Introduction to Algebraic Topology (Part – I)
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Lecture 10
Van Kampen's Theorem

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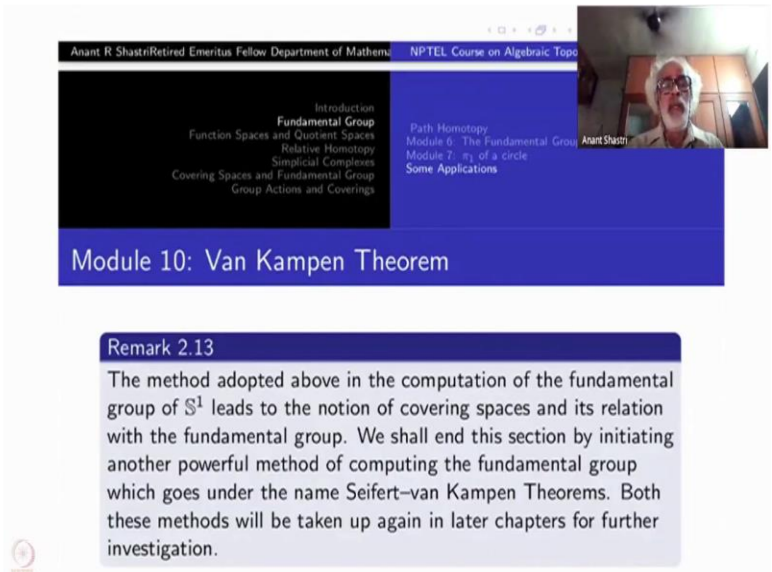

Welcome to all of you. So in module 10 as promised earlier, we will do what is called Seifert-van Kampen theorem. The simplest form will be taken today. Later on, we will see more advanced versions of this one. Seifert was a German mathematician. Van Kampen was Dutch who worked in America. (Most of the time in America this result is just called Van Kampen's theorem. They do not take the name of Seifert.) But to be precise they did it independently, ---not as a joint work. And they did it with different assumptions, in different contexts. So today, we will try to cover up simpler versions but which applies (1:22) to both the contexts. That is the whole idea.

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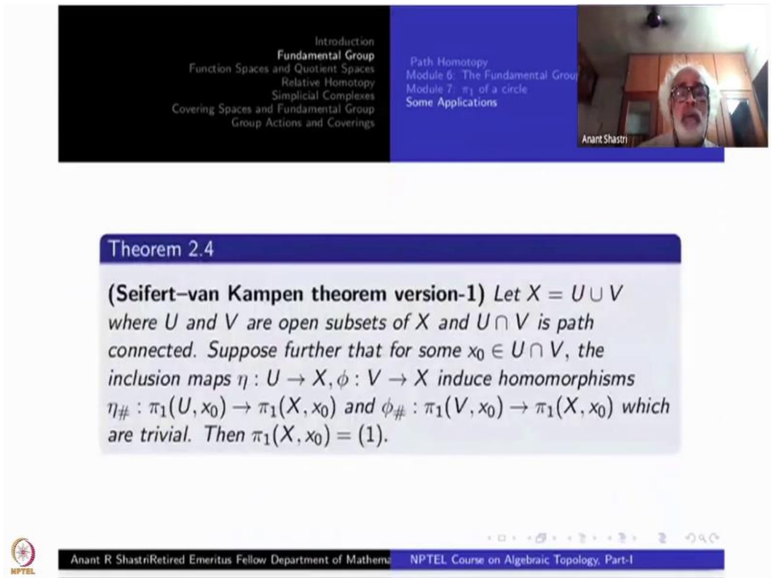

So this is the statement. Let X be written as the union of two open sets U and V. Both U and V are open and the intersection is path connected. Once you say path connected you pick up a point in the intersection, say x naught. And let us assume that the inclusion maps of U in X and V in X, η and ϕ, let us call them, they induce homomorphisms $\eta\#$ and $\phi\#$ on π_1.

They induce homomorphism $\eta\#$ and $\phi\#$ which are trivial homomorphisms; everything goes to the identity element of π_1 of X, x naught-- the trivial homomorphisms. They are not inclusion maps on fundamental groups level, they are trivial homomorphisms, $\eta\#$ and $\phi\#$. η and ϕ themselves are just inclusion maps. So this is the assumption --- $\eta\#$ and $\phi\#$ are trivial homomorphisms.

X is covered by these two open sets. Intersection is path connected. Under this, we can conclude that the fundamental group of X at x naught itself is trivial. You understand that the whole thing. For example, suppose π_1 of U x naught itself is trivial, then the homomorphism will be trivial. Similarly, π_1 of V x naught itself is trivial. Then again the inclusion induced map will be trivial. Inclusion induced map will be trivial when you pass to the fundamental group.

So if U and V are simply connected then union is simply connected, provided the intersection is path connected. So this is one way of remembering it. The theorem is slightly a generalization. U and V themselves may not be simply connected but all those loops inside U as well as inside V separately thought of as loops inside X, they are null-homotopic. So these two are different statements. U itself is simply connected then they will be null-homotopic inside U. So that is a stronger hypothesis.

So under weaker hypothesis, namely, inclusion induced maps are trivial we will get that the fundamental group of X itself is trivial. In other words, if fundamental group of X was not trivial, some element of π_1 of U, some loop in U or something in V must have been non-trivially mapped into π_1 of X. This is the meaning of all this. So let us prove it now by totally elementary methods---totally elementary methods; no covering spaces, nothing of that kind.

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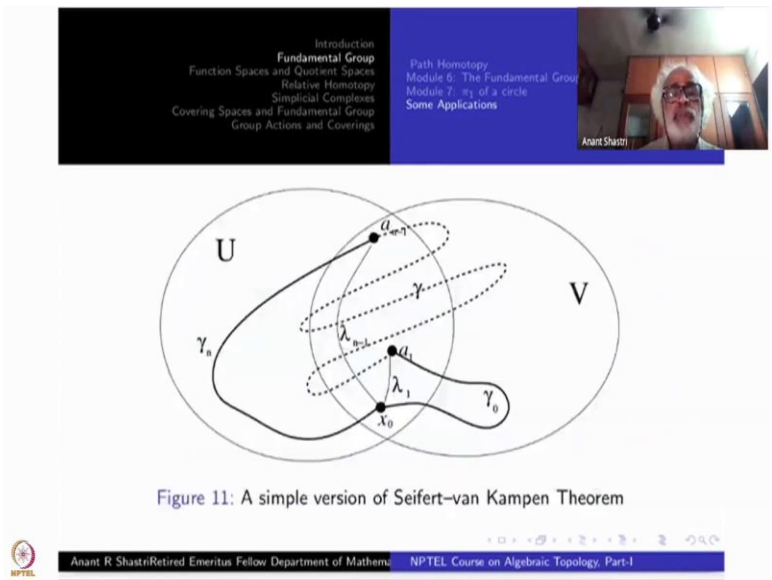

Figure 11: A simple version of Seifert–van Kampen Theorem

Here is the picture of what is happening. So U and V are open subsets like this. Their intersection, I am assuming, is path connected. The entire space is your X. Now I have chosen, in the intersection, I have chosen a point x naught. So the idea is if I take any loop inside V like this, when you use the entire space it is null-homotopic. Similarly, any loop which completely lies inside U, the inclusion map will take it to X. There, it is null-homotopic. That is the assumption.

So suppose I start with a loop which is γ, like this and I have divided, this whole thing is γ; dot, dot, dot; dot, dot, dot, and coming here and then coming back here. Only the last two portions I have drawn with thick arrows, thick lines. So this is my gamma naught, this is my γ. So what I have done is I start tracing gamma naught, this part of gamma. Now I see that it is going through, it is going into V part from U part. So here I stop and cut it, I mean, I am making a subdivision. Remember we can make subdivisions.

Now from this point onwards, I am inside U, so somewhere here, I have to stop because now I will be going inside V. So you can call this $\gamma 1$. Now I go here and call this part $\gamma 2$. Again I am entering into V so I say $\gamma 3$, $\gamma 4$, blah, blah, blah, finally γn coming back. What I have done? I have cut down the path, the loop, into subdivisions such that each path is either inside U or inside V.

Once I have that, I use the path connectivity of U intersection V. Look at this. After going here I have stopped somewhere here. Where did I stop? I have stopped inside U seeing that I am now going inside V. So this point is already in U intersection V. The first point on the path, I mean zeroth point is this one, the first; a1 is already inside U intersection V. So this $\gamma 0$ is only a path. Now I complete it to a loop by joining a1 to x naught. This entire path $\gamma 0$ composite $\lambda 1$ is a loop inside V situated at x naught.

Therefore, by my hypothesis if you think this as a loop inside X then this is null-homotopic. Because this part is null-homotopic I can just ignore all this part and just go directly from here to here and look at the rest of them. Because this part is null-homotopic. You can add or subtract, add or delete this part. There is no problem. So delete it. What has happened? The number of divisions in γ has reduced. So by induction the whole thing is null-homotopic.

What is the induction-starting point? That there is no division. That means the entire thing is inside U or inside V. Then it is null-homotopic by the hypothesis. So that is the starting point of the induction. So if I cut down this one, then I have only from here to here directly--you know this $\gamma 0$ is not there. First thing is this whole thing is $\gamma 1$. This entire thing is inside U now. Then the $\gamma 2$, $\gamma 3$, $\gamma 4$, $\gamma 5$, 1 up to n instead of 0 to n. So that is only n - 1 parts. Therefore, by induction, the proof is over.

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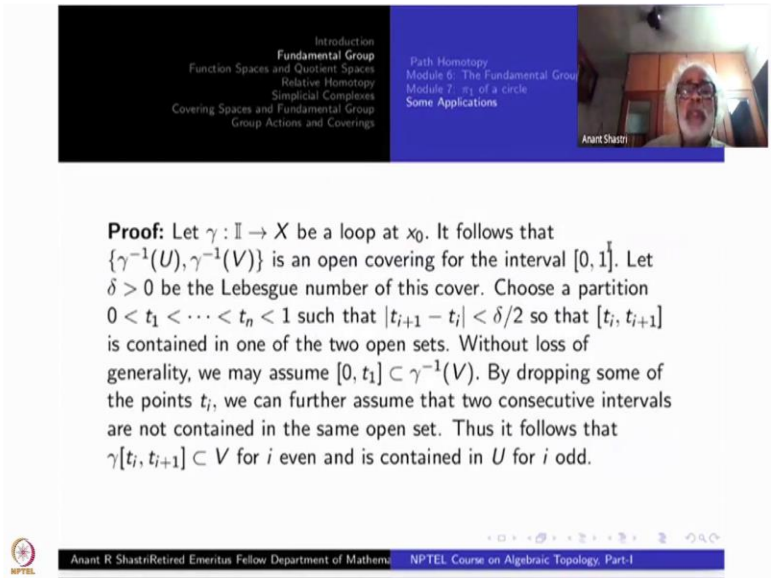

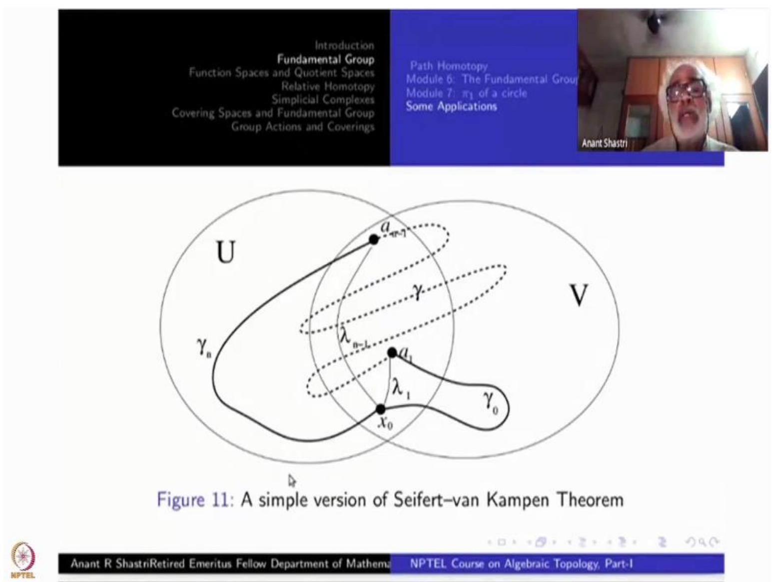

Now let me just go back here and see what details I have written, how to write down this. Start with a loop γ in X at the point x naught. U and V cover X, therefore γ inverse of U and γ inverse of V will cover the interval [0,1]. Just like we did last time, we can choose, you know, a Lebesgue number δ>0 and a number which is slightly smaller than Lebesgue number for this covering. Then I choose the length of these intervals small enough, I am cutting it into n parts, where $\frac{1}{n} < \frac{\delta}{2}$, then what happens is, the respective sub intervals $[t_{i-1}, t_i]$, they will be all of length less than $\frac{\delta}{2}$.

That would mean that each closed interval is either inside γ inverse of U or γ inverse of V. This is same thing as saying that γ restricted to each of these intervals is either inside U or inside V. The point, starting point x naught is in both of them. It is in the intersection. So the first part namely t0 to t1, this part is in one of them. Which one? Just for definiteness, I have written as gamma inverse of V on the right side, by dropping some of the points, please remember that you might have cut it too small. So t1 to t2 maybe also inside V.

Then you better take both of them together 0 to t2 all of it inside V. So like this you can combine all the consecutive divisions which are inside V till you come to a point from which the next arc is going to be inside U. You can rename t1, t2, … tn in such a way that 0 to t1 is inside V, t1 to t2, (I mean the part of the curve not the interval), t1 to t2 inside next one U and alternatively, U, V; U, V and so on. So that is just for saving some argument. So ti to ti plus 1 is inside V for all even numbers and for all odd, it is inside U, because I start with 0 to t1 inside V. The next one will be odd so it will be inside U. And so on. So, that is the picture here.

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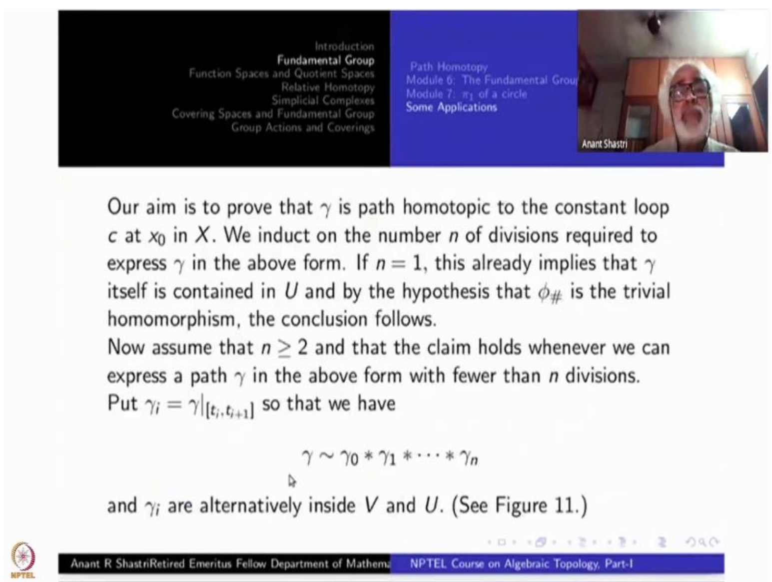

Next step is to show that γ is homotopic to a constant loop, the constant path being c at x naught. We induct on the numbers of the division required to express γ in the above form. If n is 1 this already implies that γ itself is contained inside U or V. Therefore, by hypothesis that $\phi\#$ is a trivial homomorphism, we are done. Maybe I should write this here V, so this should be V. $\phi\#$ check is from π_1 of V to this one. This must be V, this is the conclusion.

Assume now n is greater than or equal to 2, and that the claim holds whenever we can express a path in the above form with fewer than n divisions. So it is an induction hypothesis. Now put γ i equal to γ restricted to $[t_i, t_{i+1}]$. How does γ look like? It looks like γ0 star γ1 star dotdotdot γn. Looks like means what? These two are not the same paths. They are path homotopic.

You remember that. If you subdivide then each of them (you have to express all of them) in terms of paths, -paths have to be all the time parameterized on the interval [0,1]. So you have to take composite of this. Then this is path homotopic. This is what we have seen. And γi are alternatively inside V and U, V, U; V, U and so on, that is the picture that we are seeing.

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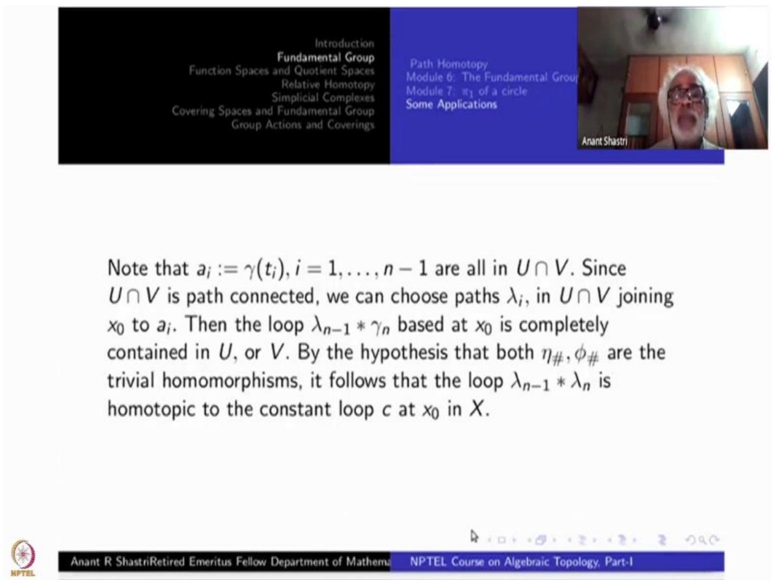

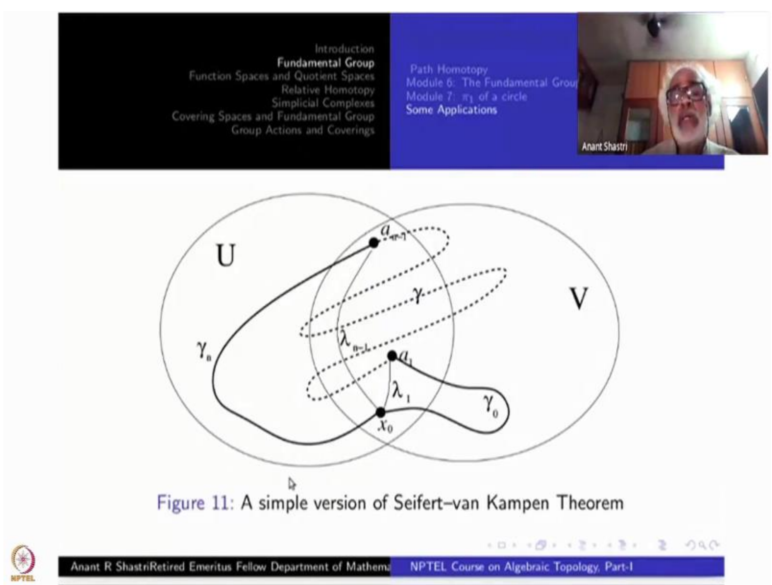

Now look at ai which is equal to $\gamma(t_i)$. ai is in the intersection of U and V. Choose a path λ_i from ai to x naught. Then what happens is: the loop $\lambda_{n-1}^{-1} * \gamma_n$ based at x naught is completely contained inside U or V. Earlier, I chucked out gamma 1 but you can chuck out the last one,--- γn here. Deliberately I have done this one in two different ways.

So here also in the picture, see λ_{n-1} goes like this, come back via γn that is null-homotopic. Therefore, by induction hypothesis, this path all the way going up blah, blah, blah uptil here, coming back, that will have only one less number of divisions. So this will be also null-homotopic. So composite of two null-homotopic things is null-homotopic. So you will get gamma itself null-homotopic. So, this is the way to write down the proof, That is all. Any questions?

Student: No

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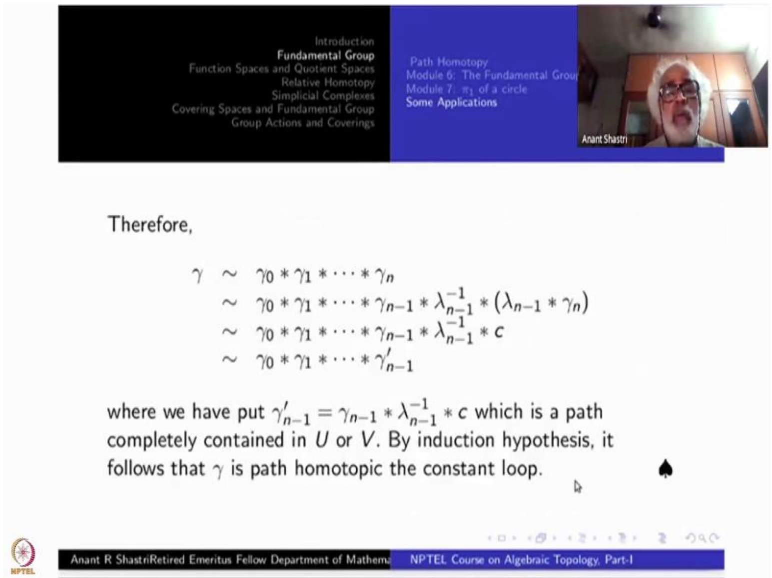

Professor: Here I have written down full detail again. Start with gamma which is divided into n parts. These paths are not necessarily loops. Their endpoints could be different. $\gamma 0$ starts at x naught, ends up somewhere in a1. You have to convert them into loops. So first do not worry about this part. Take the last part, join it with lambda n minus 1. Insert lambda n minus 1 and lambda n minus 1 inverse between γ_{n-1} and γn. This is the inverse of this. So I can insert it because this itself is null-homotopic.

Then you use associativity and put these two things in a bracket. This becomes a loop in either U or V. So you can ignore this one. This is constant, I mean homotopic to a constant. This part together, this whole thing will be inside U or V. So the rest of them together have one less division. So the thing is now in n minus 1. Therefore, by induction hypothesis, this part, viz.,

$\gamma_0 * \cdots * \gamma_{n-1} * \lambda_{n-1}$ is also null-homotopic.

So together the whole thing is null-homotopic. This is the way to write down, so proof is over.

If two open sets are such that they are simply connected and they intersect in a path connected subspace, then the union is simply connected. This is a corollary to this one, as I have indicated. Any questions?

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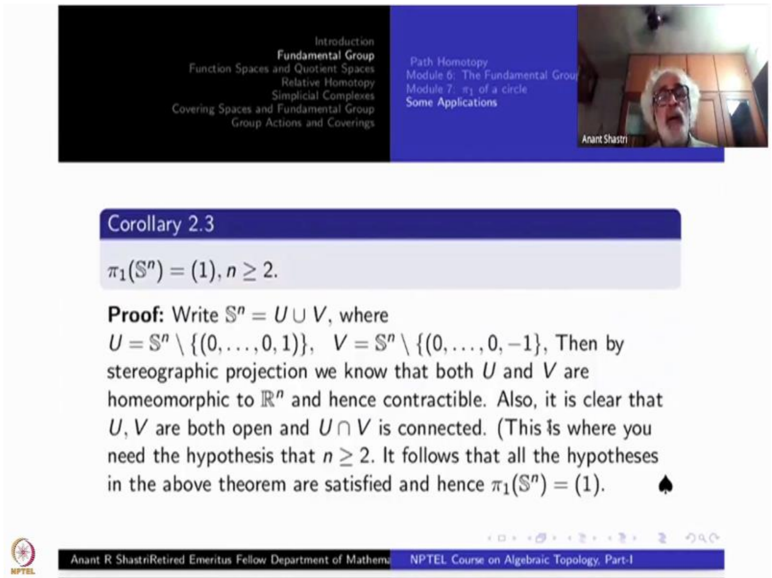

Let us now prove a very interesting result. We have computed π_1 of $\mathbb{S}^1$, now we are computing for all the spheres, π_1 of $\mathbb{S}^n$ for all n greater than 1. --- I am going to compute them in one go. What is it? All of them are simply connected. The fundamental group is trivial. Strictly speaking you should remember that I should pick up a base point here. If I am claiming that this is trivial for one base point it will be trivial for all base points. We have seen that.

Because if you change of base points, the two groups are isomorphic. That is what we have seen. So, I have not written down what the base point. That is all. It is not a mistake. This is deliberate--- just to cut down the notation, that is all. But in my mind it is there, the base point. Whatever base point you take; it is the same. That is why I have not mentioned the base point. $\pi_1(\mathbb{S}^n)$ is trivial, i.e., $\mathbb{S}^n$ is simply connected for n greater than or equal to 2.

For n equal to 1 it is infinite cyclic. For n equal to 0 what is it? For n equal to 0, S naught is not even connected. You take any connected component; again the fundamental group is trivial, so in that sense only n equal to 1 is is a distinct case. In this sense, all other spheres are `simply connected'. However, a space which is not path connected is never referred to as simply connected. In the definition of simply connectivity you first assume that it is path connected. And then put the condition that fundamental group is trivial.

Therefore, you cannot call S naught as simply connected. But fundamental group of S naught taking any point as base point is trivial. That is true. Yeah.

So what is the proof? Proof is very easy. All that you have to do is write the sphere as union of two open sets in a nice way. What do I do? I select the north pole 0 dot dot 0 1. Subtract it. Throw it away. Similarly, take the South pole 0 dot dot 0 minus 1. Throw it away. U and V are open subsets because each of them is $\mathbb{S}^n$ minus a single point. Single points in $\mathbb{S}^n$ are closed. So the complements are open. So, once these things are open the union, you have to check is whole space. U misses one point, that point is already inside V. So U union V is the whole of $\mathbb{S}^n$. That is fine. Now comes the point how does $\mathbb{S}^n$ minus a point look like? So this is where you have to know elementary topology namely, if you remove any one from a sphere, the space you get is homeomorphic to $\mathbb{R}^n$ via stereographic projection.

I hope you know these things. If some of you do not know you can ask your tutors and you must know this before the end of this course, if you have not learnt it in the beginning of the course. If you remove one point from $\mathbb{S}^1$ what space you get is is homeomorphic to $\mathbb{R}$. Same thing happens for all n. You remove one point from $\mathbb{S}^2$; you get a space homeomorphic to $\mathbb{R}^2$. And so on.

In particular U and V are simply connected because they are homomorphic to contractible spaces. So they are themselves contractible. So they are simply connected. Finally, you have to look at what? How does U intersection V looks like? U intersection V is sphere minus two points. Here you have to use the fact n is greater than or equal to 2, not in equal to 1. If n equal to 1 and remove north pole and south pole, what you get is two copies of $\mathbb{R}$, disjoint.

That will not be connected, If you remove one point from the sphere, you get a $\mathbb{R}^n, n \geq 2$ n. If you remove one more point it is still connected. Therefore, the intersection of U and V is connected, actually path connected. So all the hypothesis of the above theorem are satisfied, actually stronger hypothesis namely π_1 of U and π_1 of V are themselves trivial. So what is the conclusion? π_1 of $\pi_1(\mathbb{S}^n)$ is trivial. Is it okay?

Student: Yes

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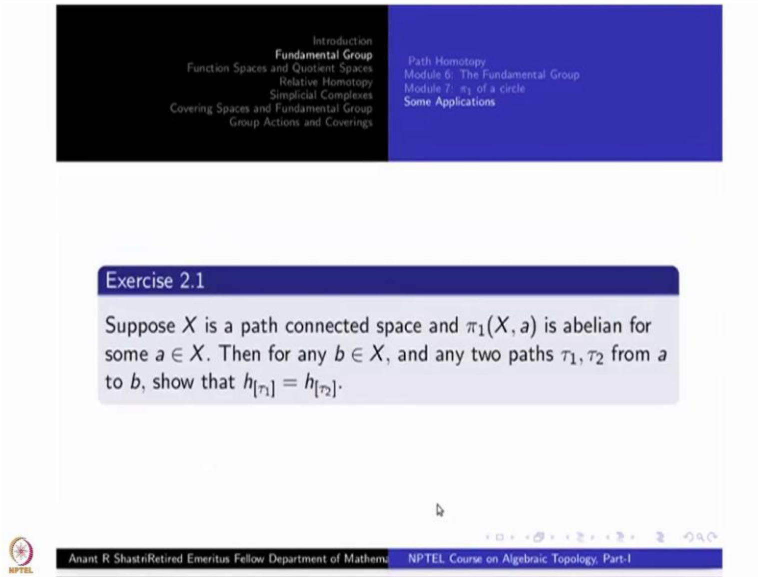

Professor: Now here are a number of exercises. Each of you should try to solve them by yourself. Then only you will know that whether you are understanding this course or not. So you have to rely more on your self-assessments here, whether you take any exam or not, so the tutors will help you in understanding things, if at all you communicate with them, by telling whether your answers are correct or not.

So let me just go through these exercises. These exercises will be separately sent to you in PDF format. Right now you do not have to write down these things. So, the first one is: suppose X is path connected and pi 1 of X, a, is abelian-- commutative group, for some a inside X, a is a some point in X. Then for any b inside X, any two paths, τ_1 and τ_2 joining a to b, your h_{τ_1} and h_{τ_2}, these two homomorphisms are the same. This is what you have to prove, h_{τ_1} of some $[\omega]$ is equal h_{τ_2} of some $[\omega]$, where $[\omega]$ is an element of $\pi_1(X, a)$.

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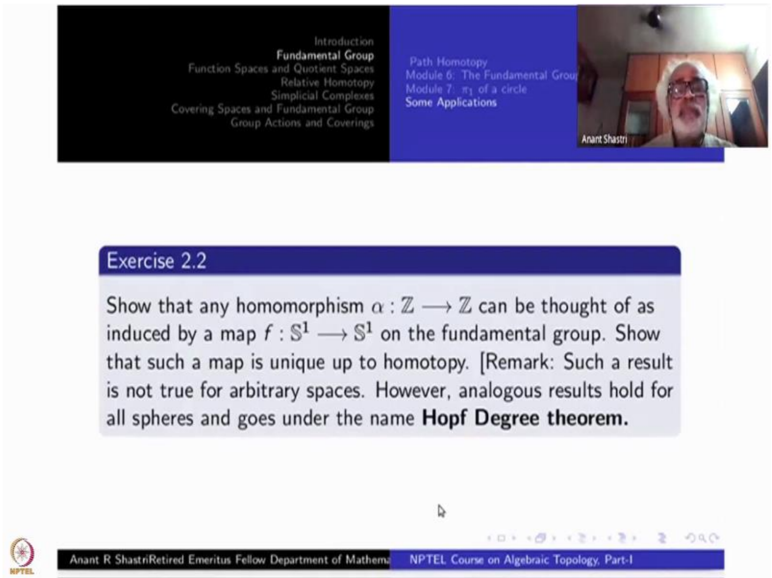

Second exercise is, take any homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$, integers to integers, group homomorphism, you can think of this as induced by a map from $\mathbb{S}^1$ to $\mathbb{S}^1$, given a continuous function from $\mathbb{S}^1$ to $\mathbb{S}^1$. you can fix up some point say, 1, you can also assume 1 goes to 1. Then pass to the fundamental group level, $\pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^1)$. But they are infinite cyclic groups, $\mathbb{Z}$ to $\mathbb{Z}$. That homomorphism will be α. Every homomorphism is given by some map. This is what you have to show.

Moreover, up to homotopy, this a map is unique. Suppose you choose two such maps, f and g such that f check and g check are the same on the fundamental group. Then I want to show, I want to claim that the maps themselves, f and g are homotopic. Then there is a remark that this result is true for all spheres, but not for arbitrary spaces. For spheres, this goes under the name `Hopf degree theorem' which we shall not be able to do in this course.

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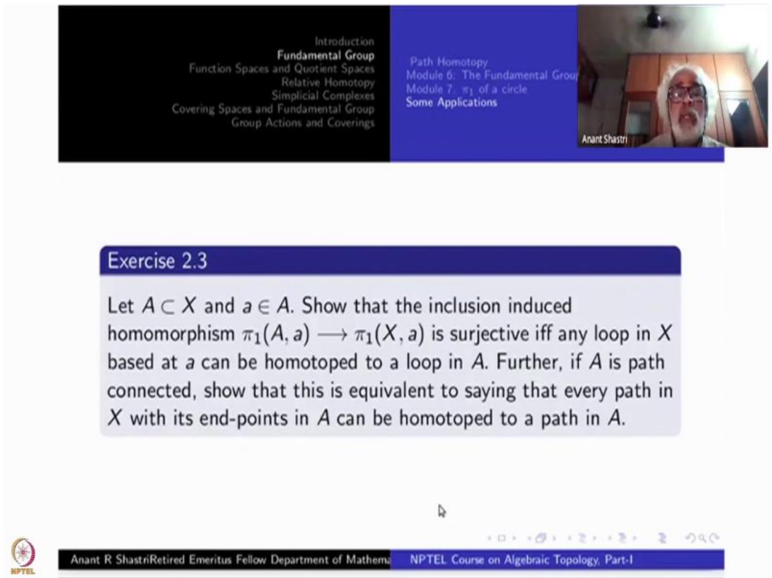

Take a subspace A contained inside X and a point a inside A. Inclusion induced homomorphism A to X is surjective if and only if every loop in X based at a can be homotoped to a loop in A. Suppose omega is a loop in X, based at a. Then there will be a loop, say tau inside A and these two will be homotopic. That is the meaning of this one. Further if A is path connected, then show that the statement that pi 1 is surjective is equivalent to saying that every path in X with endpoints in A can be homotoped to a path inside A. So, these are straight forward exercises you have to work out.

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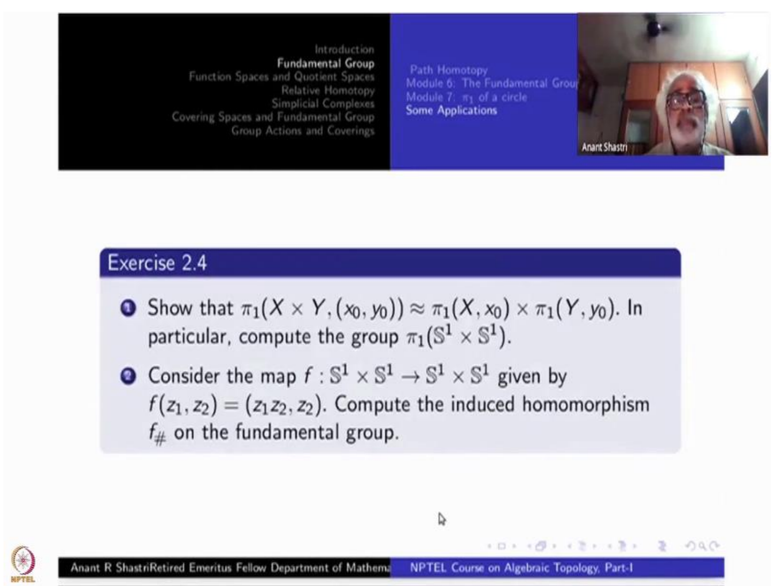

The next exercise is about the product. The product of two spaces π_1 of X cross Y comma x naught comma y naught is π_1 of X comma x naught cross pi 1 of Y comma y naught. Product of two spaces, the fundamental group is also a product of the corresponding spaces, product of the fundamental group of the corresponding spaces. In particular, you have to write down what is pi 1 of $\mathbb{S}^1 \times \mathbb{S}^1$. That is very obvious once you know the new step. This notation indicates an isomorphism of groups here.

Next, once you have computed this, the next exercise depends --- that is why they are bunched together here, depends upon this exercise. Look at a map $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ given by z1 comma z2, (these are unit complex numbers, so you can multiply them,) going to z1 z2 comma Z2. (z2 the second coordinate remains as it is and z1 gets multiplied by z2.) Look at this function. What happens to this function when you pass to the fundamental group level? What is f check? You have to compute. This you can do once you have done the previous one correctly.

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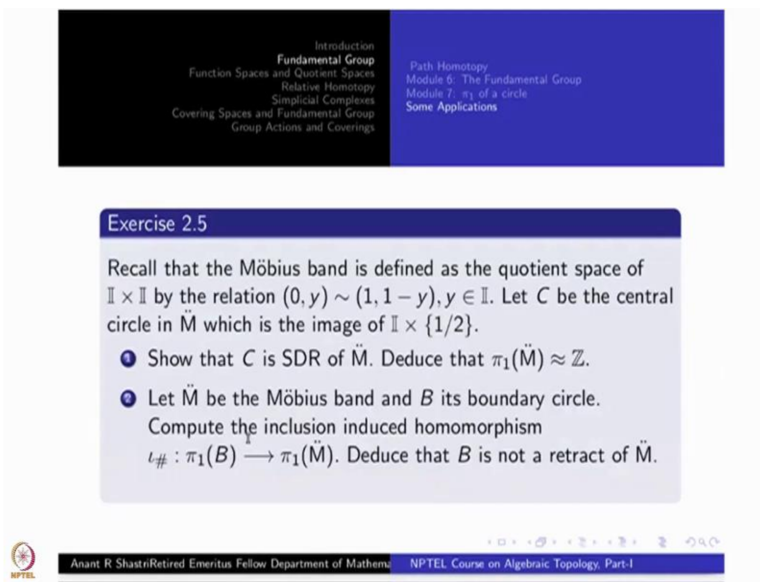

This exercise you do not have to worry about right now. Only after several, some more lectures have been done, namely after you have, you have done a live session, we can discuss the Mobius band.

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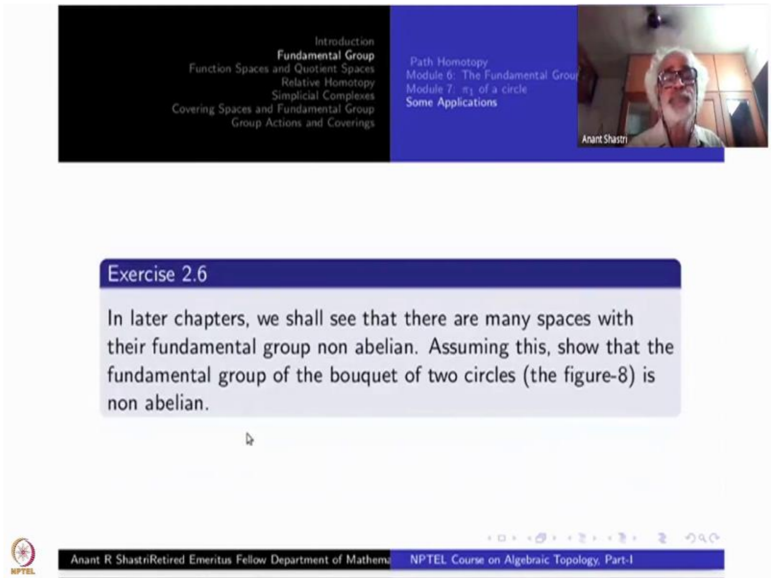

So I have told you some about of the exercises. But you can take try them all.

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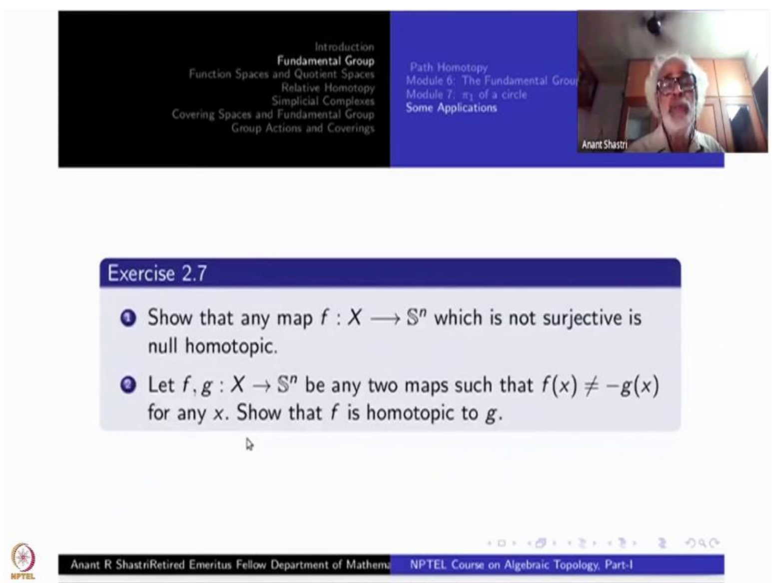

So for example this one says take two functions from X to any sphere, the sphere could be $\mathbb{S}^1, \mathbb{S}^2, \mathbb{S}^3$ whatever. These two are maps are such that f of x will never be the negative of g of x. That is for no point x in X, f x and g x are antipodal. Then f is homotopic to g.

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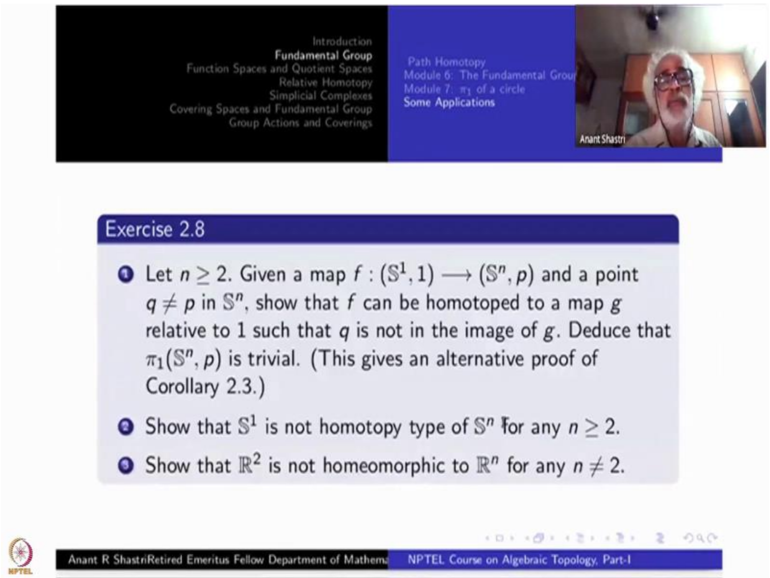

So there are enough exercises here for you to work out.

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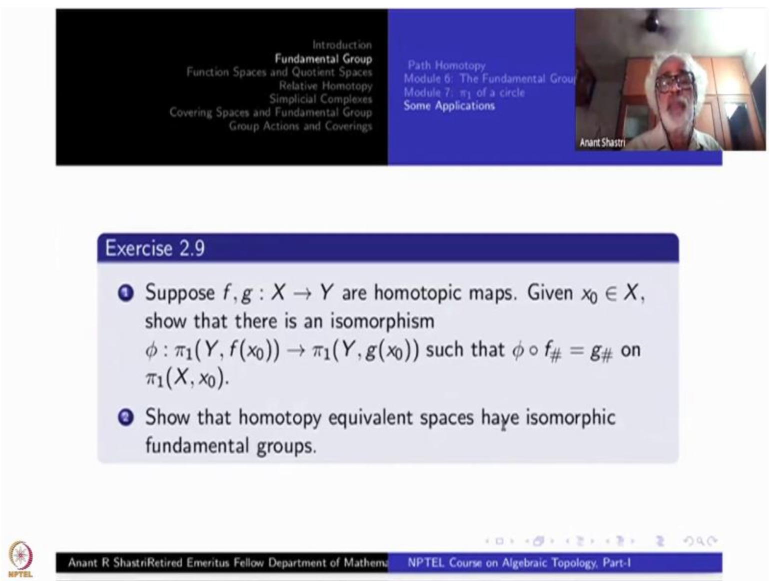

Next module will be some Set Theory, sorry Set Topology, Point Set Topology. These point set topology result, in an elementary course in point set topology, might not have been covered. Because they are somewhat advanced topics. So I will cover them to the extent we will need it in this course. Though the title of this topic is Function Space and so on, we are not going to do the entire function space theory as done in a point set topology course. That will divert the course. We will do only things which we need, rest of them, if you want more, you will have to pick it up from some Point Set Topology book. That is for the next module. Thank you.