A Basic Course in Number Theory
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Lecture 63
Tying some loose ends

Welcome back for the last lecture, this is going to be a short lecture, we are just going to tie up some loose ends, let me start right away.

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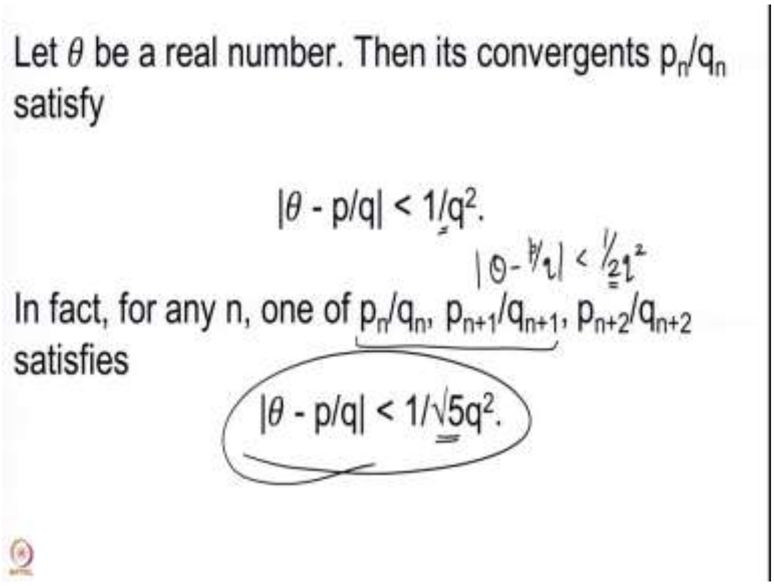

So, we have proved this that whenever theta is a real number and we take the continued fraction expansion for theta and suppose pn upon qn denote the convergents then these convergents satisfy this inequality. So, this is what we had said that the convergents give you a good approximation for theta. Moreover, we also saw this result that whenever you have any consecutive triple of convergents say pn upon qn p n plus 1 by q n plus 1 and p n plus 2 by q n plus 2, then one of these three should satisfy this improved inequality.

Theta minus p by q less than 1 upon root 5 q square, one of these three should satisfy this inequality. In fact, if you take one of the two, then we have also proved this in that theta minus p by q less than 1 upon 2q square will be satisfied by one of these. So, here we had the coefficient the multiple of Q was 1 here this is 2 and then we improved it to root 5.

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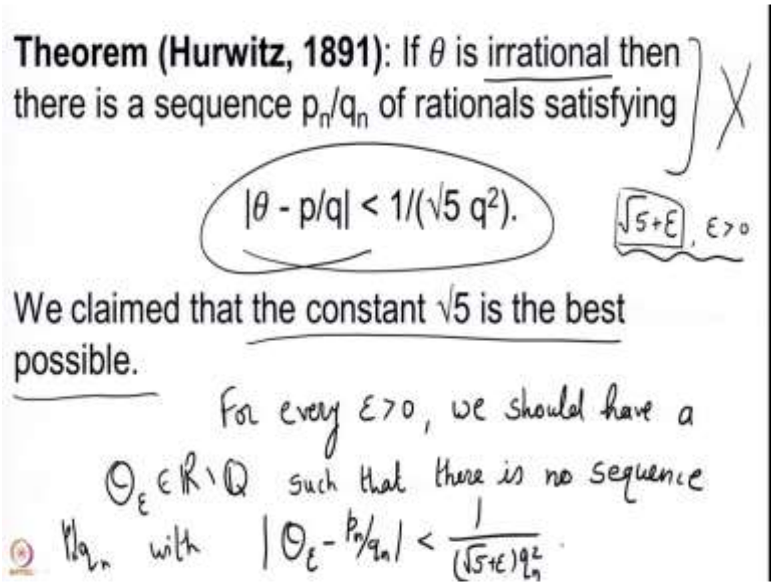

And this root 5 gave us a proof of the Hurwitz theorem, which said this is proved long time back then 1891 in the 19th century, this theorem says that, if you have an irrational number, then there is a sequence of rationals converging to theta ofcourse, satisfying this particular inequality. And then we said that this constant root 5 is the best possible constant. This is our claim that root 5 is the best possible constant you cannot improve root 5 even by any slight quantity and we are going to prove this.

So, what do we have to prove? We want to prove that this root 5 is the best possible such constant that means if we prove that if you want to change root 5 to root 5 plus epsilon for any epsilon positive, then this theorem does not hold this is what we want to prove that if you change root 5, increase the root 5 constant by any small number, positive number say epsilon, then this theorem does not hold that means we should give you an example.

So, for every epsilon positive we should have a number theta which will ofcourse depend on epsilon we need this number to be irrational. So, we have this number theta epsilon in R minus Q such that there is no sequence p by q or let us say p n by q n with modulus theta epsilon minus p n by q n less than 1 upon root 5 plus epsilon qn square, this is what we have to prove. Let us again understand what we need to prove, we want to we claim that root 5 is the best possible constant.

So, if we increase root 5 slightly by this epsilon which is a which is any positive number, you fix this positive number now, epsilon that is fixed so, we are looking at the constant to be 1 root 5 plus epsilon, this is our new constant. Then we claim that the theorem of Hurwitz does not hold for root 5 plus epsilon. What does that what would the theorem say for root 5 plus epsilon? It would say that if data is irrational, there is a sequence p n by q n such that mod theta minus p by Q is less than 1 upon root 5 plus epsilon into q n square that is what the theorem would say, we claim that this theorem does not hold.

So, it is enough to give one irrational number for which the theorem does not hold. And so far, but this has to be done for every positive epsilon. So, for every positive epsilon we should give some real number theta epsilon has to be an irrational ofcourse with the property that there is no sequence p n by q n satisfying that particular inequality what we are going to do is that we will give one theta which will work for every epsilon.

So, there are not theta epsilon for every epsilon there is one particular data that is going to be our golden ratio 1 plus root 5 upon 2. So, we are going to work with this theta and now, we want to show that for every epsilon mod theta minus pn by q n to be less than root 5 plus epsilon q n square does not hold. So, there is no sequence of pn by q n satisfying this a sequence is an infinite set, we are actually going to prove that this particular inequality holds for only finitely many rational numbers.

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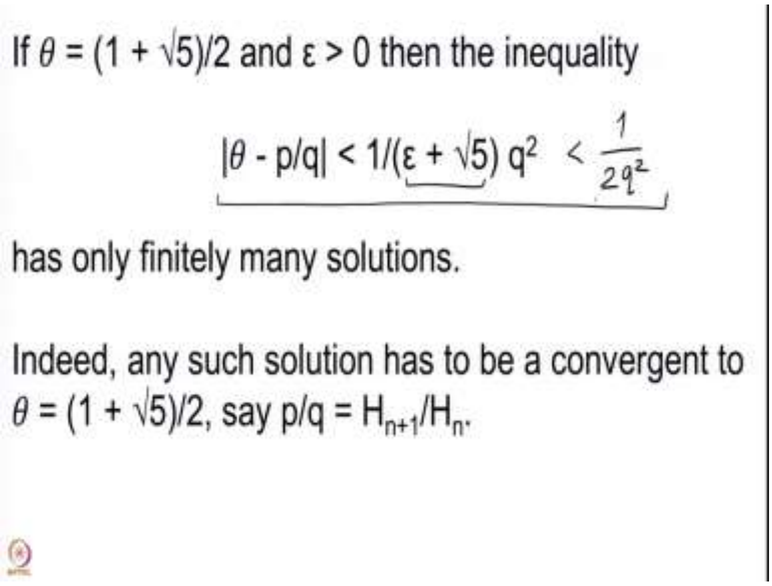

So, let us go and see what we are going to prove we prove that if theta is 1 plus root 5 by 2 and epsilon is any positive real number, then this inequality theta minus p by q less than 1 upon root 5 plus epsilon q square has only finitely many solutions. Once you prove this then there cannot be a sequence p n by q n with this possibility, because a sequence cannot by definition a sequence is a function defined from the natural numbers to reals and we would actually like this sequence to converge to our theta.

So, the Q will have to increase and therefore, there will be infinitely many such terms we claim that the particular inequality has only finitely many solutions. And therefore, it will be true therefore it will follow that Hurwitz theorem is best is possible only work with root 5 root 5 is the best possible constant with which Hurwitz's theorem holds, okay.

So, now we are looking at this particular theta which is 1 plus root 5 by 2 and we are letting epsilon to be any such positive then we have also proved that any such solution has to be a convergent because this quantity is bigger than 2 already root 5 is bigger than 2 and then you are adding a positive quantity.

So, this is going to be 1 less than 1 upon 2 q square and we have proved that whenever you have a rational number approximating any real number theta with the property that mod theta minus p by q is less than 1 upon to 2q square then this p by q has to be a convergent this p by q is trying to approximate our theta in a way better than any conversion. And that is simply not possible we have actually proved that the convergents are the best possible rational approximations to any real number theta.

So, this theta has to be a convergent to our, this p by q has to be convergent to our theta. We have also discussed these convergents already these were the Hemma Chandra Fibonacci numbers. So, instead of using the standard notation F, we will use the notation h to denote these. So, now we have that what we want to prove is the following.

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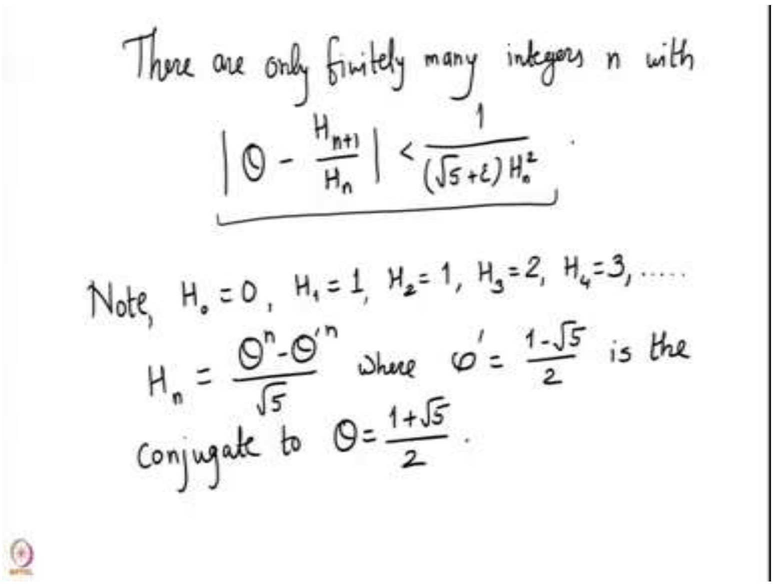

That there are only finitely many integers and with mod theta minus h n plus 1 upon Hn less than 1 upon root 5 plus epsilon Hn square. So, this is our result, we want to say that these p by q are only finitely many 1 plus epsilon plus root 5 1 upon epsilon plus root 5 this inequality holds for only finite lemony ends.

Now, this is a very explicit question that we have, we want to show that this number on the left hand side will ultimately go beyond this root 5 plus epsilon Hn square this is what we want to show. So, let me just recall these for you the Hn where H0 was 0, H1 is 1 H2 is also 1 which is the sum of 1 and 0, H3 is 2 H4 is 3 and so on. And in fact, there is a closed form formula for a Hn.

So, we have that a Hn has this formula that this is theta power n minus theta prime power n upon root 5 where theta prime is the conjugate to our theta is the conjugate to theta which is 1 plus root 5 upon 2.

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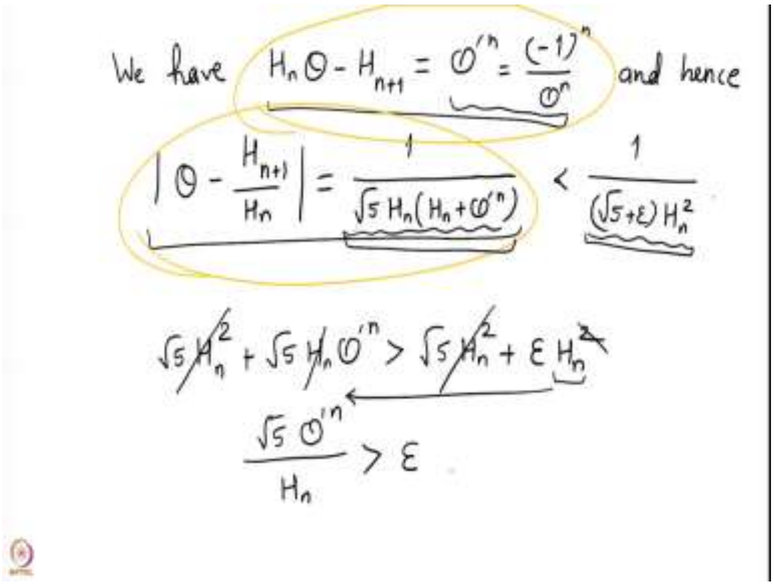

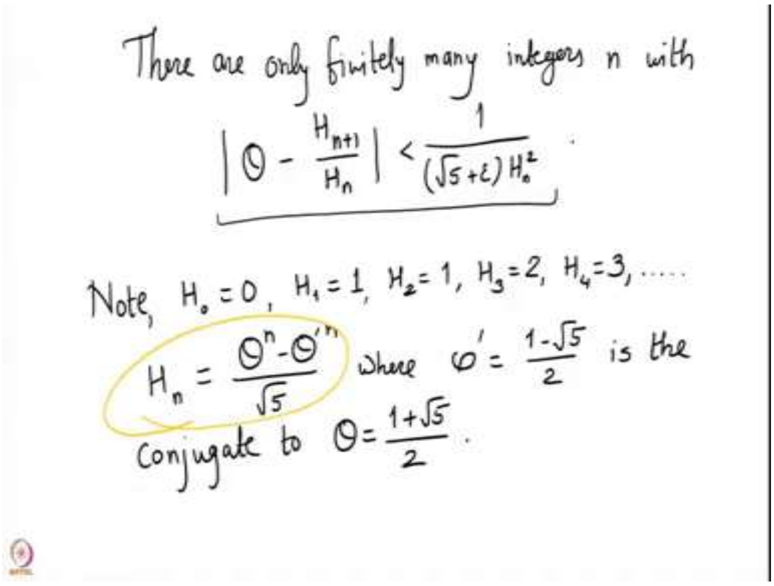

In fact, we have we have one very easy checking that a Hn theta minus Hn plus 1 is simply theta prime power n which is nothing but minus 1 power in upon theta power n and hence mod theta minus Hn plus 1 upon a Hn. This is 1 upon root 5 Hn into Hn plus theta prime power n this equality and the previous (inequality) this equality and this equality simply follow from the formula for a Hn that we have on this slide, so, this is the formula that we are going to use to get these 2 inequalities we will not prove this, but these are left to you these are simple checkings.

So, further if we are shown that this is less than root 5 plus epsilon Hn square then we get that this quantity has to be bigger than this. We are having that 1 upon some number is less than 1 upon some another number, then this number has to be bigger than or equal to bigger than this. So, we get root 5 Hn square plus root 5 Hn theta prime power n is bigger than root 5 Hn square plus epsilon h n square.

So, these root 5 Hn square are cancelled, we are able to cancel 1 Hn these are all non-zero numbers so we can cancel them and then this Hn comes to the denominator to give you root 5 theta prime power n upon a Hn is bigger than epsilon, but theta prime power n is minus 1 power n upon theta power n.

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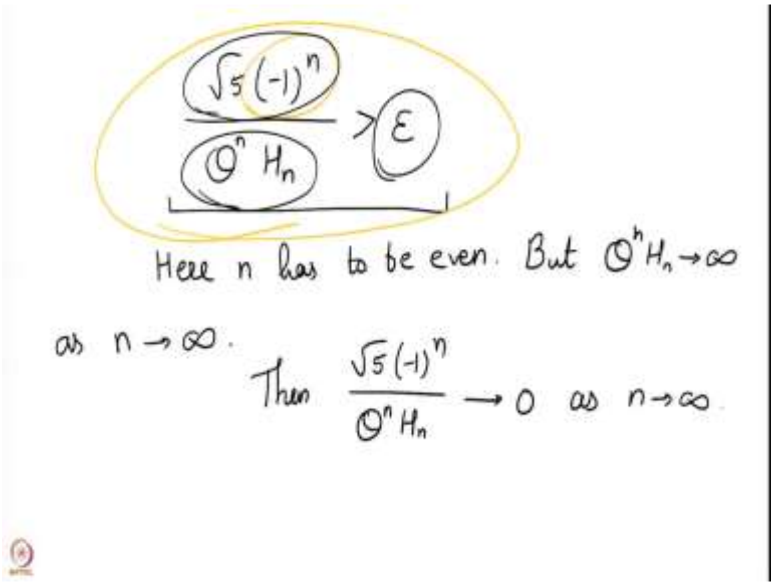

So, this will give that root 5 minus 1 power n, theta power n Hn is bigger than epsilon. Now, here n has to be clearly even, because otherwise our, this quantity will be minus 1. So, n has to be even and further we notice that the denominator numerator is bounded, but the denominator to theta power n upon Hn goes to infinity as n goes to infinity, theta is a number which is strictly bigger than 1 theta is 1 plus root 5 by 2.

So, theta power n are going to go to infinity. Similarly, h n is the nth Hemma Chandra number, and Hemma Chandra numbers are simply adding up. In fact, they go they are going to infinity faster. So, we have that set up our n into Hn goes to infinity. So, any bounded numerator upon these numbers will have to go to 0.

Therefore, the then root 5 minus 1 power n upon theta power n Hn has to go to 0 as n goes to infinity and therefore, it will mean that given any epsilon, ultimately these numbers will be smaller than epsilon what it means to that a sequence goes to 0, it means that given any quantity any small whatever small or big number you choose, ultimately the number will have to cross the sequence will have to cross that number and go nearer and nearer to 0.

Therefore, for any such epsilon, you will have at most finitely many such quantities such terms minus 1 power n into 5 into root 5 upon theta power n Hn to be bigger than epsilon. So, what we have proved is that this inequality has only finitely many solutions.

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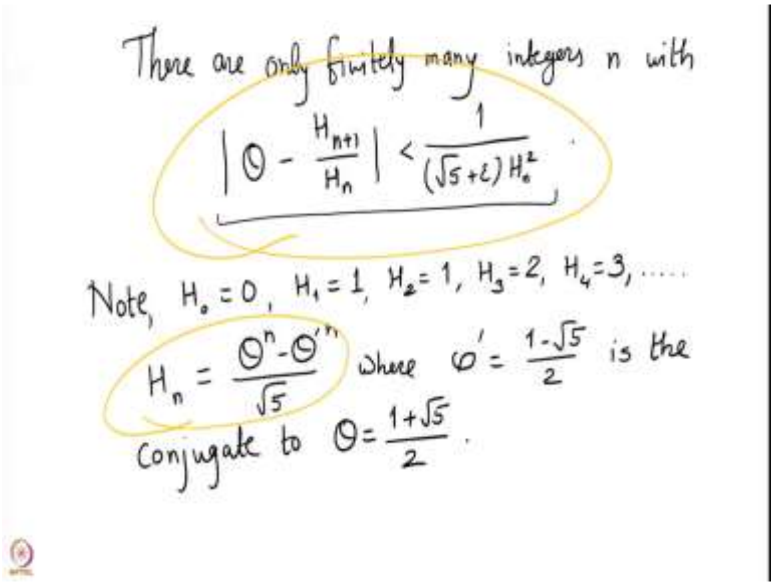

Hence, this inequality has only finitely many solutions and this is true for any epsilon, we had not fixed, we had not chosen any special epsilon, we just chose epsilon to be bigger than 0. And therefore, we have that the theorem of Hurwitz is possible only for root 5 we cannot make it to root 5 plus epsilon for whatever small number epsilon positive epsilon that you may want to choose. With this, we come to the end of our course.

It has been a pleasure to give this course I would like to thank the whole NPTEL recording team, especially Tushar for cooperation in recording these lectures.

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I would also like to thank my own team. My TAs Aakash, Anuradha, Brahadeesh and Venkatesh their inputs made this presentation more in their inputs have improved this presentation, and it would have been very difficult to conduct this course without their help. And lastly, it is a pleasure to thank you all for being interested in this course and being with me until this far. I hope to see you some other time. Thank you very much, and bye-bye.