A Basic Course in Number Theory
Professor Shripad Garge
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture 55
Convergents give better approximations

Welcome back we are studying the properties of the convergents, we have proved that any real number theta has a continued fraction expansion constructed in a very natural way, once you cut that continued fraction expansion at any finite stage we get rational numbers, we call them convergents, they are denoted by the convergents pn upon qn and towards the end of the last lecture we proved that the convergents to theta are the good rational approximations.

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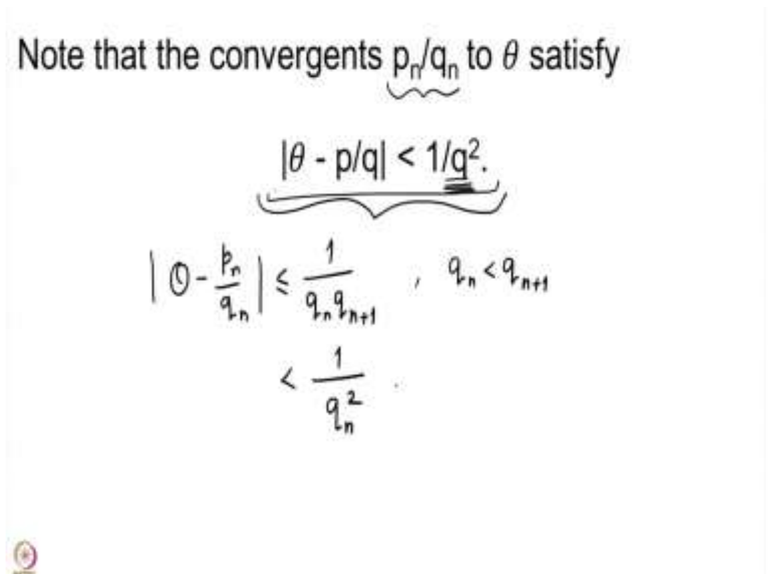

So, in fact these convergents satisfy the inequality that theta minus p by q is less than 1 by q square, the convergent’s pn upon qn satisfy the inequality that theta minus p by q is less than 1 upon q square. We are going to prove that these are the best rational approximations, but before we prove that these are better rational approximations we know what is a good rational approximation any rational p by q to theta satisfying theta minus p by q is less than 1 upon q square.

The difference is less than 1 upon the difference is less than 1 upon the denominator square, these are called good rational approximations and we are going to prove that our convergents are doing something better. So, these are not just good rational approximations, but they are better rational approximations.

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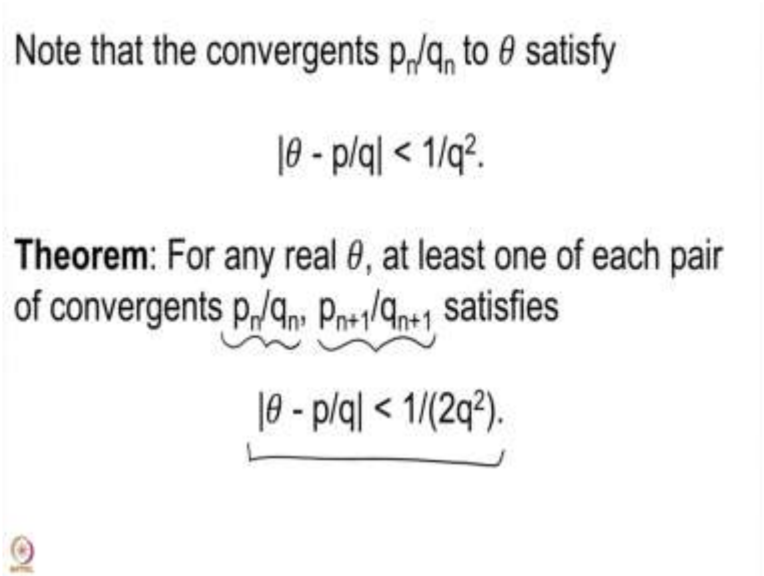

So, the result that we are going to prove is the result as follows. We have noted that our convergents are the good rational approximations, but we are going to prove that these are the better rational approximations in the following sense that if you take any pn upon qn and pn plus 1 upon qn plus 1, so take these pairs of successive convergents then one of them should satisfy that theta minus p by q is less than 1 upon 2 q square.

So, we have already noted that the convergents are closed in a to the real number theta in the sense that the difference is less than 1 upon q square, but we proved that one of these two is going to be much closer to theta, the difference is going to be less than 1 upon 2 times q square, this is what we are going to prove in this theorem. So, one of the two, pn upon qn and pn plus 1 upon qn plus 1, one of these two should satisfy this better approximation, this is what we are going to do.

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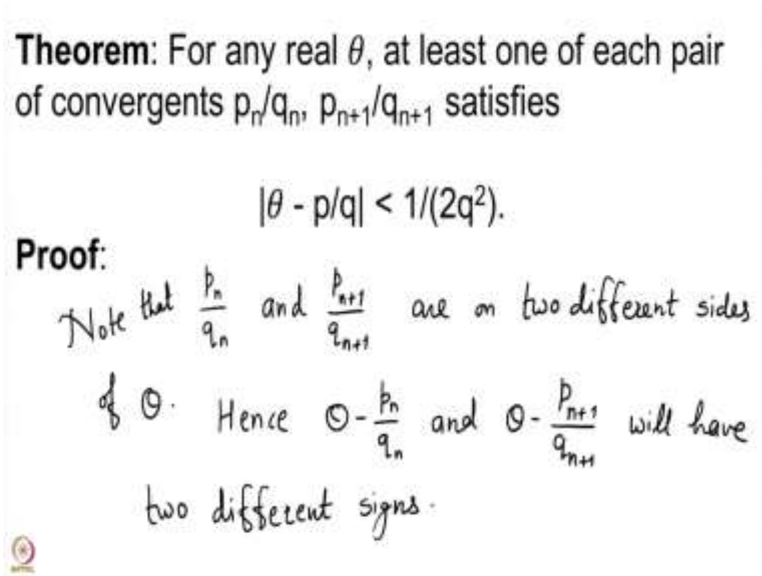

Note here that pn upon qn and pn plus 1 upon q n plus 1 these are on two different sides of theta. So, what does it say; We have these two numbers which are on two different sides of theta that means if I subtract each of them from theta then for one of them we are going to get a positive quantity and the other will be a negative quantity.

The difference will have two different signs the differences of these two successive convergents from theta will have two different signs, this is the property that we are going to have. So and if one of them is positive the other is negative that is what we have. So, we now take their sum and put a mod that is what we are going to do.

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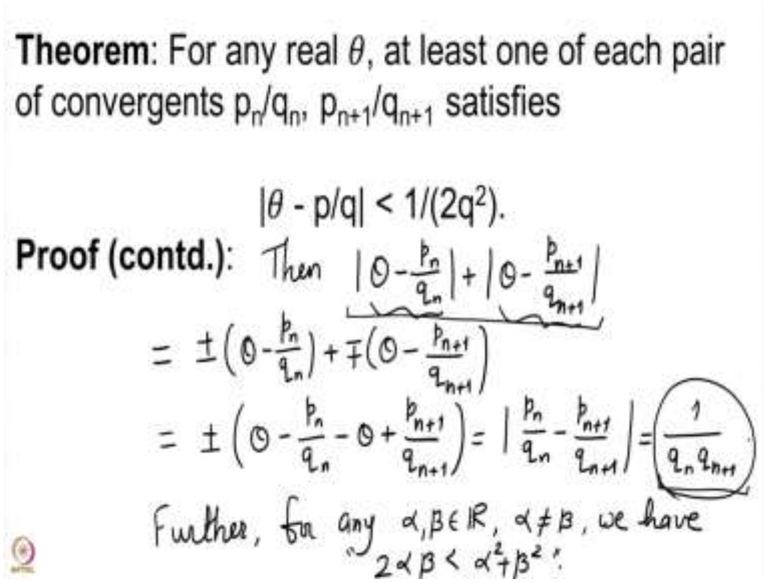

Mod of theta minus pn by qn plus mod of theta minus pn plus 1 upon qn plus 1, we consider the sum of their mods. Now, this quantity can be positive or negative, therefore the mod will be equal to that quantity or it’s negative. So, you have that this is plus or minus theta minus pn over qn and this quantity is also plus or minus its modulus, but if this is positive then this has to be negative and if the first one for the nth one is negative then the second has to be positive.

Therefore when we take the first one could be positive the second is negative, first one to be negative second is positive. So, this will give us we take the plus minus sign common and we get theta minus pn upon qn minus theta plus pn plus 1 upon qn plus 1 which gives you mod pn by qn minus pn plus 1 upon qn plus.

And we note that this is equal to 1 upon qn qn and plus 1. This is something that we have already noted because the numerator here will be pn qn plus 1 minus qn pn plus 1 that is plus or minus 1 so under the mod it is going to be 1 and the denominator continues to be qn into qn plus 1. So, we have that this sum which is the sum of the two moduli is equal to this particular positive number.

This is the thing that we have proved, further we use one inequality for any alpha beta which are distinct we have 2 times alpha beta is less than alpha square plus beta square, this can be seen because the if you take 2 alpha beta or to the other side of this inequality you are going to get alpha minus beta whole square, but alpha is not equal to beta therefore alpha minus beta is a nonzero quantity, it can be positive or negative but it's square will always be positive. So, we therefore that 2 times alpha beta is always less than alpha square plus beta square, we use this to get our final inequality by putting the alpha to be 1 upon qn and beta to be 1 upon qn plus 1.

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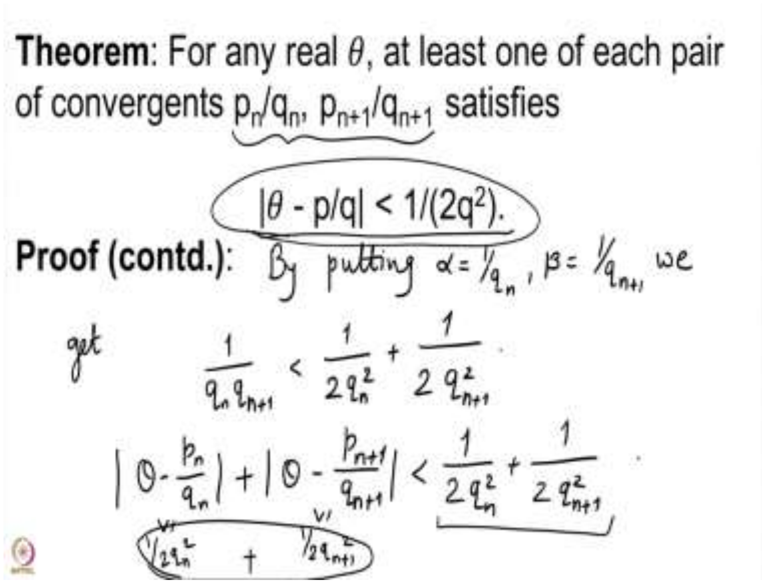

So, we get that theta minus pn upon qn in the modulus plus theta minus pn plus 1 upon qn plus 1 in the modulus is strictly less than 1 upon 2qn square plus 1 upon 2qn plus 1 square. Now, what we want to prove is that at least one of the pairs pn and qn should satisfy this inequality. If you assume that this is not true that means pn by qn and pn plus 1 by qn plus 1 both do not satisfy this inequality that would mean that for both of these we will have that this is bigger than or equal to 2qn square and this is bigger than or equal to 1 upon 2qn plus 1 square.

So, the sum will be bigger than or equal to the sum of these two, but that is a contradiction because it is strictly less than the sum. So, if you have two numbers, if you let us say have 4 numbers and you have a plus b is less than c plus b, then you should have that a is less than c or b is less than d, if you have a bigger than or equal to c and b bigger than or equal to d then a plus b is going to be bigger than or equal to c plus b.

So, when you have the sum of two positive numbers to be strictly less than sum of two other positive numbers, then at least one of the two's on the left hand side should be less than at least one of the two's on the right hand side. So, this is the inequality that we get and therefore at least one of the pair of successive convergence should satisfy this even better inequality.

So, the convergents give us a better approximation to the real number theta that we began with. So, we proved first of all that convergents to theta which we have obtained through a natural continued fraction expansion the convergence give good approximation, but they satisfy even this better result, therefore they are better approximations, ofcourse you have that one of the two will have this property, so whenever you take p0 by q0, p1 by q1, one of these two should satisfy this better one, p1 by q1 and p2 by q2 again one of these two should satisfy and so on.

So, you get a sequence of rational numbers converging to your theta giving you a better approximation than many good approximations. We can improve this result even further, we can in fact prove that the two that we had in the denominator with q square can be replaced by root 5. So, let us see what we have done so far.

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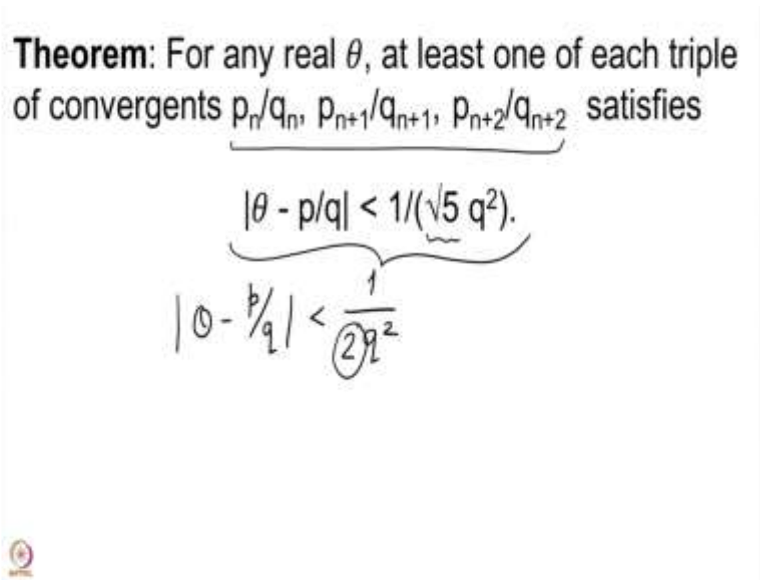

We have proved that there is a sequence given by the convergents satisfying that theta minus p by q is less than 1 upon q square, this is true for all convergents, then we prove that one of the pair of successive convergents should give even better approximations. And now we are proving that if you take the triples pn by qn pn plus 1 by qn plus 1, pn plus 2 by qn plus 2 then at least one of these should give even better approximation, that these two can be further increased to be to root 5.

So, at least one of these three successive convergents should satisfy this very strong inequality that means these convergents are really close to the real number theta they are not less than just 1 upon q square not just less than 1 upon 2q square but one of them has to be less than the distance from theta has to be less than 1 upon root 5 q square. So, this proof is slightly more complicated than the earlier proof which we proved for 2, so we are going to skip this prove for the moment, but this proves one result which was proved by Hurwitz way back in 1891.

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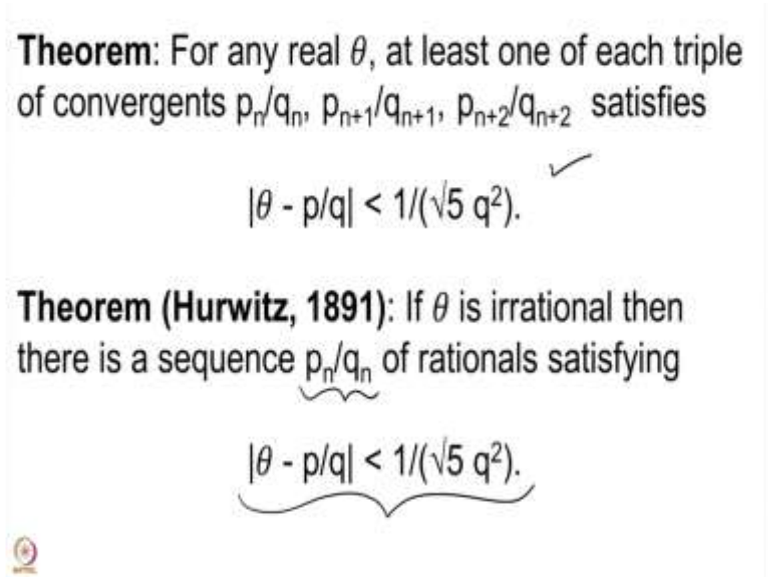

So, let me just quote this result for you. It says that if your theta is an irrational number, so this theta is an irrational number then there is a sequence of rationals satisfying this very strong inequality. This was proved in earlier by Hurwitz and what we have discussed so far if you prove this result, it will give us another proof of Hurwitz's result.

Now, you may ask, can you make this root 5 even better; is it possible to increase this constant root 5 and maybe get something better than this and you may take one of the four consecutive or one of the five consecutive or whatever you want can you improve upon this constant root 5 that we get and mathematics is interesting in both the ways, if you could improve this that would be even better or if you could prove a result saying that no, you cannot improve further this root 5 is the best constant that you get then that would also resting result.

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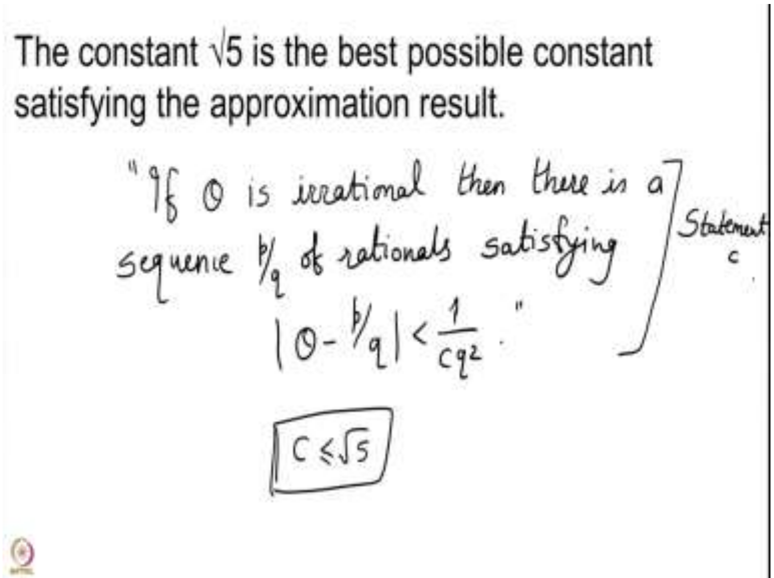

Here we have that this constant root 5 is the best possible constant that we have, so let me just explain what we mean by this, this means that consider the following statement, if theta is irrational then there is a sequence p by q of rational numbers satisfying mod theta minus p by q is less than 1 upon cq square, suppose we proved such a result for some c, so let us call this statement c.

Then the statement holds for every c up to root 5, of course we are we have proved it for root 5 and therefore it holds for every smaller v up to root 5, but it does not hold for any c bigger than root 5. And what does it mean to say that it does not hold for anything bigger than root 5, it means that there is one particular irrational for which the statement c bigger than root 5 does not hold because our statement is for all irrationals if you wanted to fix a c bigger than root 5 and you wanted to see whether that statement holds my claim is that the statement does not hold.

So I should give you an irrational for which that statement for a c bigger than root 5 should not hold and we are going to give that irrational, I will give you the irrational right away, but to prove that the statement for c bigger than root 5 does not hold for that irrational we will have to wait. So, we have not computed any nice genuine continued fraction expansion yet let us do that and compute the convergents or the continued fraction expansion to the golden ratio.

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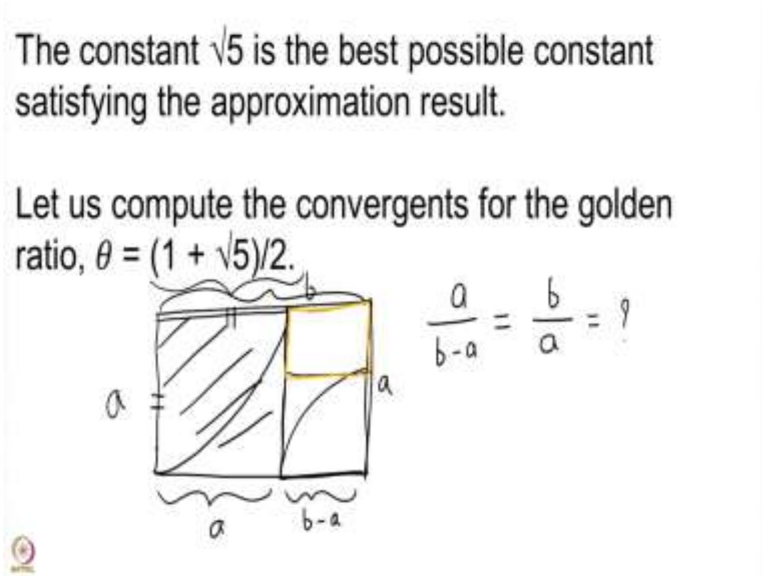

Let me explain the term golden ratio here for you first, so this was the question which was posed by some of the ancients to the other of the ancients that if you take a rectangle suppose of the sides the side is a and b and if a is let us assume that a is less than b and you cut out this square, so this is length a and this is b minus a, b is taken to be bigger than a, so the remaining is a positive quantity.

Here we get another rectangle then we assumed that this rectangle the new rectangle is congruent to the earlier rectangle, that means the sides of these new rectangle their ratio which is a upon b minus a is congruent. So, the ratio is same as the new ratio which is b upon a. So, remember here b is this full length.

And so the question is what can this ratio be if I wanted to construct the rectangles with this property that when I cut out a square obtained by taking the smaller side, so we put this thing here therefore this is equal to the length a and take it out take that square out the remaining rectangle should be congruent to the earlier rectangle, that means you expand the rectangle by a fixed quantity, that means you expand both the length and breadth of the new rectangle by a fixed quantity multiply that by a fixed quantity and you get your original rectangle.

Then what should the ratio b? And the actually the question is, does it hold again for the smaller rectangle? Now, here b minus a is smaller than a, so you cut this square from here and what you have obtained here is a smaller length, so we have this new rectangle it will turn out that this new rectangle will also be congruent to the first rectangle which we had obtained and also congruent further to the original rectangle.

Then the question is what this ratio can be and this ratio turns out to be 1 plus root 5 by 2 and therefore it is called golden ratio it is very useful in many geometry configurations, in fact there are these geometry poly-hydra the regular solids and in there construction also this golden ratio is very useful, you should perhaps see and search on internet for more material on this, we are not going to spend more time on this information about golden ratio, but we will try to now compute the continued fraction expansion for our theta which is 1 plus root 5 by 2.

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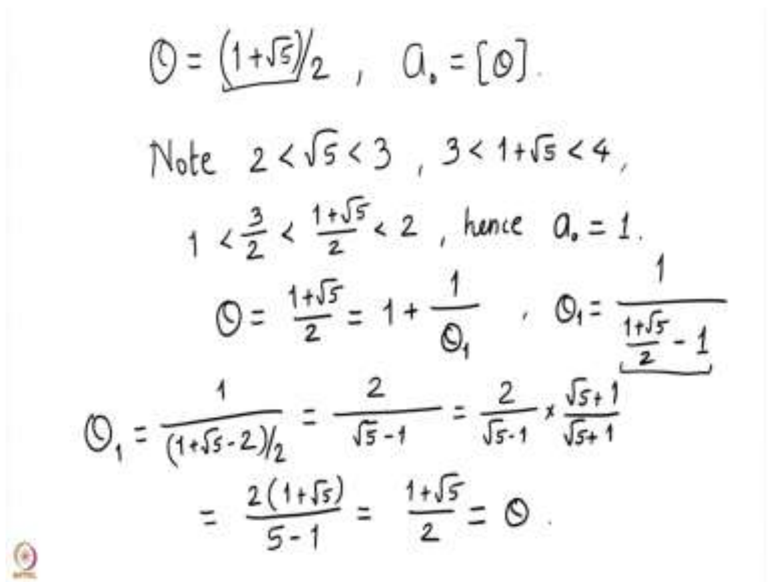

Theta is 1 plus root 5 by 2 we want to find the continued fraction expansions so we should compute a0 first which is the integral put of theta so that should be the largest integer less than or equal to the theta, how do we compute it? So, note that our root 5 sits between 2 and 3; root 5 is strictly bigger than root 4 and is strictly less than root 9, so this is between 2 and 3, we add 1 so we get that 3 is less than 1 plus root 5 less than 4.

So, at least the numerator of this quantity is between 3 and 4 divide the equation by 2 and you get 3 by 2 is less than 1 plus root 5 by 2 is less than 2, which is 4 by 2. And ofcourse we have that 3 by 2 is strictly bigger than 1 therefore a0 is equal to 1. So, we have that our theta which is 1 plus root 5 upon 2 is 1 plus 1 upon theta 1, where theta 1 is certainly bigger than 1, but we should be able to compute the quantity theta 1 from whatever we have obtained so far.

So, theta 1 its reciprocal is 1 plus root 5 upon 2 minus 1, so this is 1 plus root 5 by 2 minus ones reciprocal. So, we compute theta 1 we are going to simplify this, this becomes 1 plus root 5 minus 2 whole thing divided by 2, so you have 2 in the numerator and root 5 minus 1 in the denominator, so this root 5 has gone in the denominator we want to simplify it further by bringing the root 5 to the numerator and the standard trick to do that is to multiply both sides by what is called the conjugate of the denominator.

So, numerator is 2 times 1 plus root 5 and denominator is of the form alpha minus beta into alpha plus beta, so that is alpha square minus beta square, alpha square is (5) root 5 square, so that is 5 minus the square of 1 is 1, so this is 4 we have 2 in the numerator we get it to be 1 plus root 5 upon 2 which is the theta we started with. So, theta 1 is equal to theta.

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$$\theta = 1 + \frac{1}{\theta_1} = 1 + \frac{1}{\theta} = 1 + \frac{1}{1 + \frac{1}{\theta}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\theta}}} = [1; 1, 1, 1, \ldots].$$

$$\frac{1+\sqrt{5}}{2} = [1; \underbrace{1, 1, 1, \ldots}] = [1; \bar{1}] = [\bar{1}].$$

Therefore theta which was 1 plus 1 upon theta 1 is in fact equal to 1 plus 1 upon theta and we just keep putting these values and we get the continued fraction expansion for the number theta, so this is going to be 1, 1, 1, 1, dot dot dot the continued fraction expansion for the golden ratio is indeed one of the most beautiful continued fraction expansion that we should have which is 1, 1, 1, 1, 1 and since these ones are going to just continue as they are, it is customary to denote it by putting the bar on the top of the 1.

So, whenever you have an expression in the continued fractions expansion where the pair some certain subset of the integers that you get keeps repeating after a stage then you put a bar on the top. Ofcourse here we have taken the first one out because that was the integral part, but you may also write it in this way.

So, the golden ratio has this very beautiful continued fraction expansion it is simply given by 1 plus 1 upon 1 plus 1 upon 1 plus 1 upon dot dot dot, you can also check it by observing the quadratic equation that this golden ratio satisfies, that will also give you this particular continued fraction expansion.

But you have to be slightly careful that quadratic equation will have two roots, being a quadratic equation it will have two roots, one of them is the golden ratio which is a positive quantity, the other is going to be a negative number, in fact it is 1 minus root 5 upon to that is a negative number and that does not quite have this particular continued fraction expansion. So, do not get confused, although we will it may seem that you are using the quadratic equation to get this expansion, it does not mean that any root of the quadratic expansion any root of the quadratic equation gives you the same continued fraction expansion.

Here we have that our quadratic continued fraction expansion consists of only positive numbers and therefore it is going to converge to a positive quantity, which will therefore be the positive root of that equation which is our golden ratio. I will just quickly tell you one more thing about what are called as Hemachandra numbers or which are more commonly known as the Fibonacci numbers.

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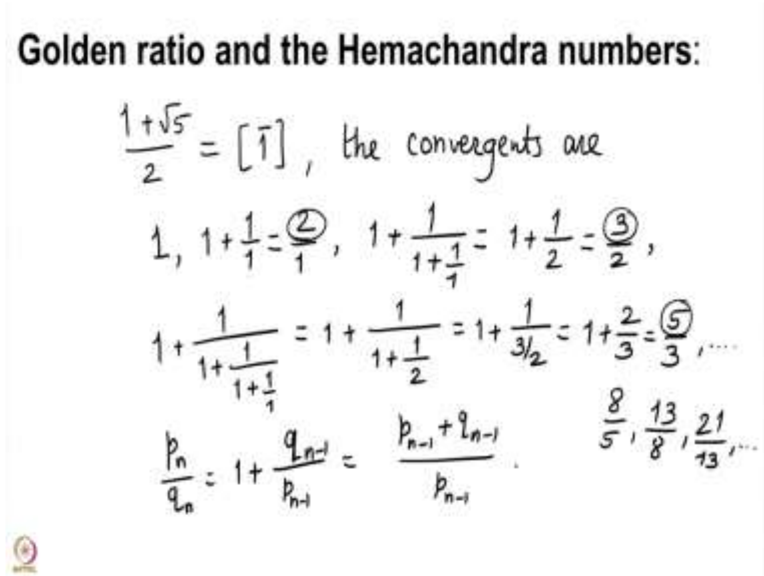

So, if you wanted to compute the convergents to this 1 plus root 5 by 2 we have noted that this is simply 1, 1, 1, 1 repeated the convergents are the first one will simply be 1, next 1 will be 1 plus 1 by 1 which are 2 comma 1 and the 1 after that will be 1 plus 1 upon 1 plus 1 upon 1 which is 1 plus 1 by 2 which is 3 by 2, the next one is let us be patient and solve this out and dot dot dot.

The reason for this is that pn by qn is going to be 1 plus 1 upon pn minus 1 and here you have qn minus 1. So, you are going to get pn minus 1 plus qn minus 1 and pn minus 1. So, the next ones the 2 that you have obtained here is 1 plus 1, the 3 that you obtained here is 2 plus 1, 5 is 3 plus 2, the next 1 would therefore be 8 upon 5 after that you are going to get 13 by 8, 21 by 13 and so on.

And these this sequence of numbers is very commonly known as Fibonacci numbers, but you should note that the Indian mathematician Hemarchandra had discovered this at least 10 years before Fibonacci and it is said that Fibonacci thought about these numbers by looking at the rabbits and the increasing population of rabbits whereas Hemachandra is supposed to have come to these numbers by observing the matras in the (())(30:02) that we have in classical music and so on.

So, Manjul bhargava, the field medallist from Indian origins has talked many times on this and I urge you to go to YouTube and find sir for Manjul Bhargava, Fibonacci and you will see videos where he explains very nicely how the attributes should really be given to Hemachandra and not to Fibonacci. We stopped at this level, but we will continue with getting the better and better approximations by the convergents and we will be doing that in the next lecture. See you then. Thank you very much.