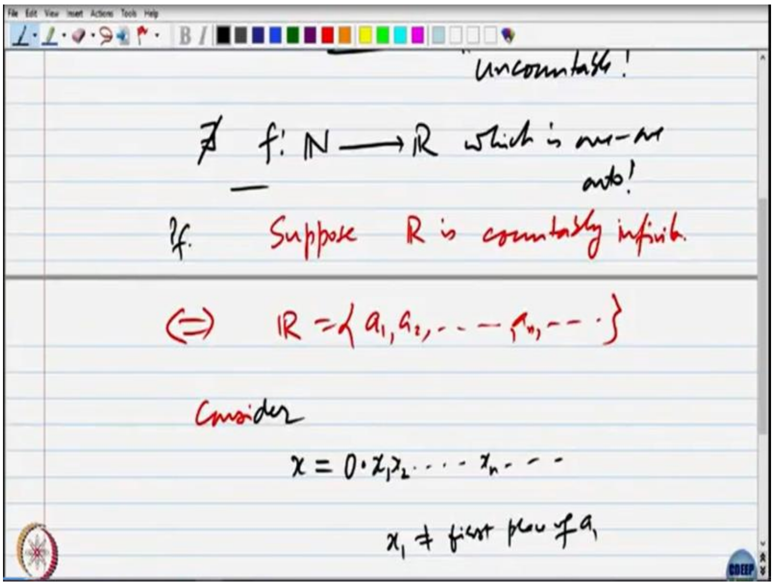
Basic Real Analysis
Professor Inder. K. Rana
Department of Mathematics
Indian Institute of Technology, Bombay
Lecture No 07
The LUB Property and Consequences – Part I

Let us place of a1. So, what will happen this number acts whatever be the decimals after the first place, it is not going to be equal to a1. x1 will be different from a1. Is that ok? x1 will be different from a1 because at the first decimal place they are different. It differs from the first decimal place of a1. So, x1 is not, x is not going to be equal to a1.

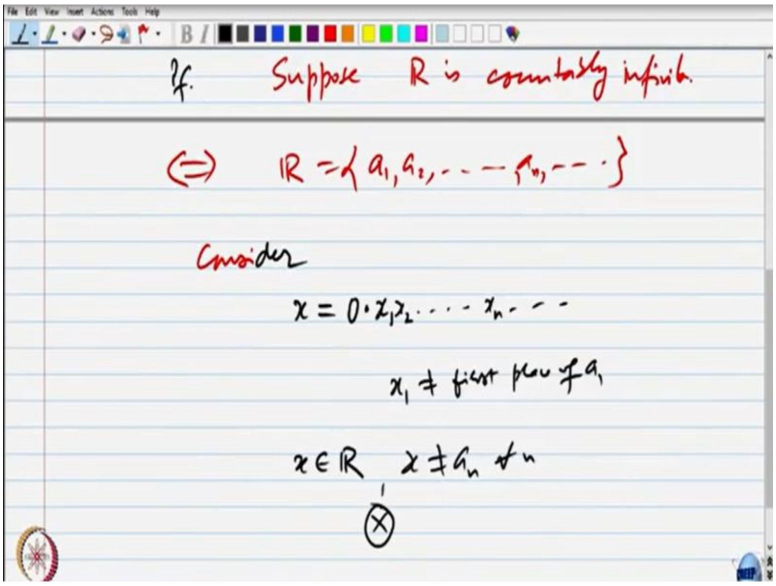
Now, I have a method of doing look at a2, look at the second place of a2 pick up some x2 which is different from the second place digit whatever appears in the decimal representation of a2. It will be 0, 1 to 9 something. So, pickup something whatever comes some different one. So, what will happen? This will also be different from a2 and I can go on doing it.

(Refer Slide Time: 1:32)



xn so what will be xn? xn will be the number the natural the non-negative number between 0 and 9 which is different from the nth place of an and go on doing it. So, I have inductive way of writing knowing the nth place I have construct the next place. Is it ok? So, what can you say about this number x?

(Refer Slide Time: 2:04)



x is a real number. Is x a real number? It is decimal number representation I have given you. I have given the decimal representation of x that is the real number x is not equal to an for every n. Because at the nth place it differs from the nth decimal, the nth place of an whatever digit comes. So, it cannot be equal. Two decimals are equal if all are equal simply. So, I have produce an x in R which is not equal to the, that is contradiction. Because every real number must be one of the an’s and we have produced real number which is not equal to anyone of the an’s. So, that solves the problem.

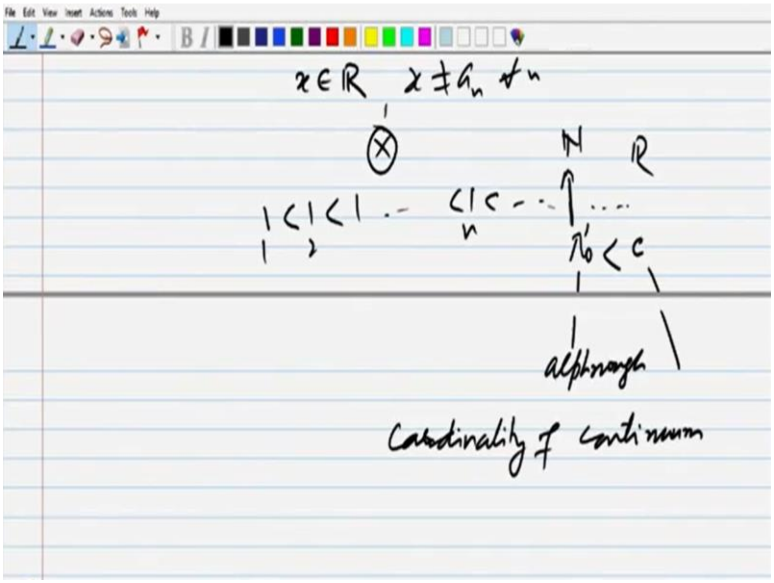
Student: Why have we taken the decimal representation? Now, what is the use of this thing? We could have use the normally taking x1 x2. I cannot get this point.

Professor: I want to construct a real number which is different from each one of a i's. So, the method states the following. Let us write a decimal representation of that number. Writing a number is same as giving decimal expression for that number, is that ok? How is the decimal representation is constructed? And I keep in mind that two real numbers are different if at some place in the decimal representation those differ.

So, keeping that in mind I am constructing x at the first place it is different from a1, at the second place it is different from a2 and so on that is all nothing more than that. That way x will be different from each one of the an’s and this 0 dot x1 x2 xn will be a real number. Because I am saying this is the decimal representation of x I am constructing.

So, it must be one the an's which is not true because it is not equal. It has to be also one of them so that is the contradiction. That is the way anything about infinity is proved. Still not?

(Refer Slide Time: 05:30)



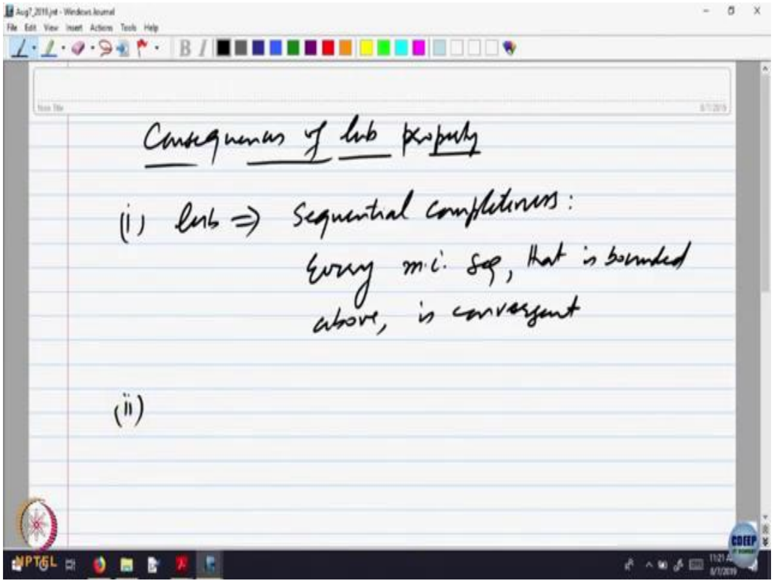
Anyway, just at this stage probably let me just so a is 1 2 n let us write something it is all the number line and the probably I should not give a picture of that let me just write 1 less than 2 less than 3 less than n. The number of elements in natural numbers let us denoted by something called it is called alphnough. So, this is called alphnough. Alph is a Greek letter like N and 0 is L. Why 0 soon will see and this thing there is no one to one correspondence between this real numbers. So, how many real numbers are there? You say uncountable.

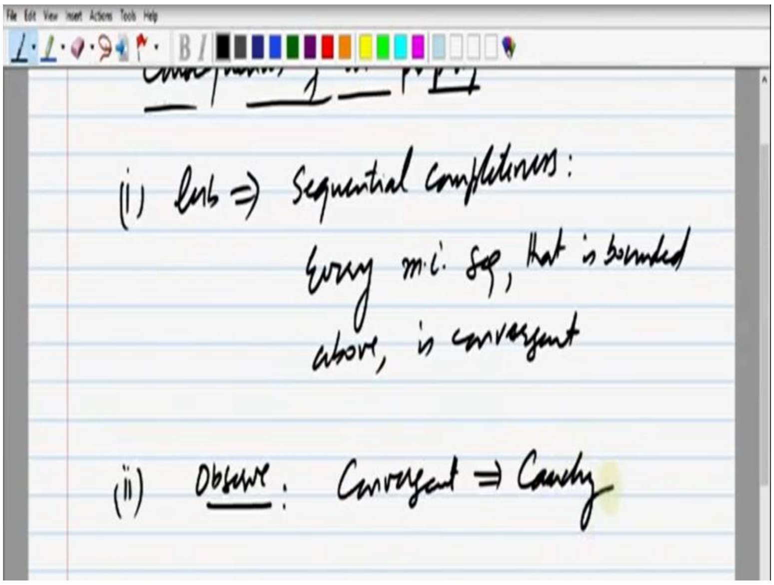
Let us given it a name let us this is denoted by small c so that is small c so that is this is N and that is R that is small c so that is Cardinality of continuum that is c. There is continues of points kind of thing and in some sense this is less than this because there is no one to one map but n sitting inside it. So, is it reasonable to say that n naught is strictly less than c? It does not make sense because both are infinities, but we can intuitively say that this is a kind of relationship between them.

Question in mathematics ever a deep one is there anything in between n and c, natural numbers one infinity, real numbers another infinity, is there any other infinity in between them or not? One question are there infinities beyond c? Are there infinities beyond c? That means is there a set whose number of elements is much more than c?

But R is sitting inside it like N is sitting inside R, R should be inside something which is much bigger. So, I think I will stop here by saying look at the power set of real numbers. How many elements do you think power set of real numbers has? So, keep thinking.

(Refer Slide Time: 8:08)





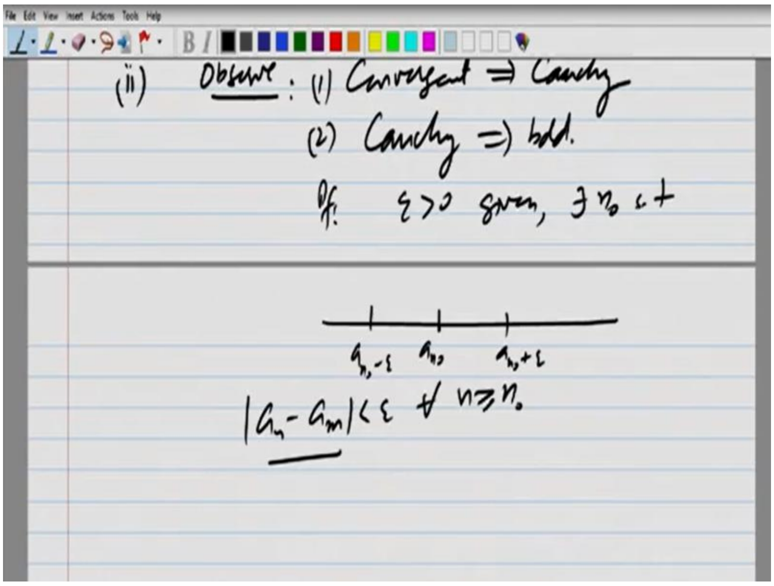
Let us comeback to our lecture for today. So, consequences of LUB Property. So, one consequence we said lub implies what we called as a Sequential Completeness that is saying that every monotonically increasing sequence, that is bounded above, is convergent. So, this is what we had proved last time.

So, second to prove the second thing first observation so observe we started looking at convergent implies it is Cauchy that we have seen last time that if a sequence is convergent, it must be Cauchy that means elements of the sequence must be coming close and close to each other.

Observe, let me give it some numbers 1 2 Cauchy implies it is bounded. What to claim like if a sequence is Cauchy, elements are coming closer. It cannot go, it cannot become very very large or very very small so intuitively every Cauchy sequence must be bounded. So, it is proved that, prove is quite so can bounded so prove.

So, let us take epsilon greater than 0 given there exist some n naught such that mod of an minus am is less than epsilon for every n bigger than or equal to n naught. That is Cauchyness. So, let us start an naught here is an naught. n naught is now known and that means for all n bigger than n naught, everything is close to each other.

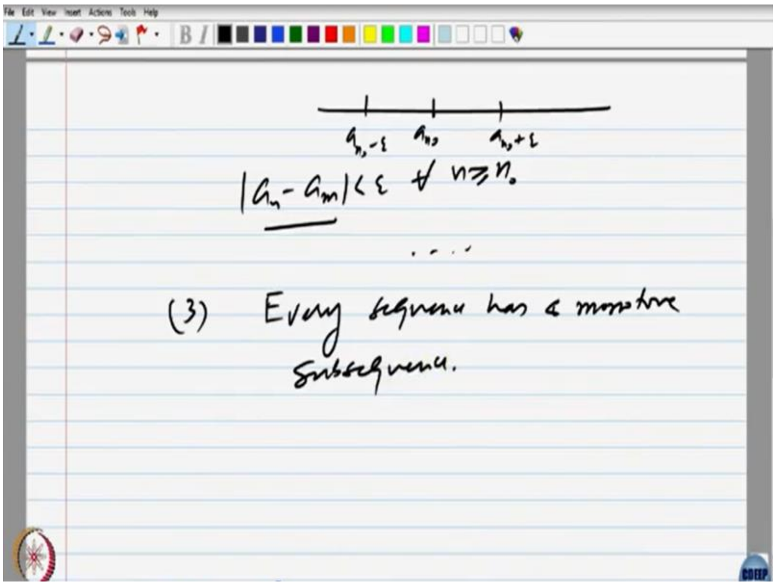
(Refer Slide Time: 10:59)



That means if I have take an naught minus epsilon and an naught plus epsilon, then everything must be inside after the say n naught, is that ok? Because they are close to each other. So, how many will be outside? Maybe a1, maybe a2, maybe an naught minus 1 again finitely many so I can take the like we have shown every convergent is bounded same proof repeated essentially saying that if is Cauchy then it must be bounded.

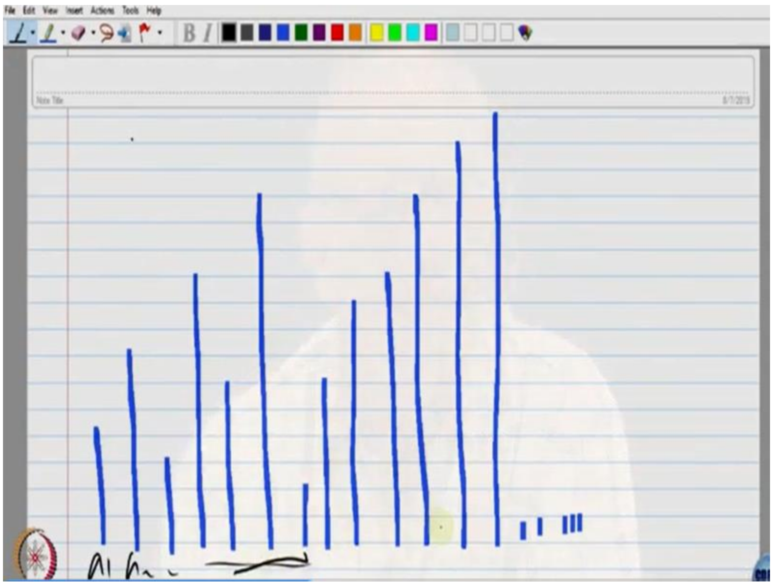
The minimum of a1, a2, an naught minus 1 and an naught minus epsilon called it alpha maximum of a1 a2 an naught minus 1 and an naught plus (alp) epsilon as beta then everything will be inside. So, let me so continue. Is that ok? Same prove basically same idea. So, Cauchy implies bounded.

(Refer Slide Time: 12:03)



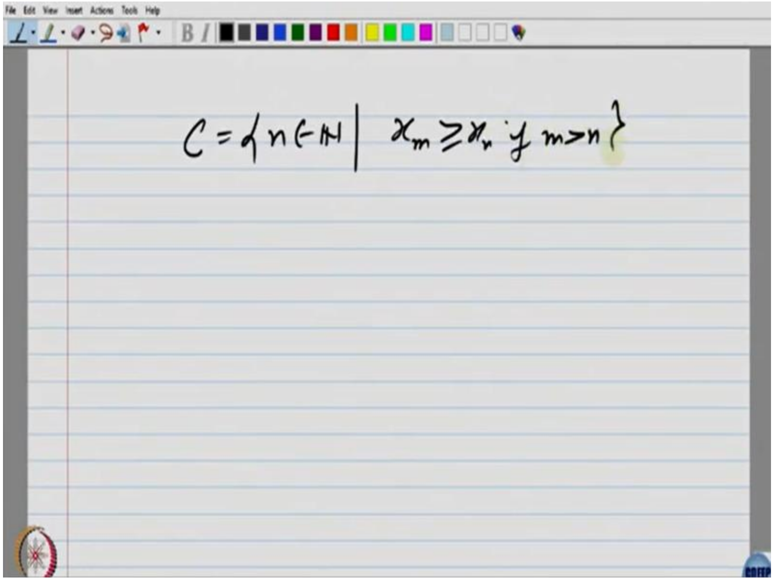
And here is the third observation, every sequence has a monotone subsequence. Given any sequence, it should have a subsequence. We define the notion of a subsequence. Picking up elements of the sequence but going ahead and ahead. So, the claim is given a sequence, there must be subsubsequence which monotonically increasing or monotonically decreasing. It should be a monotonic sequence. So, I tried to give you visualisation of this.

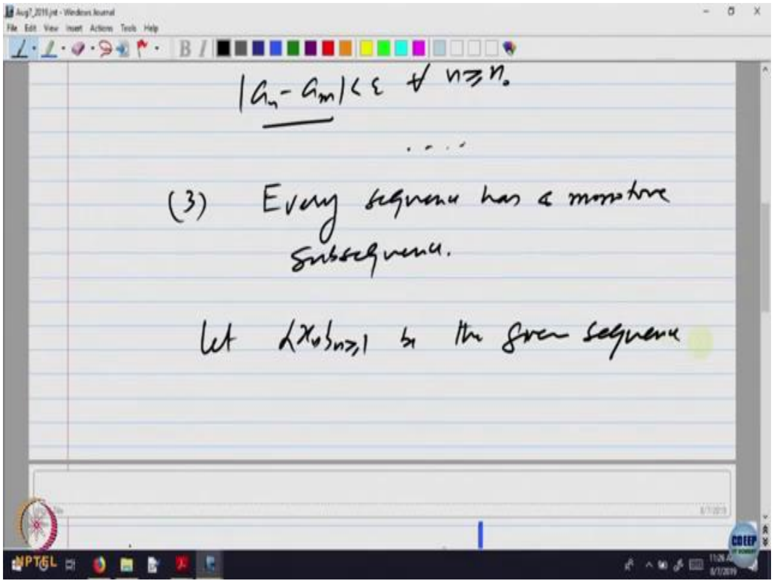
(Refer Slide Time: 12:48)



Let us imagine the sequence looks like this. This first one is a1. So, this is a1 a2 and so on. What do you observe in this picture that after this stage? I am able to see the height of every building, top of every building nothing is hidden from me. So, let us look at those numbers n, those indices n such that where we have m is bigger than n then am is bigger than an. For example, here except for this everyone else is ok.

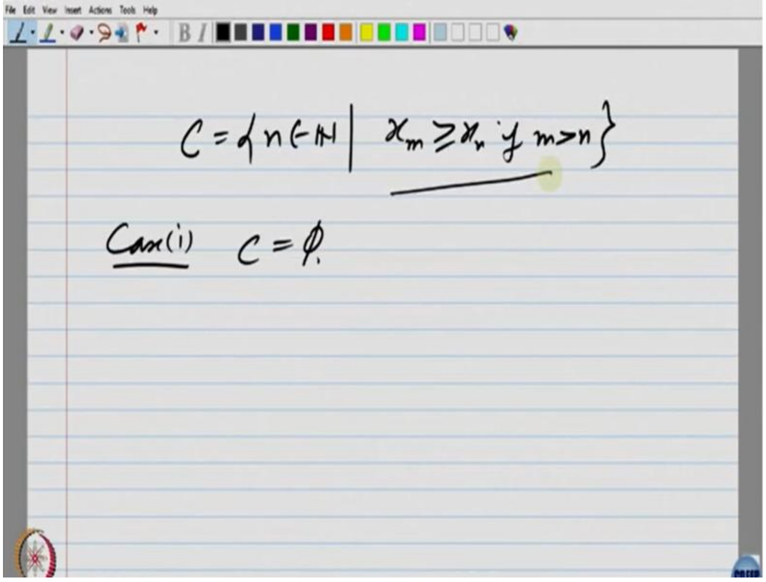
(Refer Slide Time: 13:39)





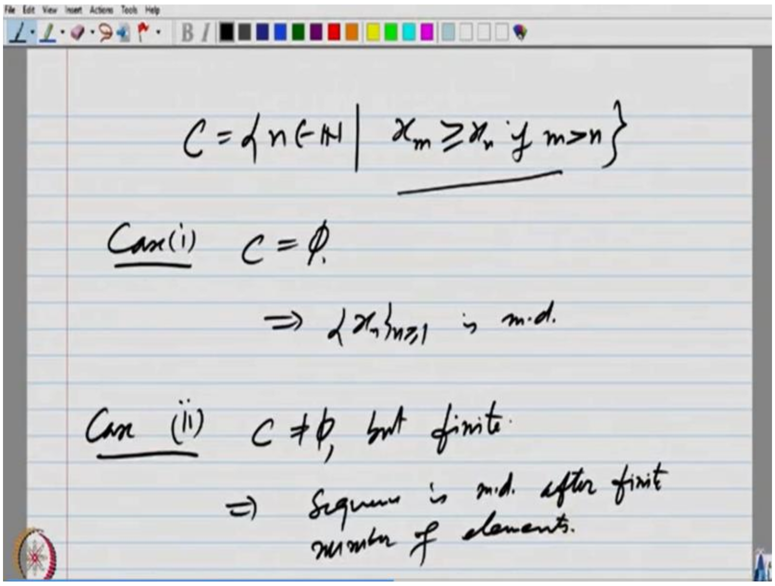
So, the set let us look at the set let us called it as C to be the set of all n such that xm what is the sequence given to us where we have written so let xn be the given sequence. Let xn be the sequence which is given to me. So, look at the indices n says that xm, xm is bigger than xn if m is bigger than n. So, what we are saying is at any place your standing look at the buildings after that you are able to see all of them, is it ok? So, this set a C is well defined set.

(Refer Slide Time: 14:39)



Now, what will be the possibilities? It may be a empty set, so case 1 C is empty. That means what? That means this statement is not true. So, whenever m is bigger than n, xm is less than or equal to xn, is that ok? But that means if I take n and n plus 1, then what is the relation between xn and xm? n plus 1 is bigger than at n, so xn plus 1 should be less than or equal to xn for every n and what does that mean? Sequence is monotonically decreasing, is ok?

(Refer Slide Time: 15:21)



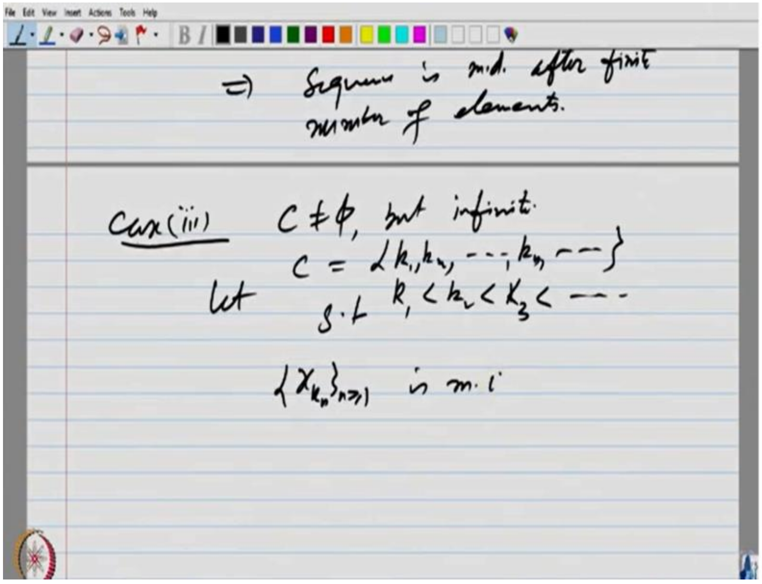
If this is so this implies xn is monotonically decreasing, sequence itself monotonically decreasing. What is the second possibility? It is a finite set, C is not empty but is finite. C is not equal to empty, but finite. If it is a finite set, it is the set of natural numbers. If it is a finite there is some largest elements in it the least upper bound property, it must have a largest element. What happens after that largest element?

This property should not be true, this property xm bigger than xn for m bigger than or, bigger than n is not true. That means what? After finite number of steps, finite number of elements after that the sequence starts decreasing. So, is that ok for everybody?

Implies that sequence is monotonically decreasing after finite number of elements. Understand what I am saying, if it is a finite set, C is a finite set, it has a largest element call it n naught something then after n naught what happens? After n naught whenever m is bigger than n, xm will be less than or equal to or strictly less than xn, is that ok? Because here we have written bigger than, negation of that.

We have collected all those for which this is true and that is a finite set. So, pick up the largest number of that, that is n naught, after that none of the indices n can be inside the set C because we have taken the largest of C that means what? That means after that stage n naught, if m is bigger than n, then xm is less than xn. That means after that stage, the sequence is monotonically decreasing. So, that is what I have written so after that stage it is.

(Refer Slide Time: 18:03)

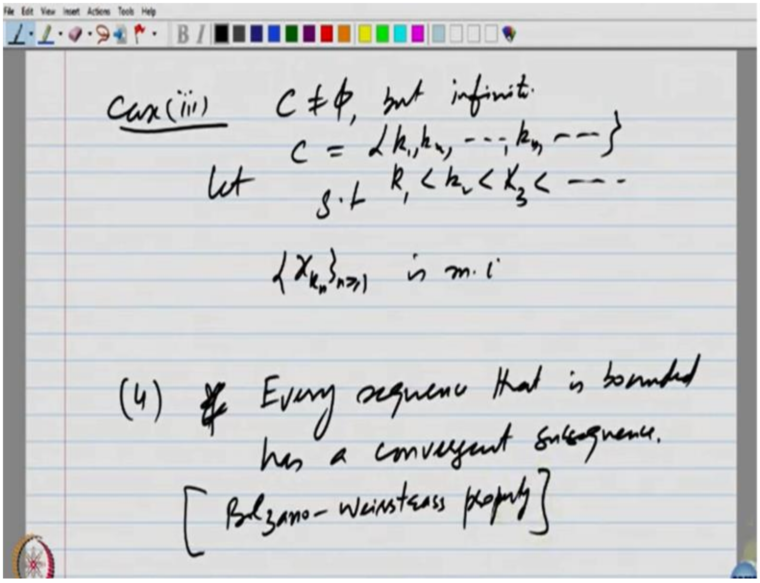


So, case 3 that C is not empty, but infinite. So, that means what? It is a infinite set. So, what does infinite mean? Whenever given any element in C, it is not bounded. There is something beyond that also possible. So, given any m in C, there is some stage kn which is also in C is that ok? Because it is infinite. So, if you like let us write it as so let C be equal to it is infinite set so let me write k1 so I can write it as k1 less than k2 less than k3 so this is the set.

I should not be write it less than that because it does not make sense of saying. You understand so let so let me write let C be equal to k1 k2 kn and so on it is a infinite set such that k1 is strictly less than k2 strictly less than, is it ok? k2 is bigger than k1 so xk2 should be bigger than xn. So, this is monotonically increase, monotonically increasing sequence is monotonically increasing is that ok? Because all elements of C kn plus 1 is bigger than kn so xkn plus 1 should be bigger than or equal to xkn because they are in C.

So, I have got a infinite number of element of C of the sequence that is a subsequence which is monotonically increasing is that ok? Everybody clear about it? So, that is something saying if you want to look at the pictures say given this I am able to see one but next one I am not able to see but I go here I am this I am able to see the next one. So, this is going to be my k1, this is going to be my k2 and such kind of things, is it ok? So, every sequence has got a subsequence which is monotonically increasing or monotonically decreasing.

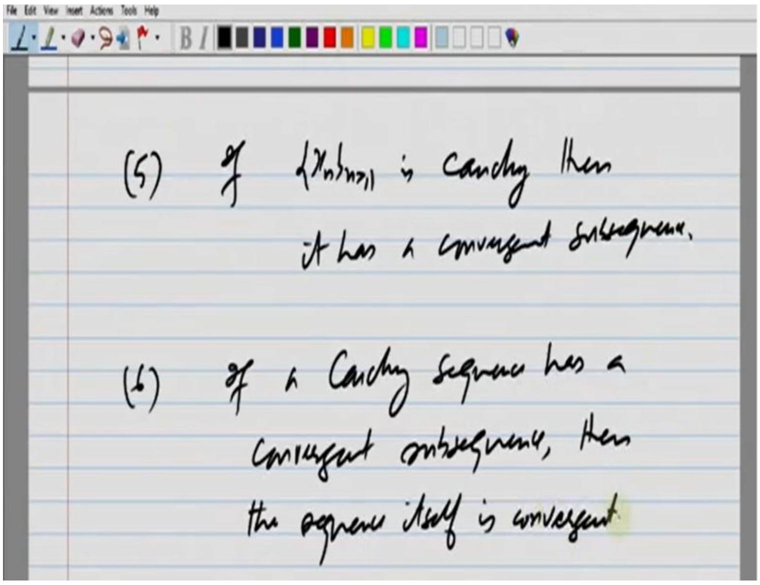
(Refer Slide Time: 20:53)



So, let us write a consequence of that if every sequence that is bounded has a convergent subsequence. So, given a sequence which is bounded by the previous property, it has a monotonically increasing or decreasing subsequence, so that is monotonically increasing or decreasing and bounded that subsequence so much converge by our earlier property sequential completeness.

So, we have to show is every sequence that is bounded has a convergent subsequence and this property goes by the name Bolzano-Weirstrass property. This goes by the name of Bolzano-Weirstrass property.

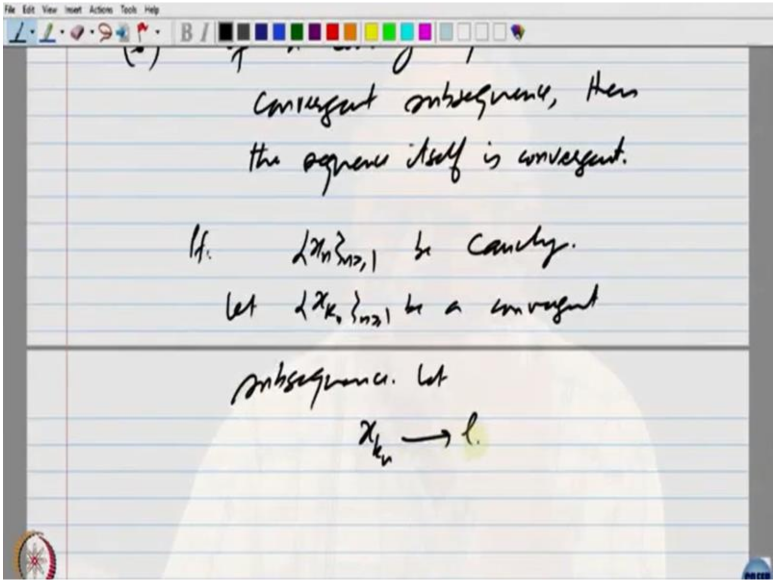
(Refer Slide Time: 22:06)

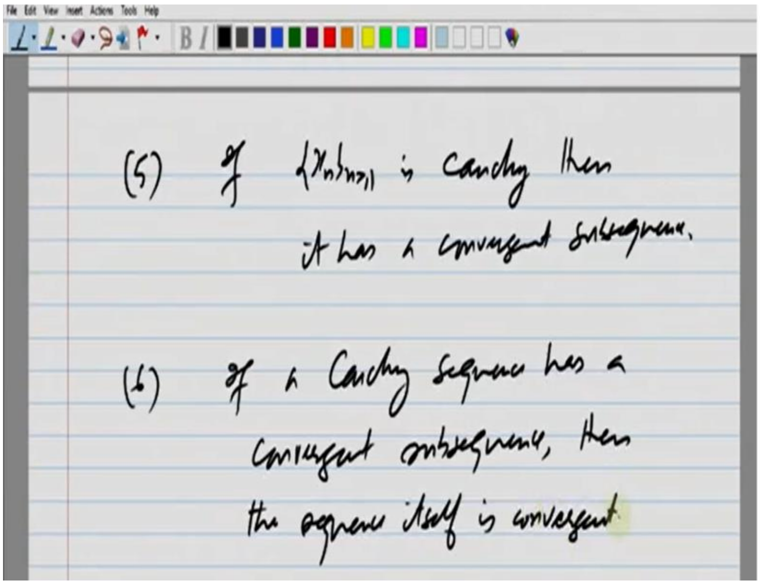


Now, let us look at the fifth consequence of what we are doing, if xn is Cauchy, then it has a convergent subsequence, why? Because we said Cauchy sequence is bounded. So, and as just now we said every bounded sequence must have a convergent subsequence. So, if a sequence is Cauchy then it has got a convergent subsequence. And sixth, if a Cauchy sequence has a convergent subsequence, then the sequence itself is convergent.

If a sequence is Cauchy, elements are coming close to each other and if it has a subsequence which is convergent is it clear that the sequence itself must converge to that limit? Not clear? So, let us try to understand so let us try to write a proof of this.

(Refer Slide Time: 24.23)



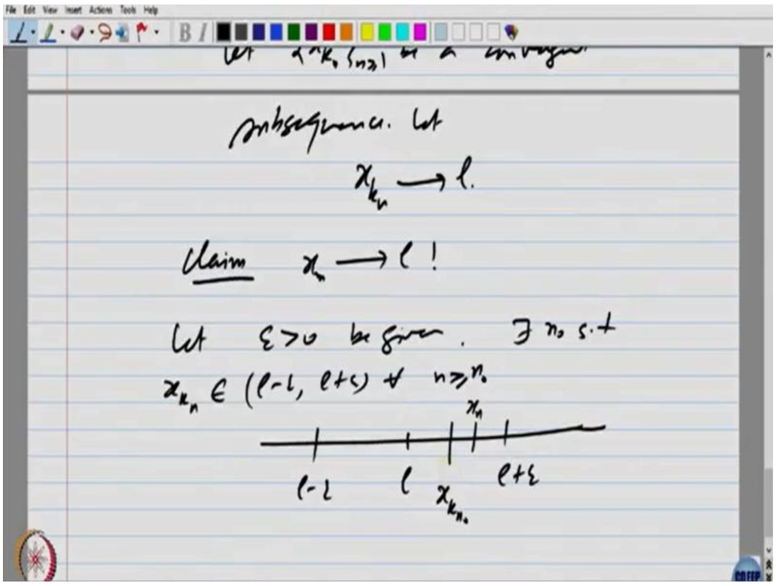


So, let us write a sequence xn be Cauchy let xkn be a convergent subsequence. I should write better be a convergent subsequence. So, let us say xkn is converge is to l. So, that is a every Cauchy sequence then it has a convergent of subsequence because Cauchy is bounded, every Cauchy sequence is bounded and every sequence has got a monotonically increasing or decreasing subsequence.

So, every Cauchy sequence, every, see if a sequence is bounded, it must have convergent subsequence. In particular Cauchy is bounded so it will have a convergent subsequence. Because every sequence has got a monotone subsequence but that monotone subsequence may not be bounded. But if a sequence is Cauchy then it is bounded. So, there are some sequence may be monotone and bounded and hence convergent. So, if a sequence is Cauchy then it has a convergent subsequence because it is bounded.

Now, what we are saying is if a Cauchy sequence we already know that it has a convergent sub sequence actually but what we are saying is that subsequence converges to l implies the sequence itself must converge to l. That is what we are trying to say. So, if a sequence xn is Cauchy let us assume it has a convergent subsequence. xnk converges to l.

(Refer Slide Time: 25:50)



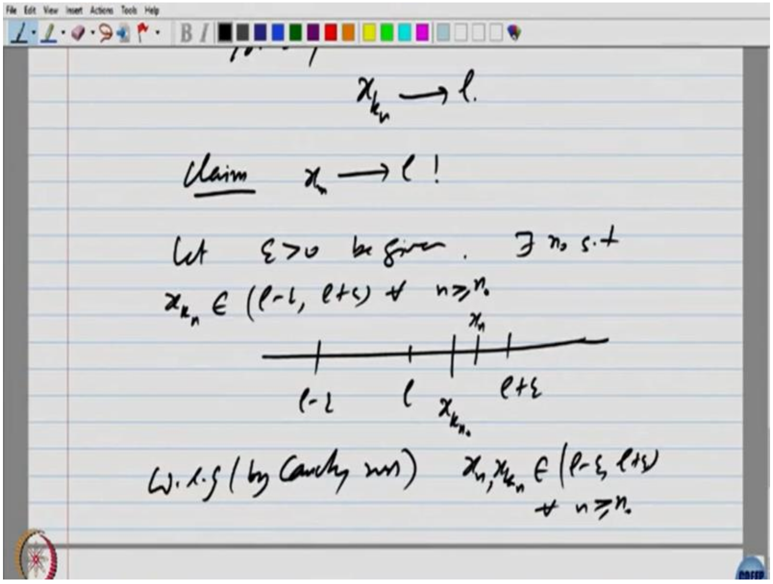
So, claim xn itself converges to l. Why is that? So, what we want to show? So, let epsilon greater than 0 be given, here is l, here l minus epsilon and here l plus epsilon. What we have to show that xn is convergent. After some stage all the xn must come inside it. But what we know elements of subsequence come inside.

So, given there exist some n naught such that what happens xkn belongs to l minus epsilon 2 l plus epsilon for every n bigger than n naught. Because the subsequence is convergent. So, here is my xkn naught. But the sequence is Cauchy, the sequence is Cauchy that means what? Elements are coming closer to each other. So, after some stage, after some stage elements will be closer to this elements of the subsequence also.

Because all elements are coming so I can say without loss of generality my xn is also close to xkn naught for n bigger than, it will be some other stage but I can take the maximum over the two stages if you want. Cauchy here says xn and xm are close by distance epsilon for n bigger than some stage n1.

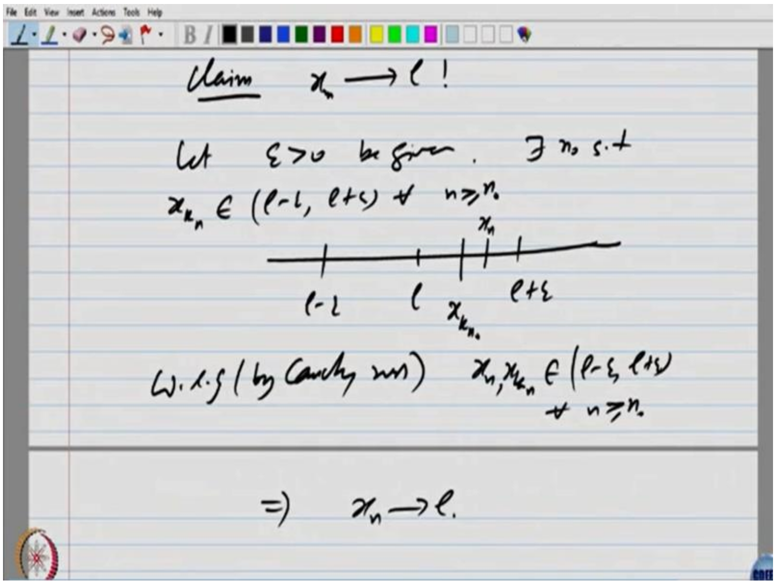
So, if I take this stage bigger than n1 and n0 that tale then what will happen? xkn will be inside l minus epsilon to l plus epsilon and xn will also be inside for n bigger than n naught and what does that mean? That means sequence is convergent. xkn is inside and xn is closer to xkn so both have to be inside that, that is all we are saying nothing more than that.

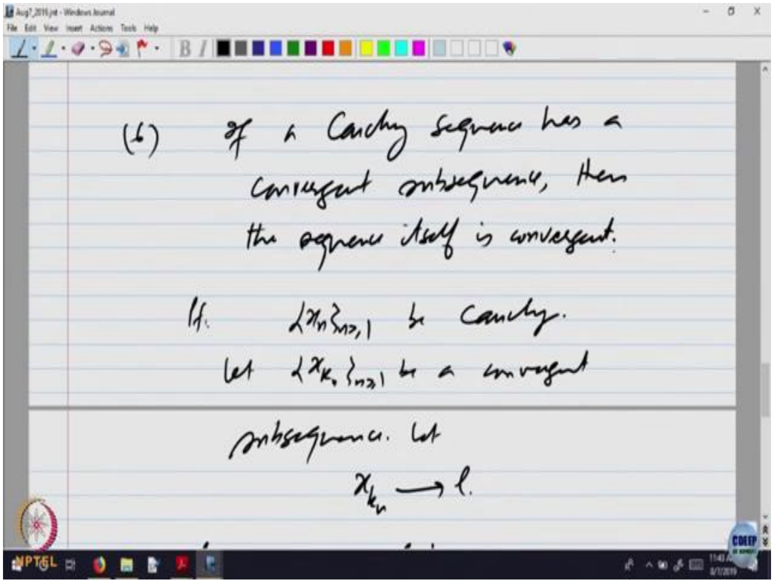
(Refer Slide Time: 28:24)



So, let us write that also. So, let us say without loss of generality by Cauchyness xn belongs to l minus epsilon, xn and xkn, xn and, xn xkn both belong to l minus epsilon to l plus epsilon for every n bigger than n naught. Is that ok? That is a Cauchyness.

(Refer Slide Time: 29:01)

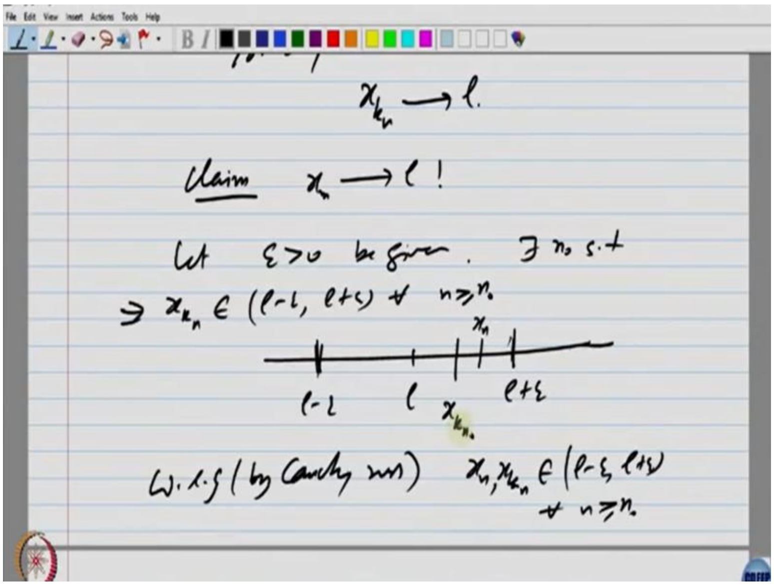




But that is same as saying implies xn converges to l because xn is coming inside l minus epsilon n l plus epsilon for n bigger than n naught, is it ok? So, that proves that this fact that if a Cauchy sequence has a convergent subsequence then the sub, then the sequence itself must converge. So, what we are saying now, take a Cauchy sequence, Cauchy is bounded, Cauchy has got a monotone subsequence so that must converge because it is bounded so Cauchy sequence also converges.

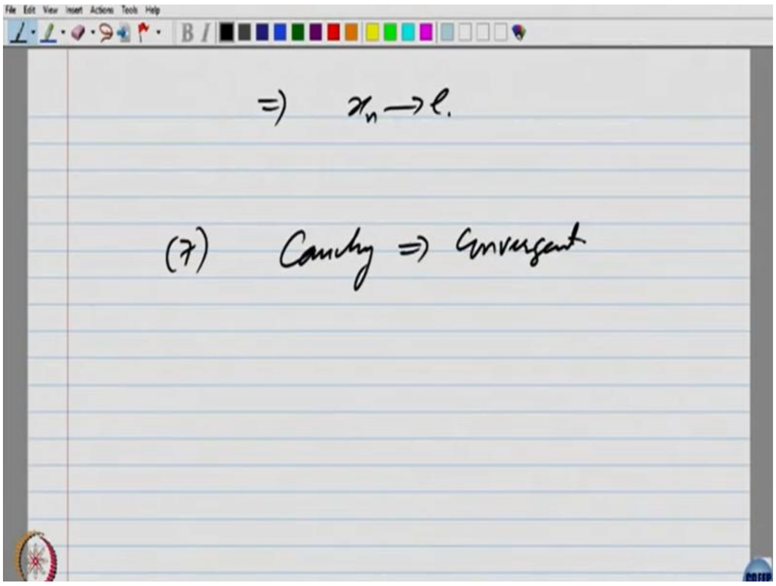
Earlier we have proved every convergent is Cauchy and now we are proving the converse that every, we proved convergent is Cauchy, now we are proving every Cauchy is also convergent. So, all this together.

(Refer Slide Time: 30:03)



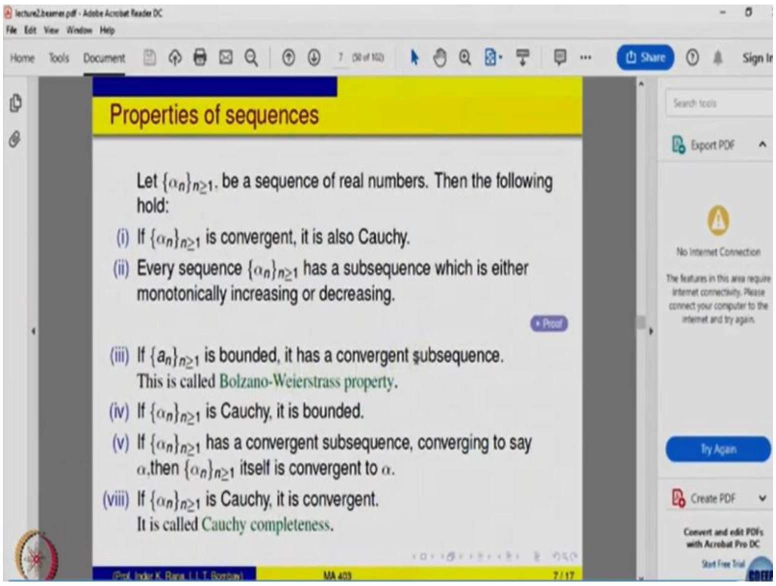
So, here is l minus epsilon here l plus epsilon by convergence of subsequence. This must have xkn inside it for n bigger than n naught. But for n bigger than n naught if you like or some other stage xn must be closer to xkn by Cauchyness. There is a stage that the tale or the sequence elements are closer. I can assume that tale is same as this one or you can take any other tale bigger than that. So, after some stage kn are close to l and xn are close to kn then where is xn? xn are close to l that is all nothing more than that, is that ok?

(Refer Slide Time: 30:57)



So, that is, so finally let us write seventh Cauchy implies convergent. So, every Cauchy sequence in the real is also convergent. Very important property of real numbers because this property is not true for rational numbers. There are sequences of rational numbers which are Cauchy but are not convergent to a rational. They will of course converge because they are, they will converge to a real number. So, let us try to see an example of that.

(Refer Slide Time: 31:37)



So, all these steps are ok? Let me just revised what we have done if this convergent then it is Cauchy. If a sequence is convergent then it is Cauchy we have proved the first thing. Every sequence has a subsequence which is either increasing or decreasing that buildings heights. Every bounded, if a sequence is bounded, it has a convergent subsequence because if a sequence is bounded by the previous one it has got a monotonically increasing or decreasing subsequence and hence it will converge.

Cauchy sequence is always bounded we proved that and if a sequence is has a convergent subsequence, converging to say then, now I think this there is a typo error. If a Cauchy sequence which has a convergent subsequence converging to alpha then this itself must converge to. Because sequence can have many different sequences convergent. So, this one, there is a step missing.

If a sequence alpha n is Cauchy and has a convergent subsequence then the sequence itself must converge that is what we have proved. So, every Cauchy sequence is convergent. So, LUB property is also equivalent to saying, every convergent sequence is Cauchy. A sequence is convergent if and only if it is Cauchy. So, this is called the Cauchy completeness of real numbers and that is equivalent to LUB property both as a consequences of that we are showing.