

Introduction to Algebraic Topology (Part I)
Professor Anant R. Shastri
Department of Mathematics
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Lecture 1
Basic Problem in Topology

(Refer Slide Time: 00:20)

The screenshot shows a video lecture interface. At the top left, a black box contains a white table of contents with the following items: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, Simplicial Complexes I, Simplicial Complexes II, Covering Spaces and Fundamental Group, and G-Coverings and Fundamental Group. To the right of this box is a video feed of Professor Anant R. Shastri, a man with glasses and a beard wearing a yellow shirt. Below the video feed is a blue box with the text 'NPTEL Course on Algebraic Topology, Part-I'. Underneath that, the text reads: 'Anant R Shastri, Retired Emeritus Fellow, Department of Mathematics, IIT Bombay'. Below this, it lists 'Team Members: B. Subhash IISER Tirupati; K. Ramesh IITM; Bidhan Paul, IITM; Siva Shankar IITM; Vinay Sipani, IITM'. At the bottom of the slide, there is a date 'November 0, 2020' and a navigation bar with icons. The NPTEL logo is visible in the bottom left corner.

I am Anant R Shastri, retired Emeritus fellow, Department of Mathematics IIT Bombay. I am going to present this course on algebraic topology, part I, under this NPTEL portal. My team members are Dr. B. Subhash IISER Tirupathi, Dr. K. Ramesh IITM, and Bidhan Paul, Siva Shankar, Vinay Sipani all from IITM. This course is presented in 62 modules of approximately 30 minutes, each in a span of twelve weeks. Along with these slides and lectures, there will be a number of live sessions for doubt clearing.

(Refer Slide Time: 01:40)

November 9, 2020

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homotopy
 Simplicial Complexes-I
 Simplicial Complexes-II
 Covering Spaces and Fundamental Group
 G-Coverings and Fundamental Group

Anant Shastri

- 1 Introduction
 - Pre-knowledge
 - Bibliography
 - Conventions and Notation
 - The Basic Problem
 - Module 1
 - Module 2 The Concept of Homotopy
 - Homotopy Types

- 2 Fundamental Group
 - Module 5 Path Homotopy
 - Module 6: The Fundamental Group
 - Module 7: π_1 of a circle
 - Some Applications



Module 17 More constructions

- 3 Relative Homotopy
 - Module 17
 - Module 19: A Typical SDR
 - Module 20

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

Anant Shastri

Module 21

- Module 22: The Harvest
- NDR Pairs
- Module 24 Some General Remarks

- 4 Simplicial Complexes-I
 - Module 25 Basics of Affine Geometry
 - Module 26 Abstract Simplicial Complexes
 - Module 27 Geometric Realization
 - Module 28 Topology on $|K|$
 - Module 29 Simplicial Maps
 - Module 30 More examples

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

Anant Shastri



Module 32 Barycentric Subdivision

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Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homotopy
 Simplicial Complexes I
 Simplicial Complexes II
 Covering Spaces and Fundamental Group
 G-Coverings and Fundamental Group

Anant Shastri

- Module 34 Simplicial Approximation
- Module 35 Sperner Lemma
- Module 36 Invariance of Domain
 - Module 37
- Module 38 Links and Stars
- Module 39 Homotopical Aspects
- Module 40 Homotopical Aspects-continued]
- Miscellaneous Exercises

7 Covering Spaces and Fundamental Group

- Module 41 Basic Definitions
- Module 42 Lifting Properties
 - Module 43 Homotopy Lifting Property
- Module 44 Relation with the Fundamental Group
 - Module 45 Regular covering

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homotopy
 Simplicial Complexes I
 Simplicial Complexes II
 Covering Spaces and Fundamental Group
 G-Coverings and Fundamental Group

- Solution of Lifting Problem
 - Module 46 Solution of Lifting Problem
- Module 47 Classification of Covering Projections
- Module 48 Classification-continued
- Module 49 Existence of Simply Connected Covering
- Module 50 Toward construction of simply connected covering
- Module 51 Common properties
- Module 52 Examples

8 G-coverings and Fundamental Group

- Module 53 G-coverings
- Module 54 Fibred Products and Pull-backs
- Module 55 Classification of G-coverings
- Module 56 Proof of Classification
- Module 57 Pushouts and Free Products

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces

Pre-requisites

- Module 55 Classification of G-coverings
- Module 56 Proof of Classification
- Module 57 Pushouts and Free Products

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology Part-I

Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes-I Simplicial Complexes-II Covering Spaces and Fundamental Group G-Coverings and Fundamental Group	Pre-requisites References Coursework The Basic Problem Module 2 The Concept of Homotopy
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- Module 58 Existence of Free Products
- Module 59 Free Products and Free Groups
- Module 60 Seifert — Van Kampen Theorem
- Module 61 Applications
- Module 62 Applications Continued

The content of the lectures can be divided into seven chapters, which you may call: (I) introduction, (II) fundamental group, (III) function space topology, (IV) the relative homotopy, (V) simplicial complexes part one and simplicial complexes part two, (VI) covering space and fundamental group, (VII) covering space and G-coverings and fundamental group.

(Refer Slide Time: 02:12)

Anant R Shastri Retired Emeritus Fellow Department of Mathematics NPTEL Course on Algebraic Topology Part-I

Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes-I Simplicial Complexes-II Covering Spaces and Fundamental Group G-Coverings and Fundamental Group	Pre-requisites References Coursework The Basic Problem Module 2 The Concept of Homotopy
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Pre-requisite

We assume that the student of this course is well acquainted with the rudiments of point set topology. This being an M. Sc. level course, a certain amount of mathematical maturity is also presumed.

I will be assuming that the student of this course is well acquainted with the rudiments of point set topology. Also this course is at the level M. Sc. Therefore, a certain amount of mathematical maturity will be presumed. We have striven hard to bring this advanced subject to your door by giving some completely elementary treatment to it and also explaining beyond all written textbooks.

The one good reference for this material is my own book on basic algebraic topology. However, I will give you a set of notes along with all the materials covered, in the PDF format in which there will be further references for further study. As far as the present course is concerned, for the duration of the present course, you can stick to those notes which will be available on the NPTEL website.

(Refer Slide Time: 03:46)

The slide is titled "Basic Problem in Topology" and is part of the "NPTEL Course on Algebraic Topology, Part-I". It features a table of contents on the left and right sides, and a central text block. The table of contents includes: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, Simplicial Complexes, Covering Spaces and Fundamental Group, Group Action and Coverings, Pre-requisites, References, Conventions, The Basic Problem, Module 2 (The Groups Of Homotopy), and Module 5 (Live session) An experiment. The central text block discusses the central problem in topology: to determine whether two given topological spaces are homeomorphic or not. It provides an example: any two open intervals in \mathbb{R} are homeomorphic to each other, since we can actually write down a homeomorphism in each case. On the other hand, we also know that a closed interval and an open interval are not homeomorphic to each other because the former is compact whereas the latter is not.

So, let us come to what is the basic problem in topology. Central problem in topology is to determine whether two given topological spaces are homeomorphic or not. For instance, hope you all know that any two open intervals in \mathbb{R} are homeomorphic to each other, since, we can actually write down a homeomorphism in each case. Given any two intervals $[a,b]$ and $[c,d]$ you can actually write down a linear homeomorphism which sends a to c and b to d .

And then you can also write down a homeomorphism from the open interval (a,b) to the entire of the real line. So, this will take care of most of the cases. For intervals, the homeomorphism problem can be settled by actually writing down homeomorphism, no problem right?

On the other hand, we also know that a closed interval and open interval are not homeomorphic to each other, because the first one (closed interval) is compact whereas the other one is not compact. So, compactness, which is a topological invariant, is good enough to distinguish between an open interval and a closed interval. There may be other methods, do not worry, this is one method. But we have been able to tell that a closed interval and an open interval are not homeomorphic.

(Refer Slide Time: 05:39)

an open interval are not homeomorphic to each other because the former is compact whereas the latter is not.

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Introduction	Pre-requisites
Fundamental Group	References
Function Spaces and Quotient Spaces	Conventions
Relative Homotopy	The Basic Problem
Simplicial Complexes	Module 2 (The Concept Of Homotopy)
Covering Spaces and Fundamental Group	Module 5 (Live session) An experiment
Group Actions and Coverings	

In general, displaying such homeomorphisms between topological spaces becomes very difficult. On the other hand, it is fruitful and easier to find out that there is no homeomorphism between two given specific spaces X and Y . The standard method is to look for a suitable 'topological invariant' such as compactness, connectedness, etc., which is present in one of the two spaces and absent in the other.

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In general, displaying a homeomorphism between topological spaces becomes very difficult, unlike the intervals-case. On the other hand, it is fruitful and easier to find out that there is no homeomorphism between two given spaces X and Y , if that is the case. Standard method is to look for a suitable topological invariant such as compactness, connectedness etc. So, these things you must have seen in a topology course. Here, I am recalling it to motivate what I am going to do in this course.

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The screenshot shows a video lecture interface. At the top, there is a header with the name 'Anant R Shastri/Retired Emeritus Fellow, Department of Mathematics' and 'NPTEL Course on Algebraic Topology'. Below this is a table of contents with two columns. The left column lists: Introduction, Fundamental Group, Function Spaces and Quotient Spaces, Relative Homotopy, Simplicial Complexes, Covering Spaces and Fundamental Group, and Group Actions and Coverings. The right column lists: Pre-requisites, References, Conventions, The Basic Problem, Module 2: The Concept Of Homotopy, and Module 5 (LIVE session) An experiment. A small video window in the top right corner shows the speaker, Anant Shastri. The main content area contains the following text:

As an example, let us show that \mathbb{R} and \mathbb{R}^2 are not homeomorphic. If

$$f : \mathbb{R} \rightarrow \mathbb{R}^2$$

were a homeomorphism then the restriction map, which we shall denote by

$$f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^2 \setminus \{f(0)\},$$

itself, is also a homeomorphism. Now, the domain of f is not connected whereas the range is. Since this is absurd, we conclude that \mathbb{R} and \mathbb{R}^2 are not homeomorphic. However, note that the connectivity could not be directly applied to the map $f : \mathbb{R} \rightarrow \mathbb{R}^2$ to arrive at this conclusion.

So, let us take another example of how to distinguish homeomorphism types using connectivity. An example, very simple example namely, let us look at \mathbb{R} and $\mathbb{R} \times \mathbb{R}$ which is \mathbb{R}^2 , real line and the plane, these are not homeomorphic. So, how do you proceed? Suppose there is a homeomorphism, let us call it f from \mathbb{R} to \mathbb{R}^2 , throw away the 0 from the domain \mathbb{R} , throw away the image of 0 from the codomain \mathbb{R}^2 .

Whatever is left out, f takes \mathbb{R} minus 0 to \mathbb{R}^2 minus $f(0)$ and the restriction itself will be a 1-1 map, continuous, inverse is also continuous, and so it is a homeomorphism. Any restriction of a homeomorphism to the corresponding subsets is also a homeomorphism. Now, what is happening? The domain of f is not connected, whereas range is connected. You must be knowing that if you throw away a single point from \mathbb{R}^2 , it is still connected, whereas from \mathbb{R} , we throw away one single point, it is disconnected. Therefore, right in the beginning, there could not have been any function f which is a homeomorphism from \mathbb{R} to \mathbb{R}^2 , okay?


(Refer Slide Time: 08:12)

Again, this method may not work very far. For example, it is not effective if the problem is to prove that \mathbb{R}^n and \mathbb{R}^m are not homeomorphic to each other for $m > n > 1$. Of course, there are purely point-set-topological proofs of this result as well. However, they are not so easy. So, one looks for other topological invariants, which are perhaps not so demanding.

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Introduction
Fundamental Group
Function Spaces and Quotient Spaces
Relative Homology
Simplicial Complexes
Covering Spaces and Fundamental Group
Group Actions and Coverings

Pre-requisites
References
Conventions
The Basic Problem
Module 2 The Concept Of Homotopy
Module 5 (Last session) An experiment

 The Role of Algebraic Topology

Again this method may not work very far, I have used connectivity, I have used compactness and so on, but you can go on using a number of them. There are situations wherein any or all of these, the so called topological invariants, may fail to distinguish between two spaces. Such a thing can happen.

Let us just think about say \mathbb{R}^n and \mathbb{R}^m , do not have to go too far, where n and m are just arbitrary. Say m is bigger than n . Can you say that \mathbb{R}^n and \mathbb{R}^m are not homeomorphic? No simple topological invariant will be able to tell you this. You know just connectivity etc, will not be enough--- you may succeed to do something with \mathbb{R}^2 , but again with $\mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^5$ etc, it will not work.

However, there are some very deep point-set topological results here. Dimension-theory will tell you that \mathbb{R}^m and \mathbb{R}^n are not homeomorphic. However, these are quite difficult results in point-set topology. They are not easy.

(Refer Slide Time: 09:43)

Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homology
 Simplicial Complexes
 Covering Spaces and Fundamental Group
 Group Actions and Coverings

Prerequisites
 References
 Conventions
The Basic Problem
 Module 2: The Concept Of Homotopy
 Module 5 (Live session) An experiment

The Role of Algebraic Topology

When we start the study of a discipline, it is good to have at least some idea of what the fundamental problems in that discipline are. In most of the cases, these problems remain unsolved. In some cases, some day one may find that the fundamental problem cannot be solved. However, that does not mean that we have come to the end of the road—there will always be some related problems or modified problems demanding our attention. And this is the case with topology in general and algebraic topology in particular.

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Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homology
 Simplicial Complexes

Prerequisites
 References
 Conventions
The Basic Problem

The role of algebraic topology is now. It is good to have at least some idea what the fundamental problems in this discipline are. Whenever you want to study some discipline you must know what it is about. In most cases, these problems remain unsolved or you will know that someday that the problem is too difficult or the problem is just unsolvable.

Even then, the concept of the central problem or fundamental problem has so much of a role to play, namely that there will always be some related problems or modified problems which will demand our attention. And this is the case with topology also in general, and algebraic topology also in particular.

(Refer Slide Time: 10:46)

Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homology
 Simplicial Complexes
 Covering Spaces and Fundamental Group
 Group Actions and Coverings

Pre-requisites
 References
 Conventions
The Basic Problem
 Module 2 The Concept Of Homotopy
 Module 3 (Last session) An experiment

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The central objects of study in topology in Main-stream Mathematics, are 'manifolds' a much modest collection than the collection of all topological spaces. Roughly speaking, an n -dimensional manifold is a topological space in which each point has a neighbourhood system consisting of open sets which are homeomorphic to open sets in a n -dimensional Euclidean space. The problem then is to determine whether any two given manifolds are homeomorphic or not.

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Introduction Pre-requisites

The central objects of study in topology in mainstream mathematics are not topological spaces, (it is too large,) but let us take what are called 'manifolds'. These manifolds are something like modeled on \mathbb{R}^n , that we are discussing. So, they are locally, i.e., in a small neighborhood of a point, look like \mathbb{R}^n . The rigorous definition will be introduced much later. The problem is then to determine whether two given manifolds are homeomorphic or not. Do not worry about arbitrary spaces--- even this restricted problem is known to be unsolvable.

(Refer Slide Time: 11:31)

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Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings		Pre-requisites References Conventions The Basic Problem Module 2 The Concept Of Homotopy Module 5 (Live session) An experiment	

The role of algebraic topology is that it associates certain algebraic objects such as groups and rings etc. rather than descriptive invariants such as compactness or separability etc., which are indeed 'more flexible' than homeomorphism invariants and in a 'natural way'. Obviously, these invariants will become additional tools for us in distinguishing two topological spaces.



So, I want to tell you that algebraic topology is associated with certain algebraic objects such as groups and rings, rather than compactness, separability etc, which are more qualitative. The algebraic invariants that we are going to introduce are much more flexible than homeomorphism invariants. Obviously, these invariants will become additional tools for us while distinguishing two topological spaces.

(Refer Slide Time: 12:10)

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Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings		Pre-requisites References Conventions The Basic Problem Module 2 The Concept Of Homotopy Module 5 (Live session) An experiment	

It is known that this classification problem for manifolds is not solvable. Even this negative result is quite valuable and let us take a few seconds to see how this result was established.



I told you that the classification problem means determining whether two given spaces are homeomorphic or not. Up to homeomorphism, we are putting them in different classes. This is known to be unsolvable. This negative result itself has some implications for us, it is useful for us. So, let us take a few minutes to look at how it was established. This is where algebraic topology comes into picture and not point-set-topology.

(Refer Slide Time: 13:05)

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<p>Introduction</p> <ul style="list-style-type: none"> Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings 	<p>Prerequisites</p> <p>References</p> <p>Coverages</p> <p>The Basic Problem</p> <p>Module 2: The Concept Of Homotopy</p> <p>Module 5 (Last session): An experiment</p>
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To each path connected space X one 'associates' a group called the fundamental group $\pi_1(X)$. This association has the property that for any map $f : X \rightarrow Y$, there is the homomorphism of groups $f_{\#} : \pi_1(X) \rightarrow \pi_1(Y)$. The association is 'natural' in the sense that if $g : Y \rightarrow Z$ is another map, then $(g \circ f)_{\#} = g_{\#} \circ f_{\#}$ and for the identity map $Id : X \rightarrow X$, we have $Id_{\#} : \pi_1(X) \rightarrow \pi_1(X)$ is the identity homomorphism. In particular, it now follows that if $f : X \rightarrow Y$ is a homeomorphism then $f_{\#}$ is an isomorphism. Thus, in order that two given path connected spaces X and Y are homeomorphic, first of all, their fundamental groups must be isomorphic.

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To each path connected space X one associates a group called the fundamental group. (We are going to study this phenomenon very rigorously in this course. So, right now, you take it for granted that this can be done.) Take a topological space which is path connected, you get a group out of it. This association has the property that if you have a function from X to Y then there will be a homomorphism from the group $\pi_1(X)$ to $\pi_1(Y)$. This association is so natural that if another map from Y to Z is there, then you have two maps $f_{\#}$ and $g_{\#}$ which are group homomorphisms, then by $(g \circ f)_{\#}$, it means you first composite g and f and then take the associated map that is nothing but the composite of the corresponding associated maps $g_{\#} \circ f_{\#}$.

Moreover, if you take the identity map from X to X , the associated map will be the identity homomorphism from $\pi_1(X)$ to $\pi_1(X)$. So, all this machinery is very, very fundamental in the construction of the fundamental group and that is a part of algebraic topology.

So, the point is once you have such an association, if f is a homeomorphism then $f_\#$ becomes an isomorphism, why? Because, look at the inverse g , f composite g is identity. Therefore, $g_\#$ composite $f_\#$ is identity. Therefore $g_\#$ becomes the inverse of $f_\#$ as a homomorphism.

So, a homeomorphism induces an isomorphism. It just means that if you have started with two spaces, which are homeomorphic, the corresponding groups must be isomorphic. Therefore, if for some reason I know that the groups are not isomorphic then you know that the corresponding spaces are not homeomorphic.

(Refer Slide Time: 15:42)

homeomorphic, first of all, their fundamental groups must be isomorphic.

Introduction	Pre-requisites
Fundamental Group	References
Function Spaces and Quotient Spaces	Conventions
Relative Homology	The Basic Problem
Simplicial Complexes	Module 2: The Concept Of Homotopy
Covering Spaces and Fundamental Group	Module 5 (Last session) An experiment
Group Actions and Coverings	

The next step is to construct a manifold such that $\pi_1(M)$ is isomorphic to a given group G . Indeed, given a group G with finitely many generators and relations, (i.e., a **finitely presented group**) it can be shown that there is a compact 4-dimensional manifold M with $\pi_1(M) = G$. The net result is that now the homeomorphism problem for compact 4-dimensional manifolds implies the isomorphism problem for finitely presented groups. This latter problem goes under the name 'word problem' and nowadays is a very specialized branch of group theory and mathematical logic.

Now, how does this help to show that the classification problem itself is unsolvable?--- by showing that the corresponding classification in group theory is not solvable. So, that result itself is a huge thing, even the restricted problem in group theory, namely, for what are called finitely generated groups with finitely many relations (they are called finitely presented groups).

Suppose, I have given you a set of generators and a set of relations that would describe a group. On the other hand, another set of generators and another set of relations that identify another group. The word problem is to determine whether these two groups are isomorphic or not. It is known that this word problem cannot be solved in general. What is the meaning of that?

Suppose you write an algorithm to solve such a thing. As soon as you display your algorithm, we can give a set of examples to show that the problem cannot be determined by the algorithm, i.e., whether the two groups are isomorphic or not cannot be solved. This is the meaning that 'the word problem in group theory cannot be solved'.

A consequence of this is that the homeomorphism problem for manifolds cannot be solved. There is a small gap here namely how does one go back to homeomorphisms?--by constructing manifolds with a particular group as the fundamental group.

So, these problems are all problems in algebraic topology--- not general-topology and unfortunately, we will not go this much deep in this course. So, this is only a trailer for you to study more and more algebraic topology.

(Refer Slide Time: 18:21)

latter problem goes under the name word problem and nowadays is a very specialized branch of group theory and mathematical logic.

The non solubility of the word problem was established in 1955 by P.S. Novikov [Novikov, 1955]. Thus we also know that homeomorphism problem for topological manifolds 'cannot' be solved.

This problem in group theory was solved by a topologist P. S. Novikov in 1955.

(Refer Slide Time: 18:37)

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Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homology Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings	Pre-requisites References Courseware The Basic Problem Module 2 The Concept Of Homotopy Module 3 (Live session) An experiment
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This, however does not close the subject altogether—topology is still a very lively subject extending a helping hand in solving problems from several areas of mathematics. A process such as associating the fundamental group to a space X as considered above is called constructing a 'functor.' Algebraic topology may be described at the outset as the study of such processes. In this course, we shall construct and study one of the most important functor called the fundamental group and see a number of applications.



This, however, does not close the subject altogether. Topology is still a very lively subject extending a helping hand in solving problems in various areas of mathematics. A process such as associating the fundamental group to a space X , as considered above, is called constructing a 'functor'. Algebraic topology may be described as, at the outset, study of such processes. In this course, we shall construct and study one most important functor called fundamental group and we will have a number of applications.

(Refer Slide Time: 20:02)

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<p>Introduction</p> <ul style="list-style-type: none"> Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings 	<p>Pre-requisites</p> <p>References</p> <p>Conventions</p> <p>The Basic Problem</p> <p>Module 2 The Concept Of Homotopy</p> <p>Module 5 (Live session) An experiment</p>
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Coming back to the fundamental problem, there are many interesting related problems, other than the central one. For instance before finding a homeomorphism $f : X \rightarrow Y$, we may want to find some map which may be defined on a part of X or may be defined all over X but does not have all the properties that we demand and hence need to be modified. This, in turn, raises many other questions. Instead of listing all these questions, we shall begin with two of the important ones:

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Coming back to the fundamental problem there are many interesting related problems, other than the central one. For example, you want to find a homeomorphism from one space to another space. Before that, you would have to find some nice functions. As a part of finding a homeomorphism, you will have to find a 'nice' function. Okay? A nice function which is continuous, maybe with some more properties such as open mapping and so on. So, instead of listing all these problems, I will list here two central problems, just two basic ones.

(Refer Slide Time: 20:11)

shall begin with two of the important ones:

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Introduction Fundamental Group Function Spaces and Quotient Spaces Relative Homotopy Simplicial Complexes Covering Spaces and Fundamental Group Group Actions and Coverings	Pre-requisites References Conventions The Basic Problem Module 2 The Concept Of Homotopy Module 5 (Live session) An experiment
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- (i) the lifting problem and
- (ii) the extension problem.

The underlying themes in these two problems occur repeatedly in the study of several basic notions in algebraic topology. So, it may be worthwhile to get some familiarity with these concepts.



So, they are, I recall, one is called the lifting problem, the other one is called the extension problem. The underlying themes in these two problems occur repeatedly in the study of basic notions of algebraic topology. So, let us get familiar with these concepts slowly.

(Refer Slide Time: 20:37)

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Consider a triangle of maps as represented in the figure below.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow h & \swarrow g \\ & & Z \end{array}$$

Figure 1: Triangle of maps

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So, start with a triangle of functions: X, Y, Z are sets, f, g and h are functions, what is the meaning that this triangle is commutative. Starting from X , you can go to Y and then come

to Z , the arrows indicate where to go from where, the arrow is a function. So, X to Y , Y to Z .

But you can also go directly from X to Z , these two functions must be the same, namely $g \circ f$ must be h . That is the meaning of a diagram like this is commutative. You will have lots of such diagrams, not necessarily triangles, may have rectangles, may have pentagons and so on.

In particular, if X, Y, Z are topological spaces and f and g, h are continuous functions, then you would like to have maps, but commutativity is just a set theoretic notion: g composed with f must be h .

(Refer Slide Time: 21:46)

Such a diagram is called a commutative diagram, if $h = g \circ f$. Given any two of the three maps, we can consider the problem of finding one or all maps which fit the third arrow in the diagram. Naturally, this problem can be broken up into three cases, out of which one case is too easy, viz., if f, g are given then h can be taken to be $g \circ f$ and nothing else.

Such a diagram is called a commutative diagram. Given two of the three maps, the question is: consider the problem of finding one or all maps which fit the third arrow. So, this is the general problem. If f, g are given, then h can be taken to be $g \circ f$. There is no other choice. But suppose h is given and f is given. Then for g , there would be many possibilities.

Similarly, if h is given and g is given, there may be many possibilities for f . There may not be any possibilities for f . This is the problem that I want to attack now. So, let us make two different cases.

(Refer Slide Time: 22:36)

So, we consider the other two cases which we reformulate as follows:

Q. Given maps, $p : E \rightarrow B$ and $f : X \rightarrow B$, does there exist $g : X \rightarrow E$ such that, $p \circ g = f$? The map g is called a **lift** of f through p and this problem goes under the name **lifting problem**.

The first case is now--- (I am going to give specific names here, so that we remember it again.) Instead of X, Y, Z , now, I will take X, E and B, E to B there is a function p , there is a map and concentration is on this map. Given any function f from X to B , can you find a function g such that $p \circ g$ is equal to f ?

This is called the lifting problem: f being lifted to g via or through p --- p composite g must be f . So, the question is on this dotted line. Other things are data. Does g exist is the question. That is why I put the dotted line. Understand this one?

(Refer Slide Time: 23:38)

Anant R Shastri/Retired Emeritus Fellow, Department of Mathematics | NPTEL Course on Algebraic Topology

Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homology
 Topological Groups
 Covering Spaces and Fundamental Group
 Group Actions and Countings

Pre-requisites
 References
 Conventions
The Basic Problem
 Module 2: The Concepts Of Homotopy
 Module 5 (Live session): An experiment

Anant Shastri

Q. II Given maps $\eta : A \rightarrow X$ and $f : A \rightarrow Y$ does there exist $\hat{f} : X \rightarrow Y$ such that $\hat{f} \circ \eta = f$?
 Question II has two important special cases:
 (a) η is a quotient map; this goes under the name 'factorization problem'.
 (b) η is an inclusion map; this goes under the name 'extension problem'.

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The second question is the other way round namely, now I have $A \rightarrow X$ a map η , sorry, a map f , so concentration is on $A \rightarrow Y$, this η is fixed. Now $A \rightarrow Y$ there is some function f is given. Can we have a map down here from $X \rightarrow Y$? So, this is called factoring. So, this f is factored through η . (Earlier we had lifting problems.) Here, it is factoring problem, but it is actually, this can be thought of as an extension problem, especially when $A \rightarrow X$, the map η is an inclusion map of topological spaces.

Then what is the meaning of having a function f on a subspace and having function \hat{f} on the larger space? So, η is the inclusion map, this is a very special case. But the special case name is given to the entire problem. You can say (a) part is a factorization problem and (b) Part is an extension problem.

The factorization comes when instead of an inclusion map you have surjective map and η is a quotient map. Both these cases are, you can call them, extension problems. Factorization problem is easier, the extension problem is difficult. So, you will concentrate on the extension problem later on. So, the factorization problem will be decided very easily in this course. However, the extension problem is a problem that needs our attention.

(Refer Slide Time: 25:19)

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Introduction
 Fundamental Group
 Function Spaces and Quotient Spaces
 Relative Homotopy
 Simplicial Complexes
 Covering Spaces and Fundamental Group
 Group Actions and Coverings

Prerequisites
 References
 Conventions
The Basic Problem
 Module 2: The Concept Of Homotopy
 Module 5 (Live session): An experiment

Clearly, these are purely point-set-topological problems which may or may not have satisfactory solutions. It turns out that (a) has an easy answer (see Subsection on quotient spaces) and so we need to bother about (b) only. However, the problem (b) turns out to be too formidable as it stands. Algebraic topology steps in here by bringing a big twist to these questions. We shall take up this point in the next module.

Anant R Shastri/Retired Emeritus Fellow, Department of Mathematics NPTEL Course on Algebraic Topology, Part-I

Algebraic topology steps in here by bringing a big twist to these questions, which we shall take in the next module. These questions will be discussed again, but with a twist namely, what algebraic topology is going to do. This is a point set topology problem: given a subset and a function and a larger space can you extend the function to the whole space and so on. It is just a point-set problem. So, what is algebraic topology going to do here? That we will discuss in the next lesson. Thank you.