Introduction to Algebraic Topology (Part I)
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Lecture 1
Basic Problem in Topology

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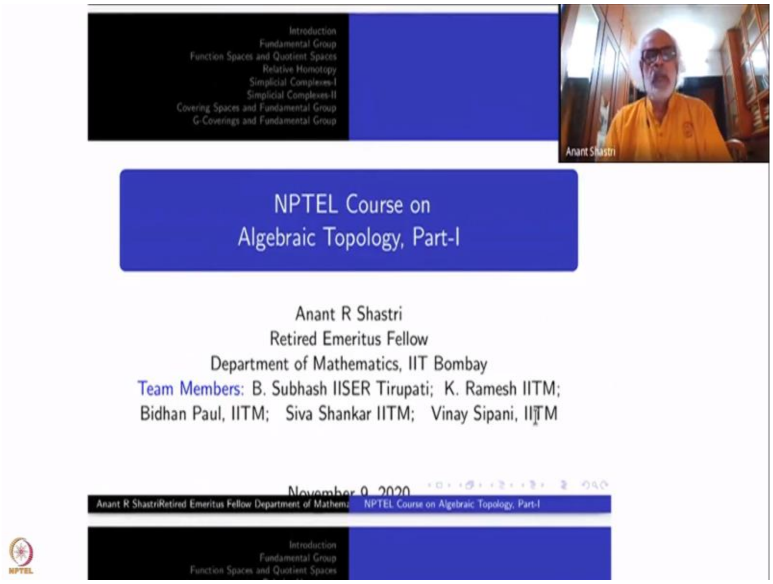

I am Anant R Shastri, retired Emeritus fellow, Department of Mathematics IIT Bombay. I am going to present this course on algebraic topology, part I, under this NPTEL portal. My team members are Dr. B. Subhash IISER Tirupathi, Dr. K. Ramesh IITM, and Bidhan Paul, Siva Shankar, Vinay Sipani all from IITM. This course is presented in 62 modules of approximately 30 minutes, each in a span of twelve weeks. Along with these slides and lectures, there will be a number of live sessions for doubt clearing.

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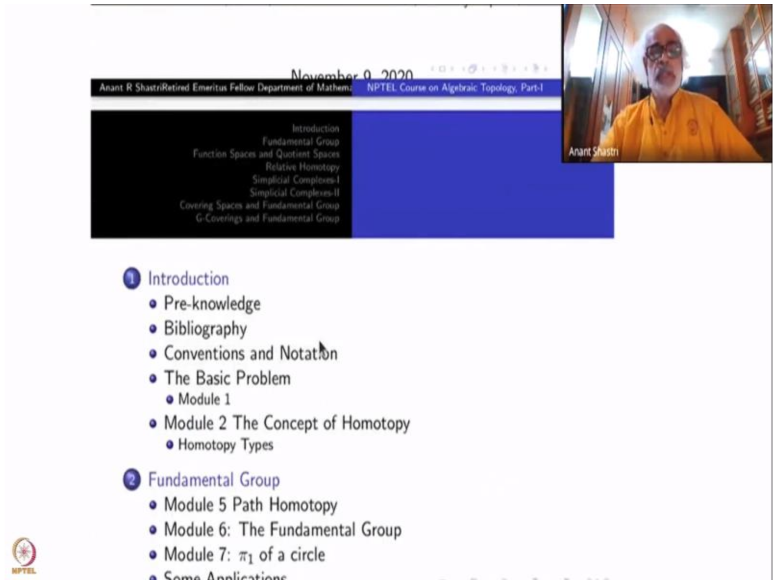
November 9, 2020
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Function Spaces and Quotient Spaces
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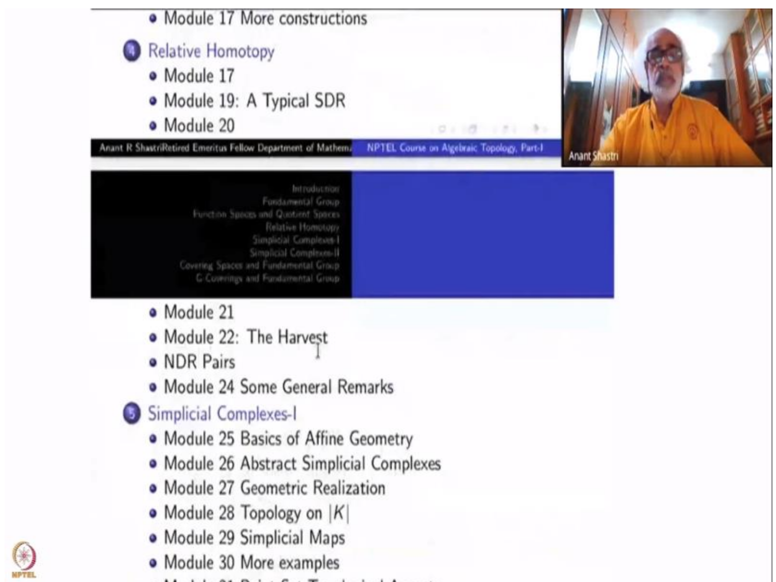
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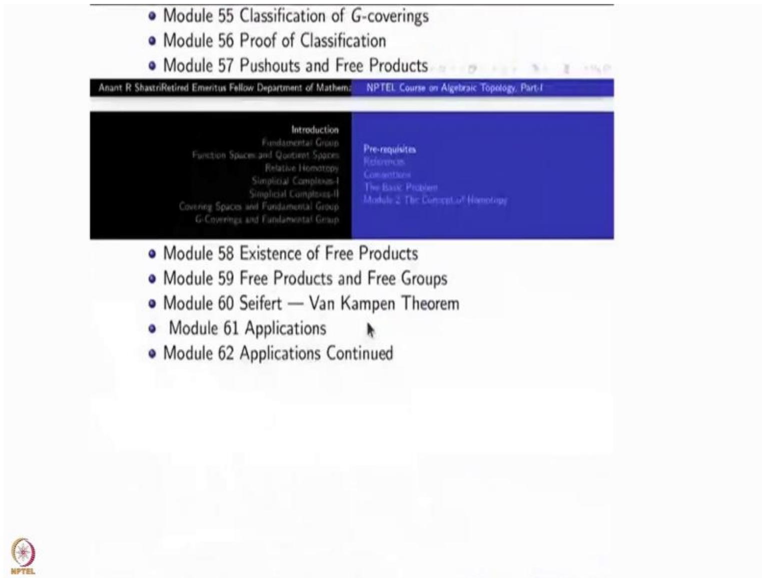

The content of the lectures can be divided into seven chapters, which you may call: (I) introduction, (II) fundamental group, (III) function space topology, (IV) the relative homotopy, (V) simplicial complexes part one and simplicial complexes part two, (VI) covering space and fundamental group, (VII) covering space and G-coverings and fundamental group.

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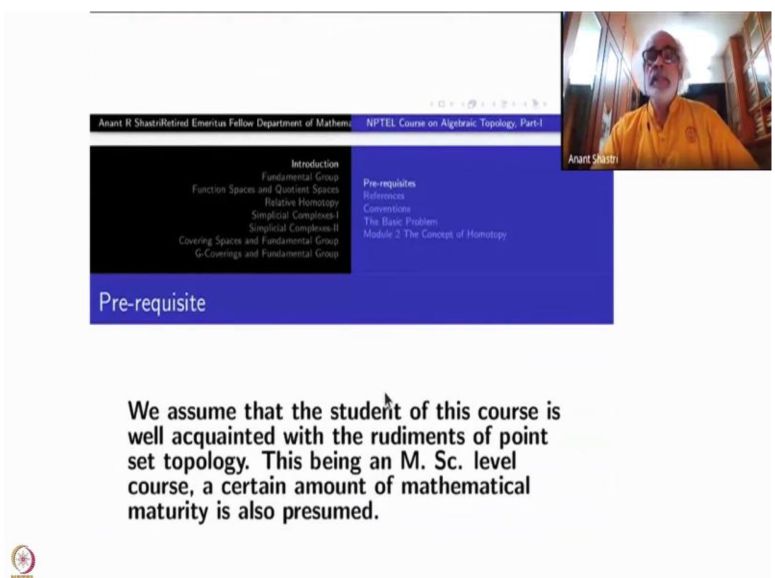

I will be assuming that the student of this course is well acquainted with the rudiments of point set topology. Also this course is at the level M. Sc. Therefore, a certain amount of mathematical maturity will be presumed. We have striven hard to bring this advanced subject to your door by giving some completely elementary treatment to it and also explaining beyond all written textbooks.

The one good reference for this material is my own book on basic algebraic topology. However, I will give you a set of notes along with all the materials covered, in the PDF format in which there will be further references for further study. As far as the present course is concerned, for the duration of the present course, you can stick to those notes which will be available on the NPTEL website.

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So, let us come to what is the basic problem in topology. Central problem in topology is to determine whether two given topological spaces are homeomorphic or not. For instance, hope you all know that any two open intervals in $\mathbb{R}$ are homeomorphic to each other, since, we can actually write down a homeomorphism in each case. Given any two intervals [a,b] and [c,d] you can actually write down a linear homeomorphism which sends a to c and b to d.

And then you can also write down a homeomorphism from the open interval (a,b) to the entire of the real line. So, this will take care of most of the cases. For intervals, the homeomorphism problem can be settled by actually writing down homeomorphism, no problem right?

On the other hand, we also know that a closed interval and open interval are not homeomorphic to each other, because the first one (closed interval) is compact whereas the other one is not compact. So, compactness, which is a topological invariant, is good enough to distinguish between an open interval and a closed interval. There may be other methods, do not worry, this is one method. But we have been able to tell that a closed interval and an open interval are not homeomorphic.

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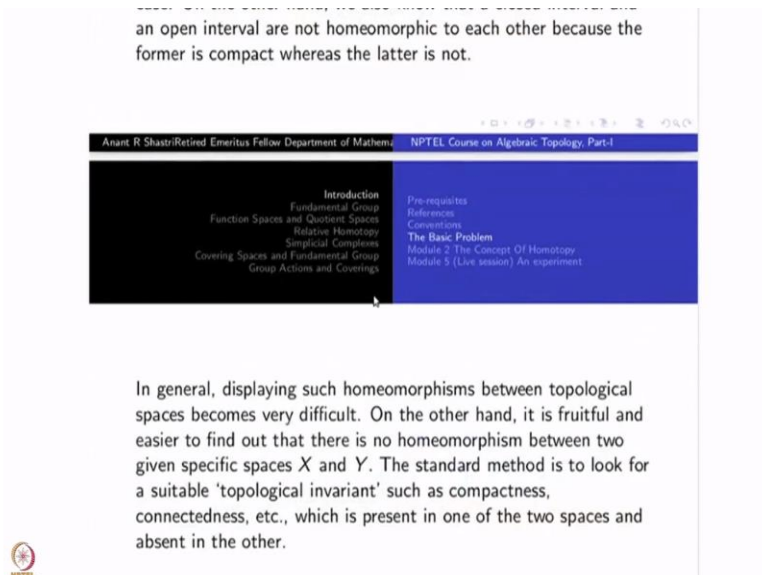

In general, displaying a homeomorphism between topological spaces becomes very difficult, unlike the intervals-case. On the other hand, it is fruitful and easier to find out that there is no homeomorphism between two given spaces X and Y, if that is the case. Standard method is to look for a suitable topological invariant such as compactness, connectedness etc. So, these things you must have seen in a topology course. Here, I am recalling it to motivate what I am going to do in this course.

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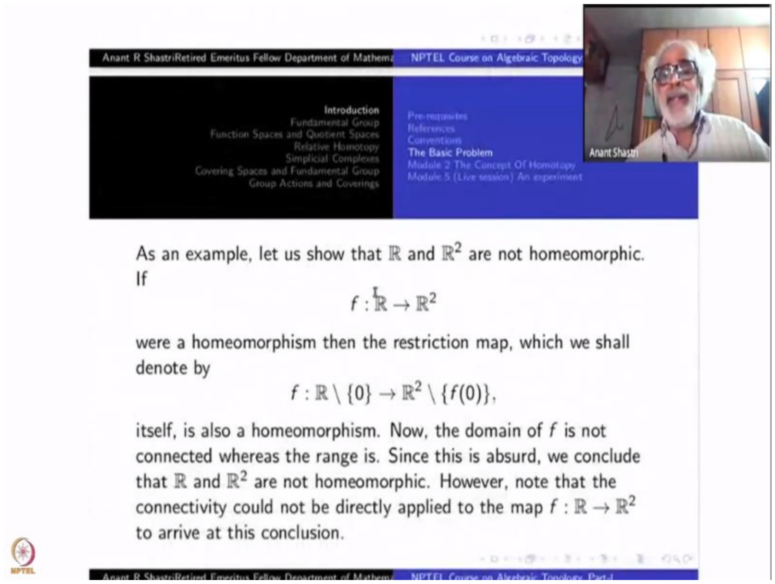

So, let us take another example of how to distinguish homeomorphism types using connectivity. An example, very simple example namely, let us look at $\mathbb{R}$ and $\mathbb{R}\times\mathbb{R}$ which is $\mathbb{R}^2$, real line and the plane, these are not homeomorphic. So, how do you proceed? Suppose there is a homeomorphism, let us call it f from $\mathbb{R}$ to $\mathbb{R}^2$, throw away the 0 from the domain $\mathbb{R}$, throw away the image of 0 from the codomain $\mathbb{R}^{2}$.

Whatever is left out, f takes $\mathbb{R}$ minus 0 to $\mathbb{R}^2$ minus f(0) and the restriction itself will be a 1-1 map, continuous, inverse is also continuous, and so it is a homeomorphism. Any restriction of a homeomorphism to the corresponding subsets is also a homeomorphism. Now, what is happening? The domain of f is not connected, whereas range is connected. You must be knowing that if you throw away a single point from $\mathbb{R}^{2}$, it is still connected, whereas from $\mathbb{R}$, we throw away one single point, it is disconnected. Therefore, right in the beginning, there could not have been any function f which is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}^2$, okay?

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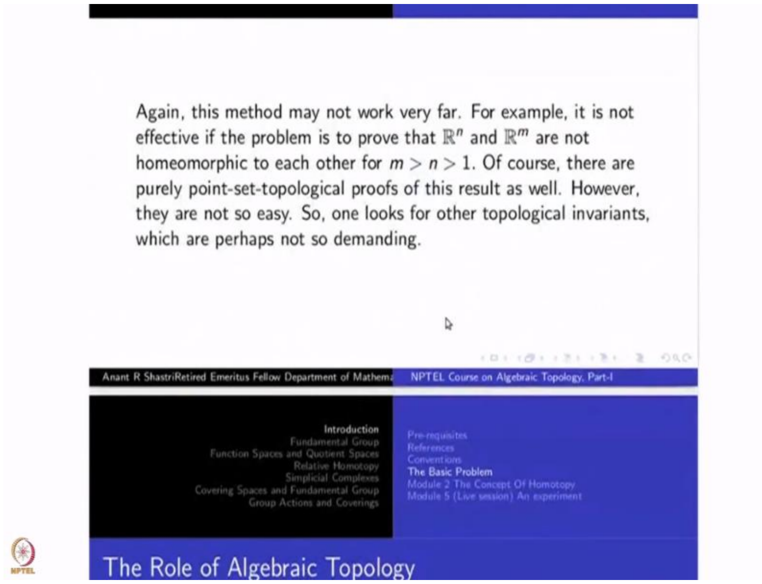

Again this method may not work very far, I have used connectivity, I have used compactness and so on, but you can go on using a number of them. There are situations wherein any or all of these, the so called topological invariants, may fail to distinguish between two spaces. Such a thing can happen.

Let us just think about say $\mathbb{R}^n$ and $\mathbb{R}^m$, do not have to go too far, where n and m are just arbitrary. Say m is bigger than n. Can you say that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic? No simple topological invariant will be able to tell you this. You know just connectivity etc, will not be enough--- you may succeed to do something with $\mathbb{R}^2$, but again with $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$ etc, it will not work.

However, there are some very deep point-set topological results here. Dimension-theory will tell you that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic. However, these are quite difficult results in point-set topology. They are not easy.

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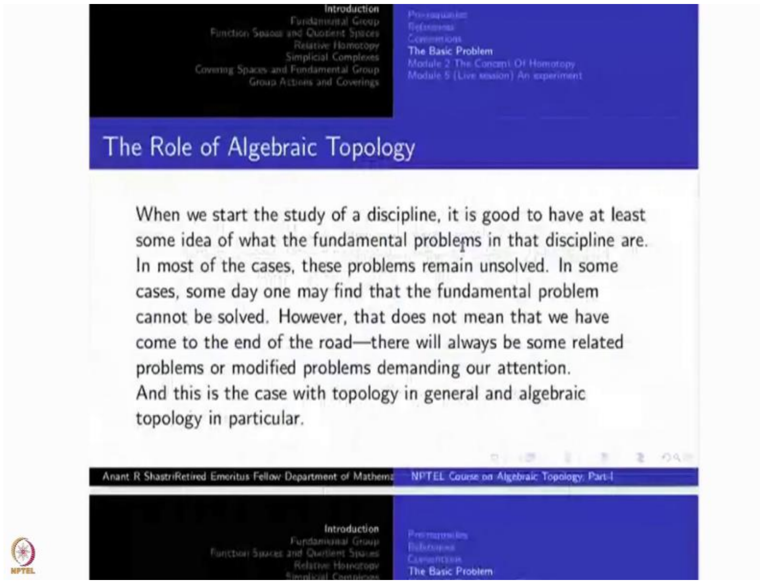

The role of algebraic topology is now. It is good to have at least some idea what the fundamental problems in this discipline are. Whenever you want to study some discipline you must know what it is about. In most cases, these problems remain unsolved or you will know that someday that the problem is too difficult or the problem is just unsolvable.

Even then, the concept of the central problem or fundamental problem has so much of a role to play, namely that there will always be some related problems or modified problems which will demand our attention. And this is the case with topology also in general, and algebraic topology also in particular.

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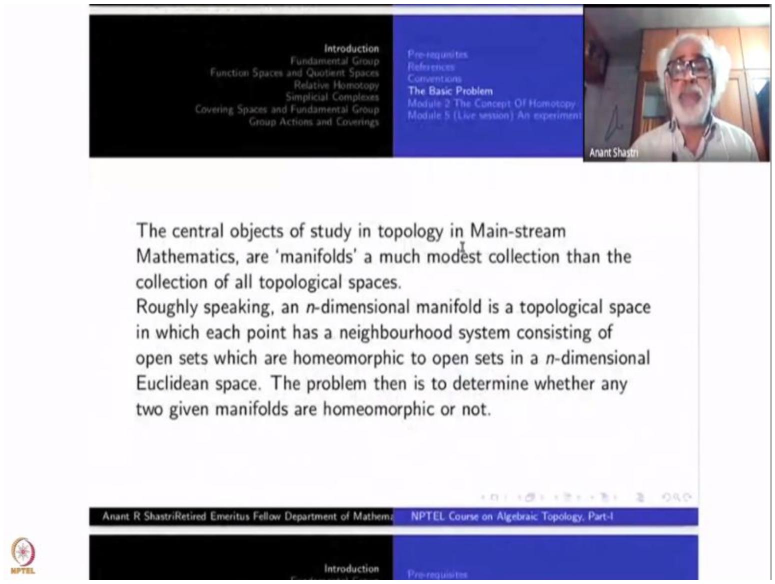

The central objects of study in topology in mainstream mathematics are not topological spaces, (it is too large,) but let us take what are called `manifolds'. These manifolds are something like modeled on $\mathbb{R}^n$, that we are discussing. So, they are locally, i.e., in a small neighborhood of a point, look like $\mathbb{R}^n$. The rigorous definition will be introduced much later. The problem is then to determine whether two given manifolds are homeomorphic or not. Do not worry about arbitrary spaces--- even this restricted problem is known to be unsolvable.

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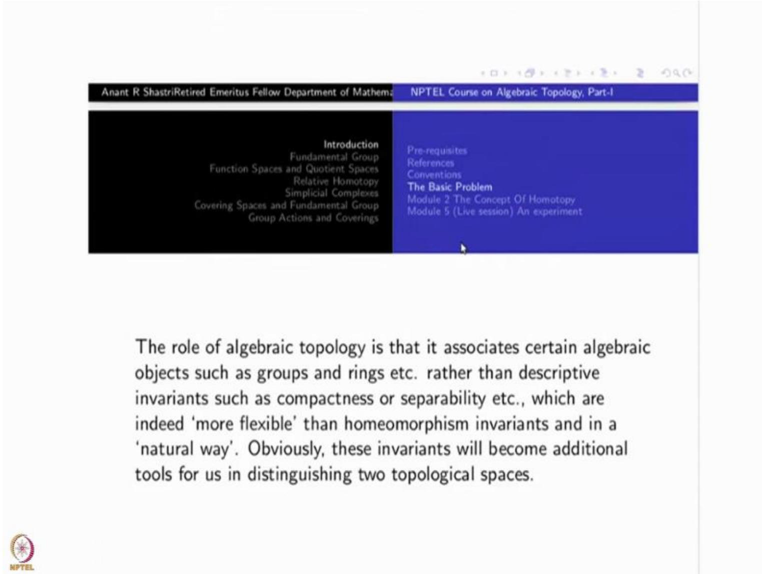

So, I want to tell you that algebraic topology is associated with certain algebraic objects such as groups and rings, rather than compactness, separability etc, which are more qualitative. The algebraic invariants that we are going to introduce are much more flexible than homeomorphism invariants. Obviously, these invariants will become additional tools for us while distinguishing two topological spaces.

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I told you that the classification problem means determining whether two given spaces are homeomorphic or not. Up to homeomorphism, we are putting them in different classes. This is known to be unsolvable. This negative result itself has some implications for us, it is useful for us. So, let us take a few minutes to look at how it was established. This is where algebraic topology comes into picture and not point-set-topology.

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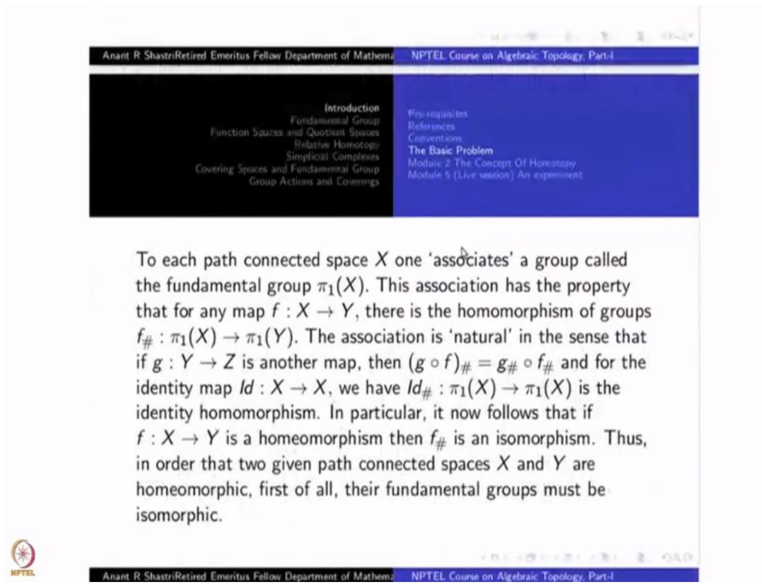

To each path connected space X one associates a group called the fundamental group. (We are going to study this phenomenon very rigorously in this course. So, right now, you take it for granted that this can be done.) Take a topological space which is path connected, you get a group out of it. This association has the property that if you have a function from X to Y then there will be a homomorphism from the group $\pi_1(X)$ to $\pi_1(Y)$. This association is so natural that if another map from Y to Z is there, then you have two maps $f_\#$ and $g_\#$ which are group homomorphisms, then by $(g \circ f)_\#$, it means you first composite g and f and then take the associated map that is nothing but the composite of the corresponding associated maps $g_\# \circ f_\#$.

Moreover, if you take the identity map from X to X, the associated map will be the identity homomorphism from π_1(X) to π_1(X). So, all this machinery is very, very fundamental in the construction of the fundamental group and that is a part of algebraic topology.

So, the point is once you have such an association, if f is a homeomorphism then $f_\#$ becomes an isomorphism, why? Because, look at the inverse g, f composite g is identity. Therefore, $g_\#$ composite $f_\#$ is identity. Therefore $g_\#$ becomes the inverse of $f_\#$ as a homomorphism.

So, a homeomorphism induces an isomorphism. It just means that if you have started with two spaces, which are homeomorphic, the corresponding groups must be isomorphic. Therefore, if for some reason I know that the groups are not isomorphic then you know that the corresponding spaces are not homeomorphic.

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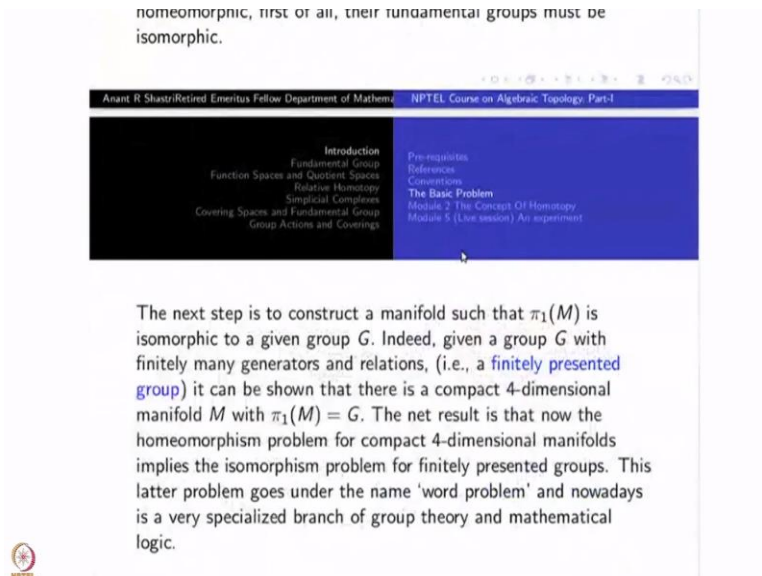

Now, how does this help to show that the classification problem itself is unsolvable?--- by showing that the corresponding classification in group theory is not solvable. So, that result itself is a huge thing, even the restricted problem in group theory, namely, for what are called finitely generated groups with finitely many relations (they are called finitely presented groups).

Suppose, I have given you a set of generators and a set of relations that would describe a group. On the other hand, another set of generators and another set of relations that identify another group. The word problem is to determine whether these two groups are isomorphic or not. It is known that this word problem cannot be solved in general. What is the meaning of that?

Suppose you write an algorithm to solve such a thing. As soon as you display your algorithm, we can give a set of examples to show that the problem cannot be determined by the algorithm, i.e., whether the two groups are isomorphic or not cannot be solved. This is the meaning that `the word problem in group theory cannot be solved'.

A consequence of this is that the homeomorphism problem for manifolds cannot be solved. There is a small gap here namely how does one go back to homemorphisms?--by constructing manifolds with a particular group as the fundamental group.

So, these problems are all problems in algebraic topology--- not general-topology and unfortunately, we will not go this much deep in this course. So, this is only a trailer for you to study more and more algebraic topology.

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This problem in group theory was solved by a topologist P. S. Novikov in 1955.

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This, however, does not close the subject altogether. Topology is still a very lively subject extending a helping hand in solving problems in various areas of mathematics. A process such as associating the fundamental group to a space X, as considered above, is called constructing a 'functor'. Algebraic topology may be described as, at the outset, study of such processes. In this course, we shall construct and study one most important functor called fundamental group and we will have a number of applications.

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Coming back to the fundamental problem there are many interesting related problems, other than the central one. For example, you want to find a homeomorphism from one space to another space. Before that, you would have to find some nice functions. As a part of finding a homeomorphism, you will have to find a `nice' function. Okay? A nice function which is continuous, maybe with some more properties such as open mapping and so on. So, instead of listing all these problems, I will list here two central problems, just two basic ones.

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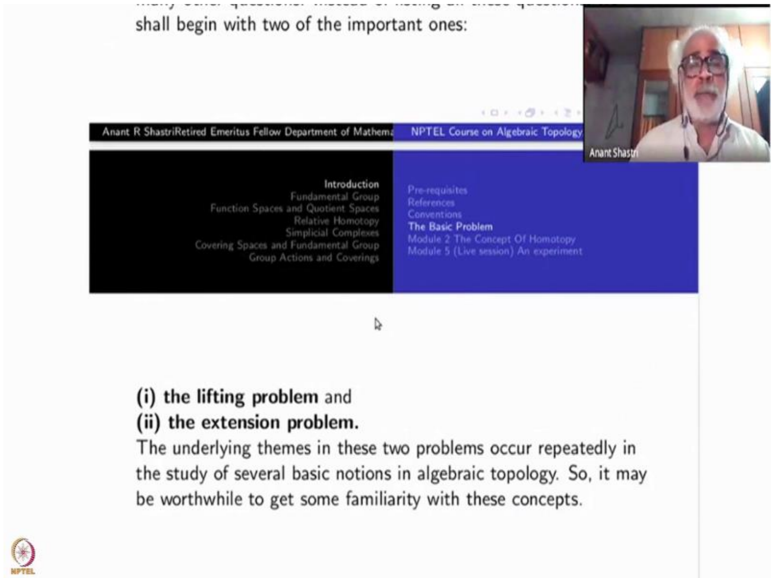

So, they are, I recall, one is called the lifting problem, the other one is called the extension problem. The underlying themes in these two problems occur repeatedly in the study of basic notions of algebraic topology. So, let us get familiar with these concepts slowly.

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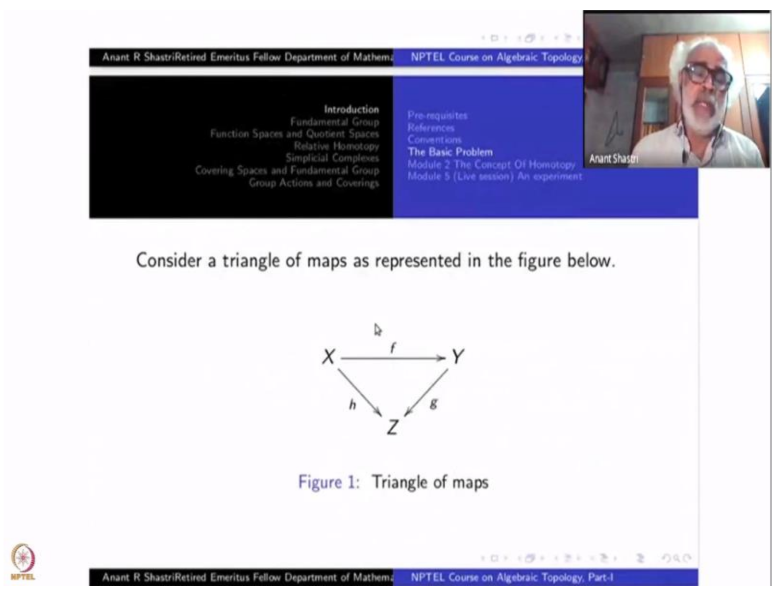

So, start with a triangle of functions: X, Y, Z are sets, f, g and h are functions, what is the meaning that this triangle is commutative. Starting from X, you can go to Y and then come to Z, the arrows indicate where to go from where, the arrow is a function. So, X to Y, Y to Z.

But you can also go directly from X to Z, these two functions must be the same, namely g composite f must be h. That is the meaning of a diagram like this is commutative. You will have lots of such diagrams, not necessarily triangles, may have rectangles, may have pentagons and so on.

In particular, if X,Y,Z are topological spaces and f and g, h are continuous functions, then you would like to have maps, but commutativity is just a set theoretic notion: g composed with f must be h.

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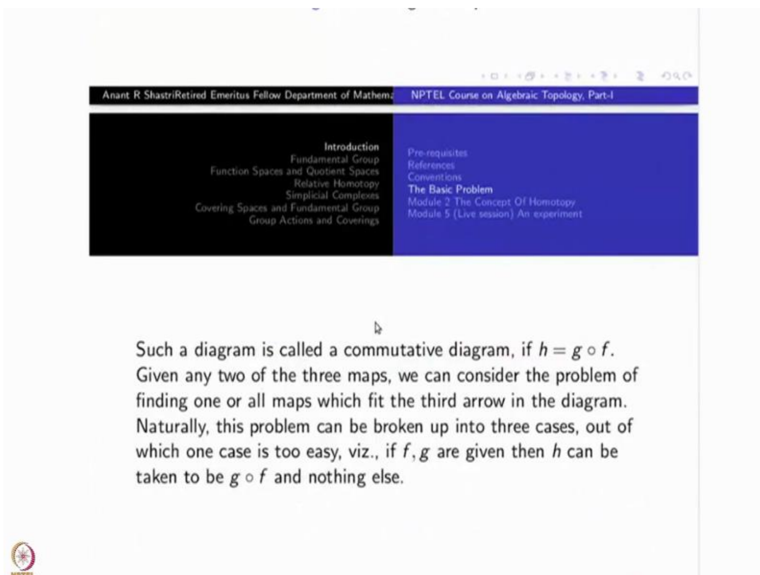

Such a diagram is called a commutative diagram. Given two of the three maps, the question is: consider the problem of finding one or all maps which fit the third arrow. So, this is the general problem. If f, g are given, then h can be taken to be g ○ f. There is no other choice. But suppose h is given and f is given. Then for g, there would be many possibilities.

Similarly, if h is given and g is given, there may be many possibilities for f. There may not be any possibilities for f . This is the problem that I want to attack now. So, let us make two different cases.

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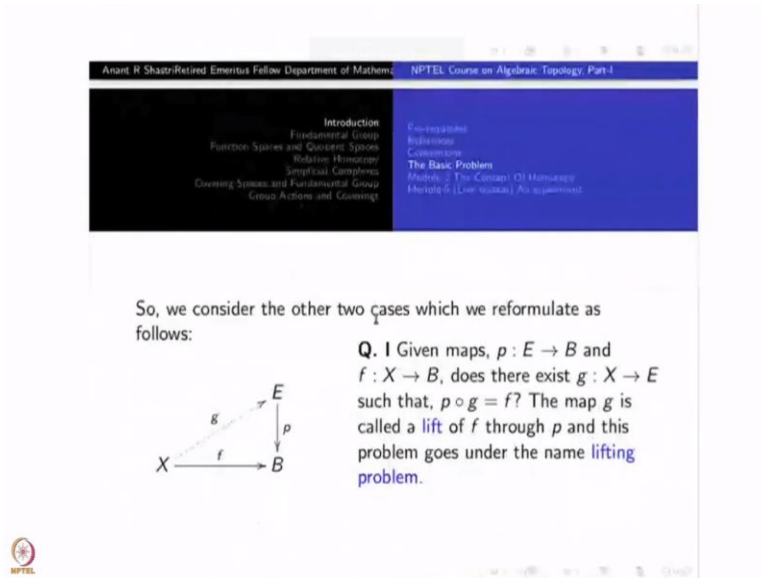

The first case is now--- (I am going to give specific names here, so that we remember it again.) Instead of X, Y, Z, now, I will take X, E and B, E to B there is a function p, there is a map and concentration is on this map. Given any function f from X to B, can you find a function g such that p ○ g is equal to f?

This is called the lifting problem: f being lifted to g via or through p--- p composite g must be f. So, the question is on this dotted line. Other things are data. Does g exist is the question. That is why I put the dotted line. Understand this one?

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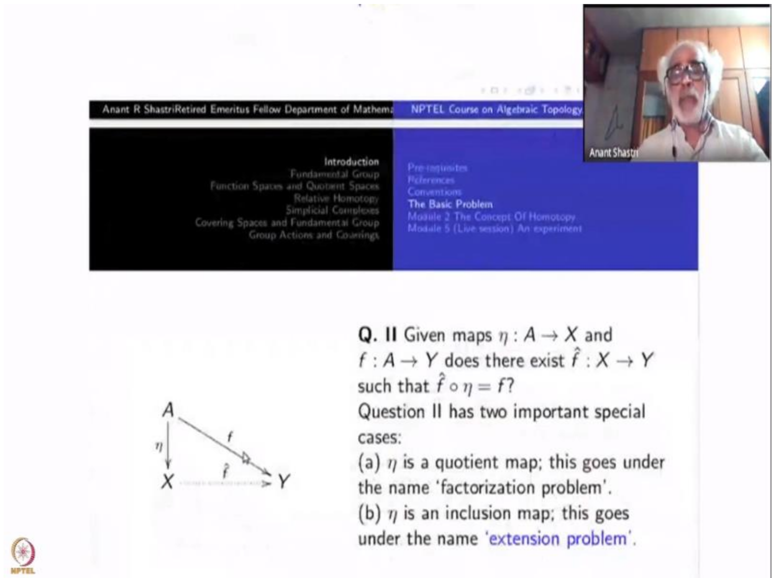

The second question is the other way round namely, now I have A to X a map f , sorry, a map η, so concentration is on A to X, this η is fixed. Now A to Y there is some function f is given. Can we have a map down here from X to Y? So, this is called factoring. So, this f is factored through η. (Earlier we had lifting problems.) Here, it is factoring problem, but it is actually, this can be thought of as an extension problem, especially when A to X, the map η is an inclusion map of topological spaces.

Then what is the meaning of having a function f on a subspace and having function $\hat{f}$ on the larger space? So, η is the inclusion map, this is a very special case. But the special case name is given to the entire problem. You can say (a) part is a factorization problem and (b) Part is an extension problem.

The factorization comes when instead of an inclusion map you have surjective map and η is a quotient map. Both these cases are, you can call them, extension problems. Factorization problem is easier, the extension problem is difficult. So, you will concentrate on the extension problem later on. So, the factorization problem will be decided very easily in this course. However, the extension problem is a problem that needs our attention.

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Algebraic topology steps in here by bringing a big twist to these questions, which we shall take in the next module. These questions will be discussed again, but with a twist namely, what algebraic topology is going to do. This is a point set topology problem: given a subset and a function and a larger space can you extend the function to the whole space and so on. It is just a point-set problem. So, what is algebraic topology going to do here? That we will discuss in the next lesson. Thank you.