ahead. So I will leave it to you as an exercise you have to spend some time to get familiar with what is going on ---playing with these exponential function alone. So you can check that psi inverse is also continuous.

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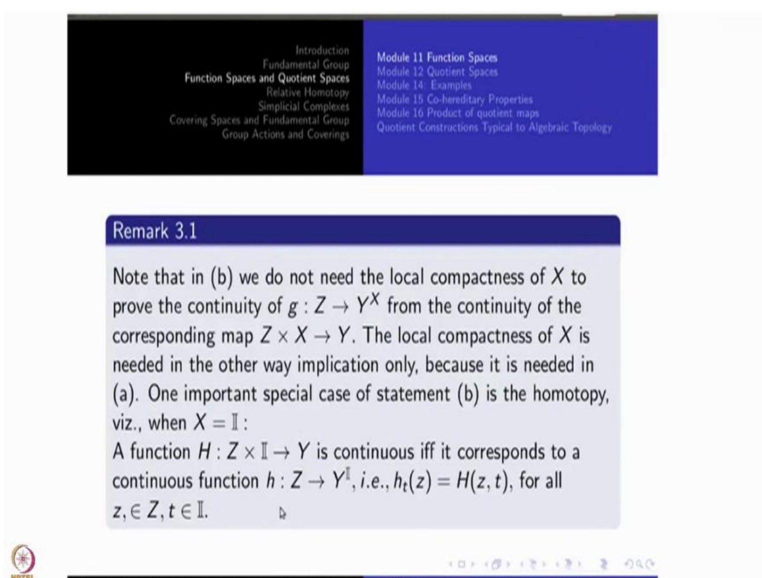

Note that in (b) we do not need the local compactness of X to prove that the map g from Z to Y power X, under the continuity continuity of the corresponding map. The local compactness of X is needed in the other way implications because it is needed in part (a). One important special case of statement (b) is the homotopy version which we will be using all the time. This is the special case where X itself is the interval I.

Z cross I to Y in place of Z cross X to Y. Z cross I to Y is a homotopy. This is continuous if and only if it corresponds to the function, little h I put, little h here remember I put little h t, little h t are all functions from Z to Y. So now this family is, whole family is a function from I to Y. So h itself is a function from Z to Y raise to I. So H is a function H of z is a function from I to Y operating on point t is H(z,t). so that is the meaning of this.

This is a very special case and of course, the suffix up here must be locally compact but we actually have a compact Hausdorff. So of course, it is locally compact Hausdorff. Therefore, joint continuity of this one ensures you that a family like this is just one single function, continuous function from Z into the function space Y raise to I, set of all paths is in Y with compact open topology. I think we will stop here because the next one is a different topic namely, quotient spaces, which we will discuss in the next module.