Basic Real Analysis
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Lecture 69
Alternating series and Power series

So, we were looking at a series of non negative numbers and we looked at various tests of convergence. For example, comparison test, root test and integral test. We start looking at series which are not necessarily non negative.

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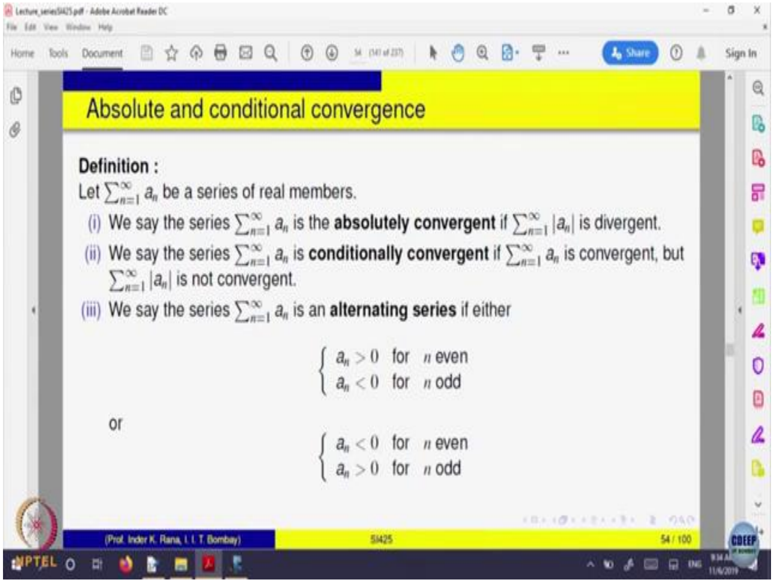

So, let us define a series an to be absolutely convergent, if you take the absolute value of each term and make a new series that is convergent. This is big typo here, is absolutely convergent if this is convergent. So, this last word divergent should be convergent. And, we say the series is conditionally convergent, if the series is convergent, but it is not absolutely convergent. So, sigma an is convergent, but sigma mod an is not convergent, then we say the series is conditionally convergent.

And there is a particular type of series in which the terms become alternatively positive and negative. So, either first term is positive, second term is negative and so on or other way around. So, such series are called alternating series. So, for example, if you remember the series minus 1 to the power n divided by n, that was a series whose first term was equal to minus 1, second was minus 1 by 2, plus 1 by 2 and so on. So, that was called alternating series. And, so what we are going to look at is some of the properties of the series.

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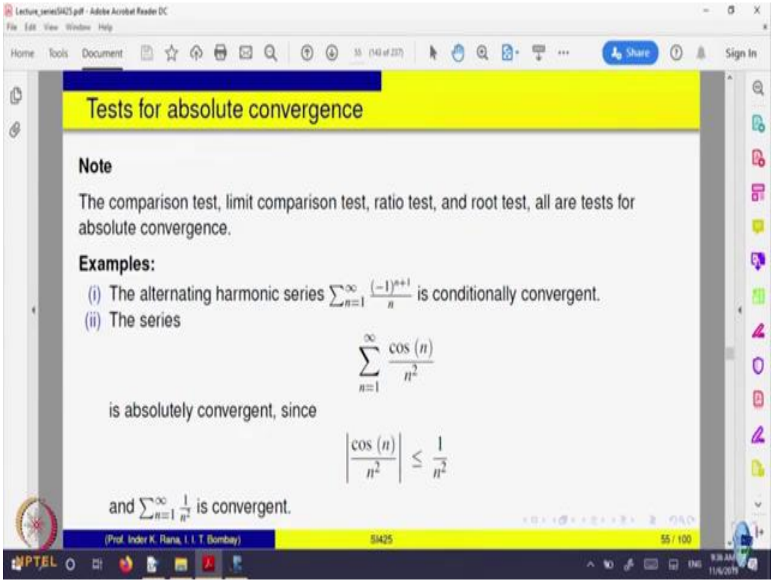

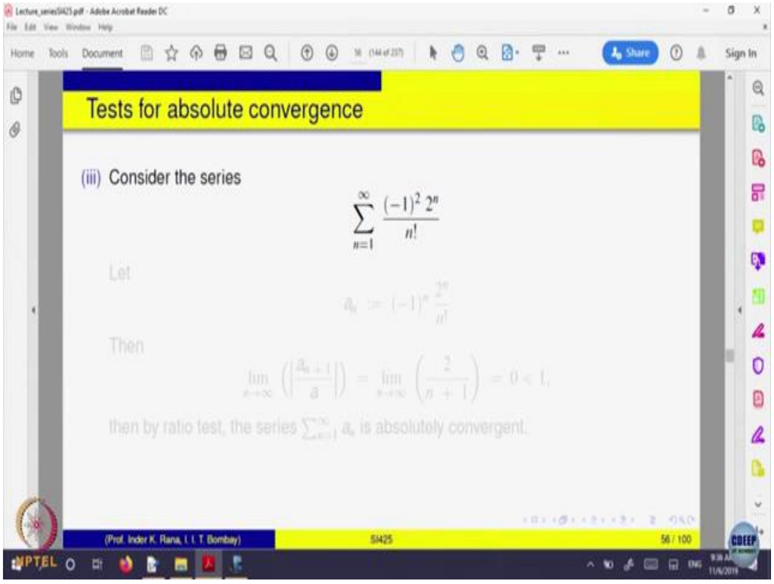

So, already we have seen one example that the series, alternating harmonic series, that is conditionally convergent, obviously we proved actually it is a convergent series and if it take the absolute value that becomes 1 over n, sigma 1 over n that is not convergent. So, this is a series which is conditionally convergent. And if you look at the series, cos n by n square, you can compare it with 1 over n square because cos is bounded by 1.

So, by comparison test absolute value of cos n divided by n square is less than 1 over n square, so an less than bn and bn is convergent, so an will be also convergent. So, this is a series which is absolutely convergent.

And you can give more examples like that. The series can be absolutely convergent and…So basically, for a series to be absolutely convergent, the tests are basically that of non negative terms because absolute value of each term is a non negative term. So, checking whether a series is absolutely convergent or not, you have to apply the test for non negative term series, like root test, comparison test and so on. So, here is a theorem which says, that in a series absolutely convergent, then it is also convergent. So, there is only one way obviously, that a series is absolutely convergent then it is also convergent.

So, what we are going to do is, to prove that it is convergent will compare it with a series which is convergent.

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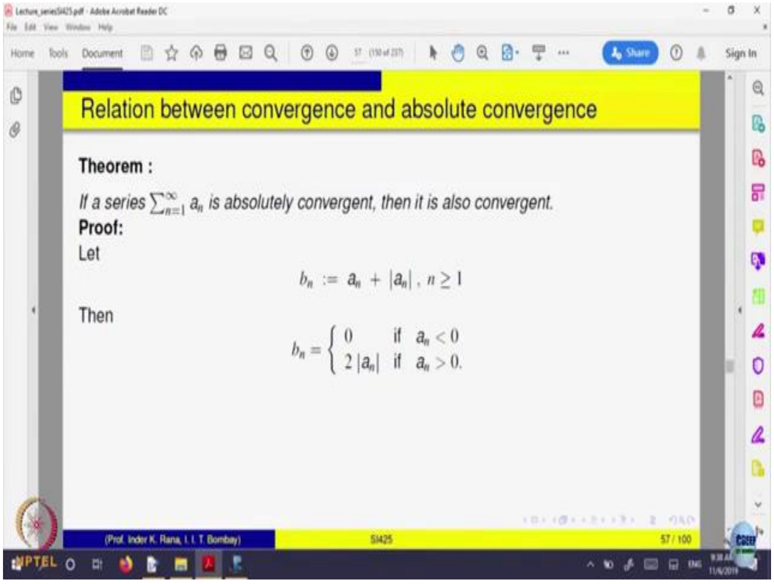

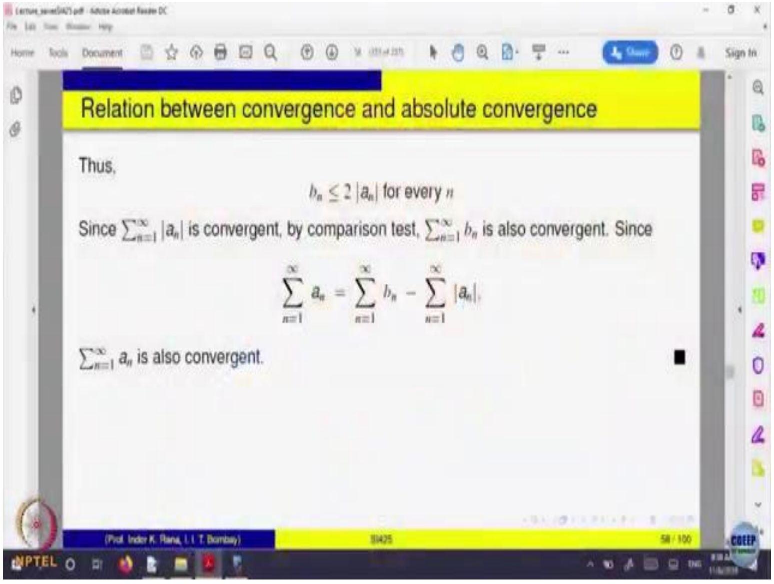

So, let us very simple constructions, let us define bn to be an plus mod of an. So, if an is less than 0, what is bn, if an is less than 0, then mod of an will be equal to minus, so this will be 0. And if an is bigger than 0, then this will be, mod of an is an itself, so it is 2 times that. So, this is a simple observation that bn is equal to 0 or bn is equal to 2 of mod an. Can you say that the series bn is convergent? Because an is given to be absolutely convergent. So, bn is either 0 or 2 mod an, so we can compare it with mod an, which is absolutely convergent.

So, series bn is convergent. So, Series bn is less than equal to mod an which is absolutely convergent, so series bn is convergent. And what is bn, what is an, an is bn minus mod of an, that is how we define. So, if you go back, an is equal to bn minus mod of an and we have already proved algebra of series if an and bn are 2 series which are convergent, then their difference some, they are all convergent. So, using that fact, we get that an is sigma bn minus sigma mode an. So, this also a convergent series, the simple thing that define bn to be equal to an plus mod an and compare with mod bn.

So, it says that if a series is absolutely convergent, it is also convergent, it is a necessary condition. Obviously, it is not sufficient, because a series may be absolutely convergent, maybe convergent, but not absolutely convergent, that alternating series example, alternating series is convergent, but not absolutely convergent, there is only conditionally. So, there is simple result, which relates the 2.

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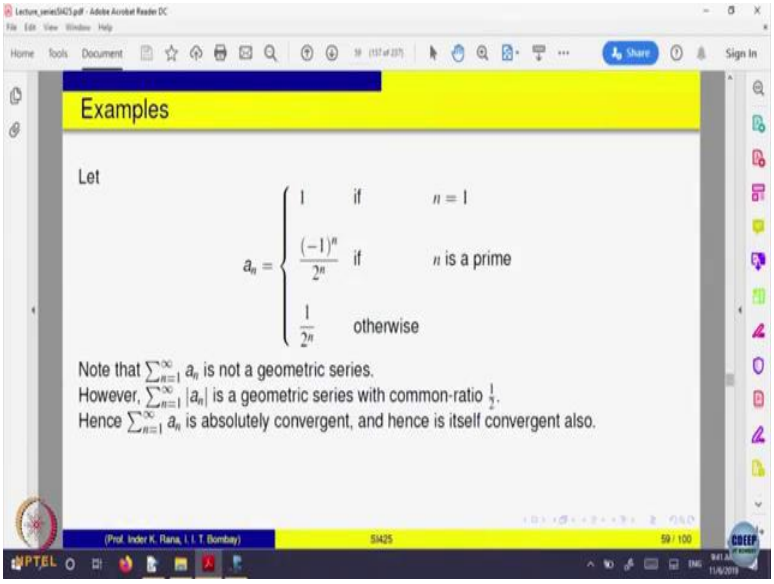

And, you can have more examples like this, looks quite complicated. It says an is equal to 1 if n is 1, it is minus 1 to the power n by 2n if n is a prime and 1 over 2 to the power n otherwise.

It looks like a geometric series but only when n is a prime, it is not 1 over 2 to the power n, it is minus 1 to the power n. So, that will be negative term there. But if you look at the absolute value of this series that is convergent, whether it is a geometric series, if you take the absolute value of the series, so mod an that will be 1 over 2 to the power n, for all n.

So, that is a convergent series because that is a geometric series with common ratio less than 1, so that is convergent. So, this is a series which is absolutely convergent and hence it must also be convergent. So, this is one way of analyzing series, that sometimes the absolute convergence series you have to prove than the series being convergent directly. So, you prove it is absolutely conversion, as a consequence the series becomes convergent. So, that is applicable here, so geometric series is convergent, so (())(8:08).

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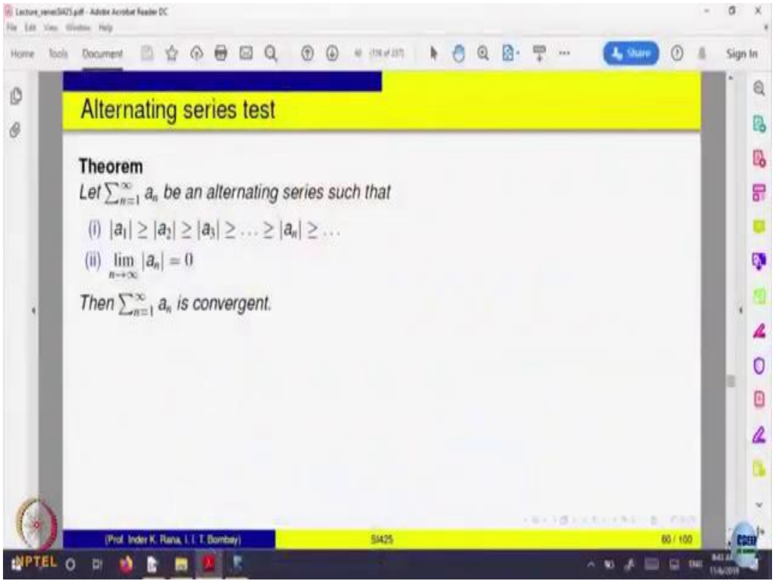

So here is a test, especially applicable for alternating series when the terms are alternatively positive and negative, only for those series. It is a useful theorem, it says that suppose an is an alternating series. So, alternatively either first time is positive, second is negative, third is positive and or other way around, the first one is negative and positive. Says the 2 conditions must be satisfied, the terms of absolute values of the terms of the series should be a decreasing sequence. Mod a 1 is bigger than mod a 2 and bigger than mod a 3 and so on.

So, it is a decreasing sequence. Not only it is decreasing, we should also have that limit of mod an is equal to 0, it is decreasing to 0, then the series is convergent. So, this is a test for alternative series, we will not go into the proof of this, we will assume it. So, for example, let us look at remember we proved that alternating harmonic series is convergent. If you look at the terms what are the terms, minus 1 to the power n plus 1 divided by n.

So, you take absolute value that is equal to 1 over n, so that is a decreasing and goes to 1 over n, mod an goes to 0. So, this theorem is applicable for that alternating series. So, as a consequence you can say, consequence of alternating series test that alternating harmonic series is convergent. We proved it by definition itself. But this is a consequence of this theorem also, now this theorem is useful. So, this is alternating series test you may come across these things.

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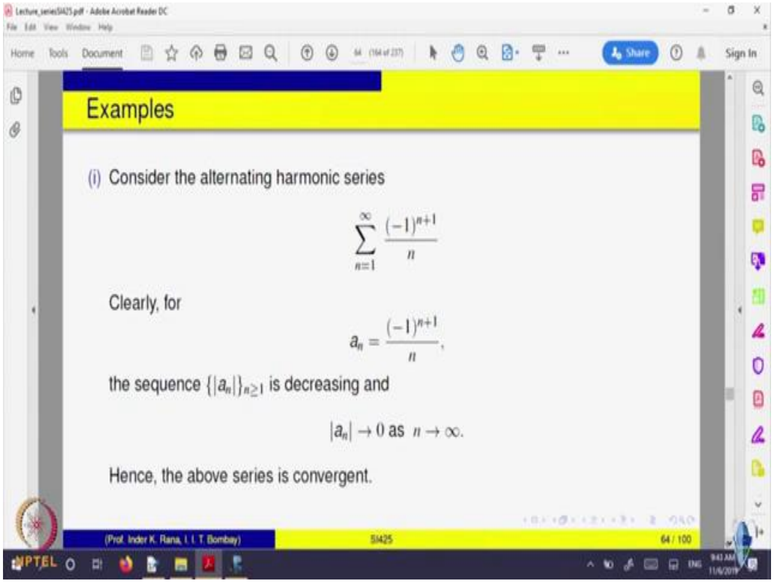

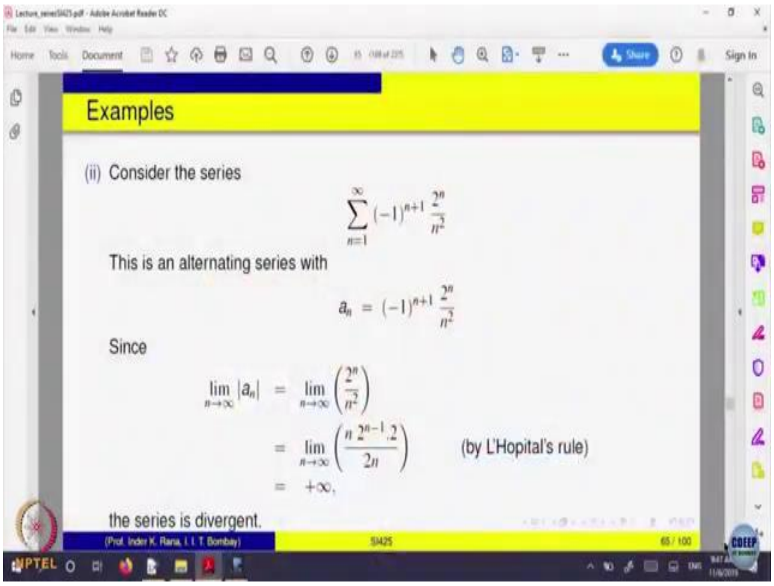

So, proof is slightly long, so we will not go through the proof. So, then for example, this alternative series, the absolute values are decreasing and goes to 0, so that is convergent. Somehow, we can have more examples like this, let us look at this series. We want to know whether this is convergent, this is alternating series, minus and plus terms are coming. Let us look at mod an, absolute value, so this will go off, so it is 2 to the power n divided by n squared. What is the limit of that, that it is a sequence.

So, sequences go to the power n divided by n square, what do you think should will be the limit of that? Let us try to make a guess. That is what important is, then you can prove it or disprove it whichever way you want it. 2 to the power n divided by n square, let us do first few terms and try to make a guess. So, n equal to 4, for example, so 2 be power 4. So, what will be 2 to the power 4, 16 and 4 to the power 2 also is 16, so that is 1. So, let us look at 5. So, as n becomes larger and larger, 2 to the power n starts becoming much bigger than n square.

So, that is how you should try to think of, the numerator is becoming larger and larger at a rate much faster than the denominator, so this will not converge. So, it should converge to plus infinity. So, that is a guess because numerator is becoming larger and larger compared to the denominator. So, to prove that one way is you apply the L'Hopital's rule.

So, when you apply the L'Hopital's rule, you get this and that clearly says that 2n cancels that it is same as a limit of 2 to the power n minus 1. So, that goes to infinity. So, what is the consequence of it, so this alternating series cannot converge because mod of an goes to plus infinity. Even you can apply the nth root, nth term test, if a series is convergent then the nth term an must go to 0 and mod an is going to plus infinity. So anyway, an cannot go to 0, because if an goes to 0, then mod of an also will go to 0 anyway, so that cannot happen. So, either way you can say this series is not convergent.

Student: We can say this by using previous theorem also.

Professor: You can apply. No, previous theorem is only one way necessary. These are sufficient conditions, the alternating series test, all tests are giving you sufficient conditions they are not necessary conditions. Test anywhere, whether in calculus or in series or anywhere, those tests always give you the sufficient conditions but not necessary. For example, maxima minima if something happens, if the second derivative is bigger than 0 then it is a local maximum, not the other way around. So, that may not be true.

So, alternating series test also is a test which is giving sufficient conditions for something to happen, namely alternative series. If mod an is decreasing and decreases to 0 then it is convergent, that does not mean other way around also is true. So, always be careful, so this is.

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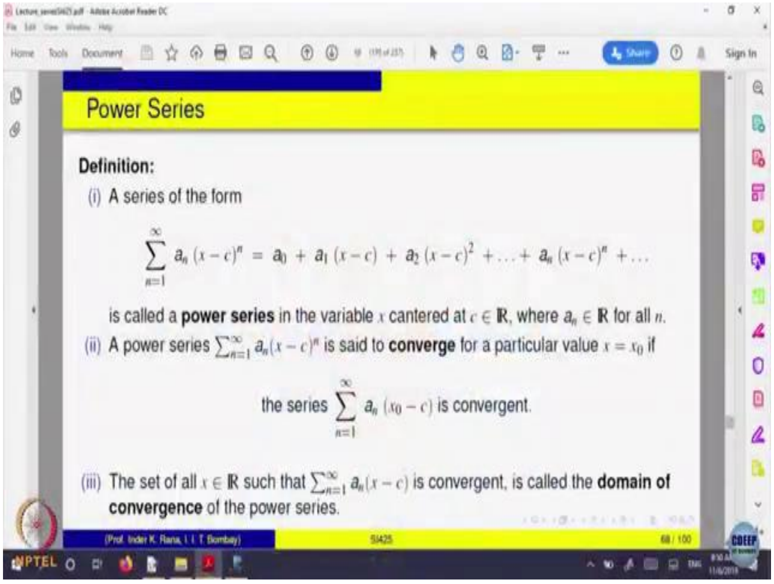

So, there are more examples that you can study later on, let us not discuss. There is something in the series, which in some one way probably some of you might have already come across. A series of the form sigma an x minus c raise to power n. So, what is this, we are trying to add up terms, a 1 x minus c raise to power 1 and so that is a0. You can also have a0 if you like, you can take n equal to 0 also here, this is n equal to 0, I can put a constant term also.

So, what are the terms look like, they look like some constant and nth terms looks like a constant an x minus a scalar c which is fixed raise to power n. So, it looks something like geometric series you can think of, where the coefficient an's are also varying, the powers are increasing. This x, what is x, x is a variable which can take any real value.

So, such a series is called a power series because it is x minus c, we say this power series is centered around the point c. So, this is the power series with the variable x centered around the point c. So, we want to say that this series is convergent or not, we want to analyze the convergence or divergence, when x is varying. But, we will only know when x is fixed, when x is fixed as a real number then it is a series of real numbers. So, we can analyze convergence or divergence of this. So, we say that the series converges at the point x is equal to x0, if you put the value of x equal to x 0 that is a series of numbers now, that is convergent.

So basic problem is to analyze, for what, given a series power series like this, for what values of x it will converge, how to analyze that?

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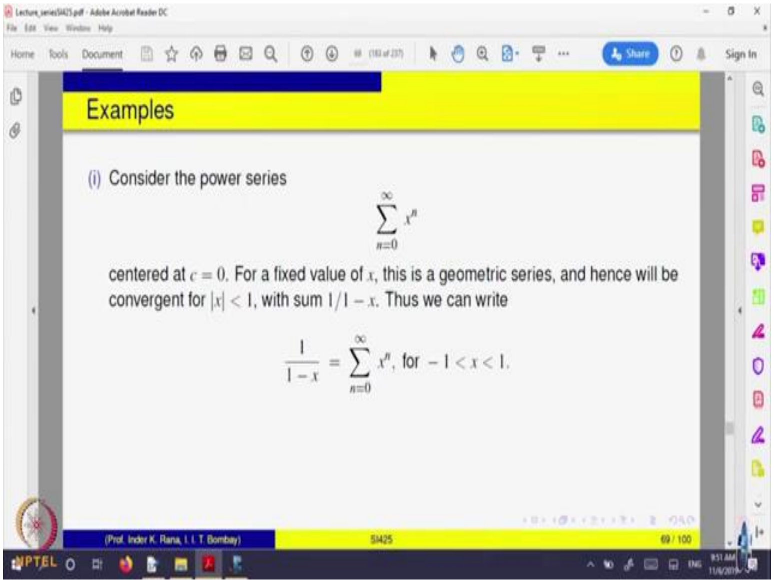

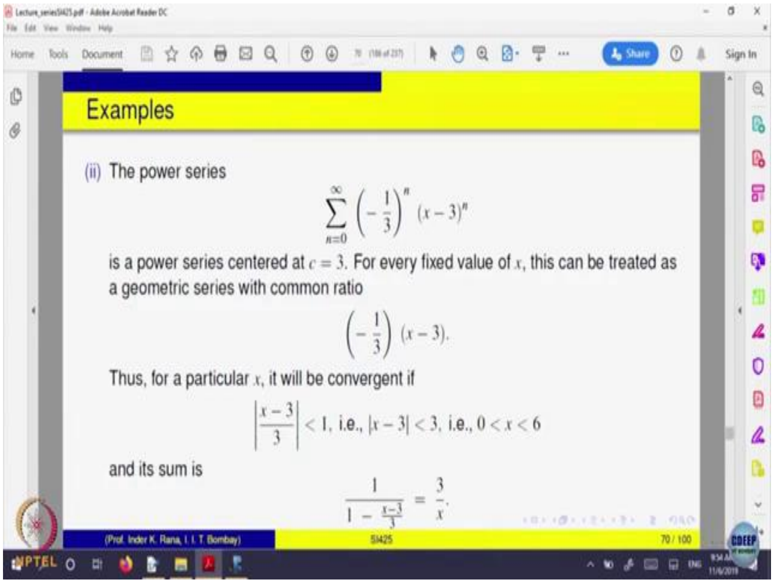

So, to analyze that, the simplest thing is, let us look at x to the power n. This series x to the power n, so at centered around x minus 0 to the power n. Where centered around x equal to 0 or c equal to 0. x to the power n, so it is a geometric series with common ratio x. So, we can analyze for what values of x this will converge that we have already seen. So, it converges for mod x, strictly less than 1. So, this series is convergent for mod x less than 1, strictly less than 1.

And we know what is the sum, sum of geometric series, 1 over 1 minus common ratio, so 1 over 1 minus x, so this is a geometric series which is convergent. So, this power series is convergent for the, all the values x between minus 1 and 1 and the sum is equal to 1 over 1 minus x.

So, now, look this, this looks like a functional equation for the domain x is equal to minus 1 to 1, the function f of x is equal to 1 over 1 minus x is equal to x to the power n for x between minus 1 and 1. So, this looks like a function being defined by a series. And is a important question in mathematics, you will see it comes at various places, when can a function be represented in the form of a series? So, will one form, one particular case, we will study, we will come across today but others are quite important.

They are something called Fourier series, there is something called in statistics probability will come across characteristic functions of distributions and so on, trying to express a function f in terms of a series. So, here is the simplest case of power series. So, this is an example, x to the power n is convergent power series with whenever mod x is between 0 and minus 1 to 1.

So, look at this for example, is a power series centered at x is equal to 3 because x minus 3 raise to power n, they giving very simple examples. So, how do you analyze this convergence of this, it is again a geometric series. It is again a geometric series, what is the common ratio, minus 1 by 3 into x minus 3. So, we now apply the geometric series test, so this will be convergent, if and only if the common ratio is between minus 1 and 1. So, minus 1 by 3 multiplied by x minus 3, that should be less than mod of that should be less than 1 common ratio. So, that gives you the range for which values of x.

So, if you analyze that, so it says x is between 0 and 6, mod of 1 by 3 x minus 3 less than 1. So, simplify that is that. So, that means this series, the given series is convergent when x lies between 0 and 6. Simple geometric series we are looking at, so that is advantage of power series that you can apply in one way or the other geometric series and get some results. And the sum is 1 over 1 minus x, so you can find out the sun.

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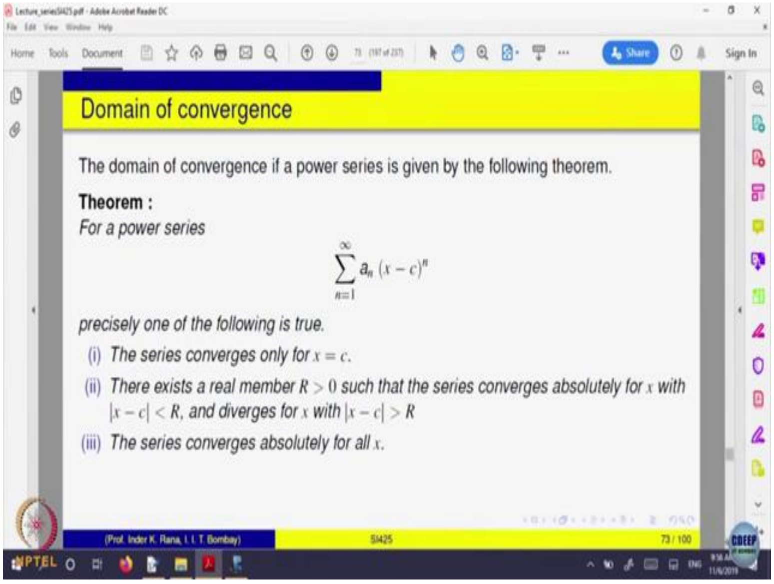

So, there are more examples you can look at. Now, I think probably, so it makes sense to define what is called the domain of convergence of a power series. So, given a power series an x minus c raise to power n, one of the things will happen. So, what will happen, the series converges only when x is equal c or there is a number R, such that it converges for all values x minus c less than R and diverges for mod of x minus c strictly bigger than R, strictly less than Rn, strictly bigger than R or the serious converges absolutely, that is also a possibility, the series converges absolutely.

So, one can prove that only one of these possibilities can hold and one will hold. Again, the argument why one of them only will hold, I will not go through that. So, where a power series, the possibilities there, the series converges for only one point x is equal to c or there is a interval around x, x minus R 2, x plus R, so that this is convergent. And if the value is strictly bigger like geometric series is mod x less than 1, strictly less than 1 it was convergent, bigger than or equal to 1 or divergent.

So, it says bigger than R, it is divergent, equal to R we do not know what can happen. So, that may depend upon the series and converge is absolutely for all values that is another possibility, so 3 possibilities. So, given these 3 possibilities, let us keep the proof.

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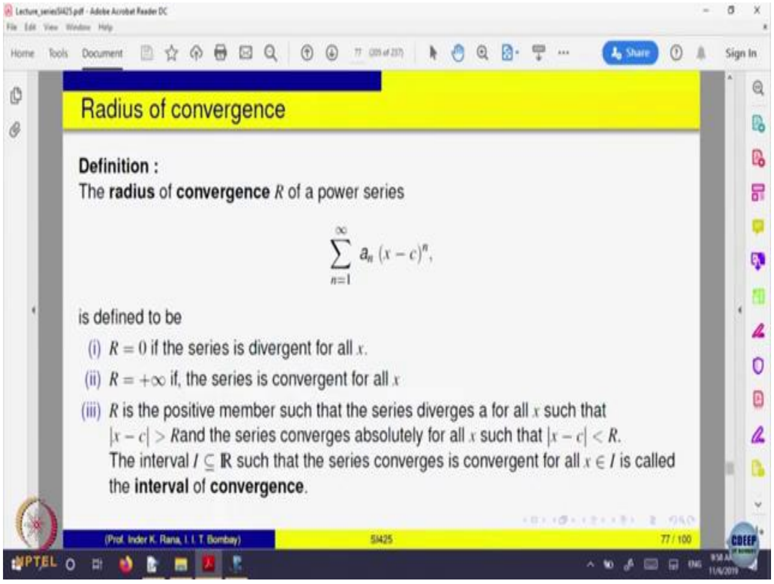

We define, what is called the radius of convergence of a power series. So, what is a radius of convergence? First possibilities was, it was convergence only at one point. So, in that case, we say that the radius of convergence is 0, only at x is equal to c, it is convergent around x equal to c there is no interval only that point. So, there is no length of the interval it is a, only one point in that interval. So, and if it converges for all, then you say R is equal to plus infinity, is all real life, so that is plus infinity.

And if there is a positive number R, such that for x minus c, bigger than R it diverges and converges absolutely for x minus c less than R, then you say this R is called the radius of convergence. And, x minus R2 x plus R is called the interval of convergence, the open interval, it is called x minus c, that means it is a interval around c, c minus R2, c plus R, x lies between that, so this says that.

So, that is called the interval of convergence. So, either it will be 0, R is equal to 0 only one point. R is a whole of real line or it is the interval, open interval, in which surely it will converge. But, we do not know what happens at the end points of that interval, for example geometric series when x is equal to plus 1 or minus 1, the series diverges.

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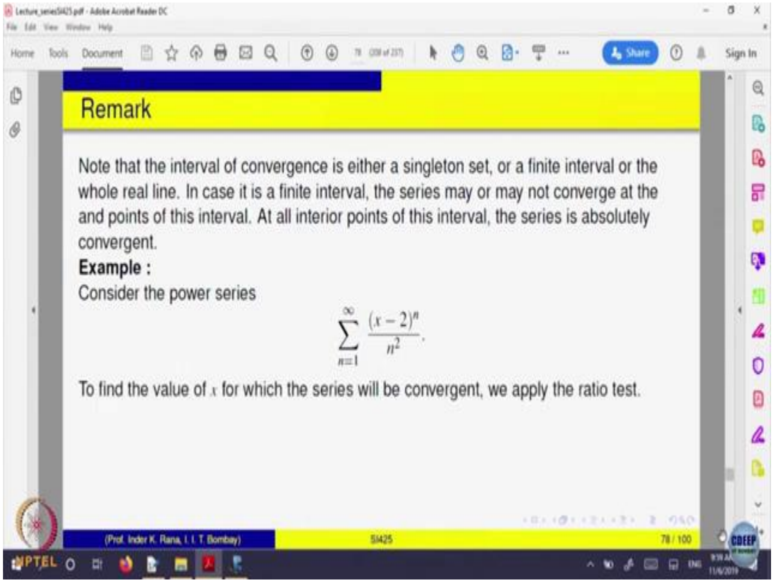

So, at n points, anything is possible. So, let us look at one example, look at the power series x minus 2 raise to the power n divided by n square. So, how do you analyze convergence of this series, power n, n square. So, which test do you think will be more suitable? I can not apply comparison test here because our numerator denominator both are increasing and becoming larger, is a power n, so it looks more suitable to take the ratio test here. When I take the ratio, powers will tend to cancel out. So, let us apply the ratio test to this and see what happens.

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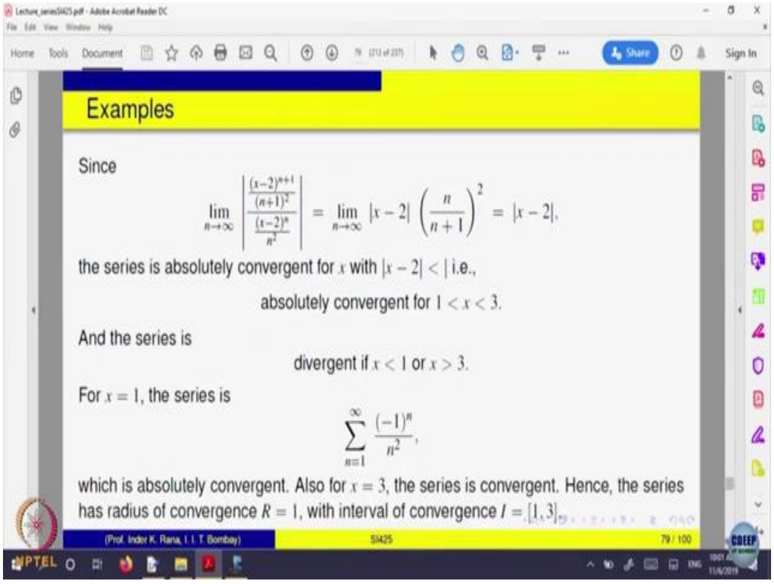

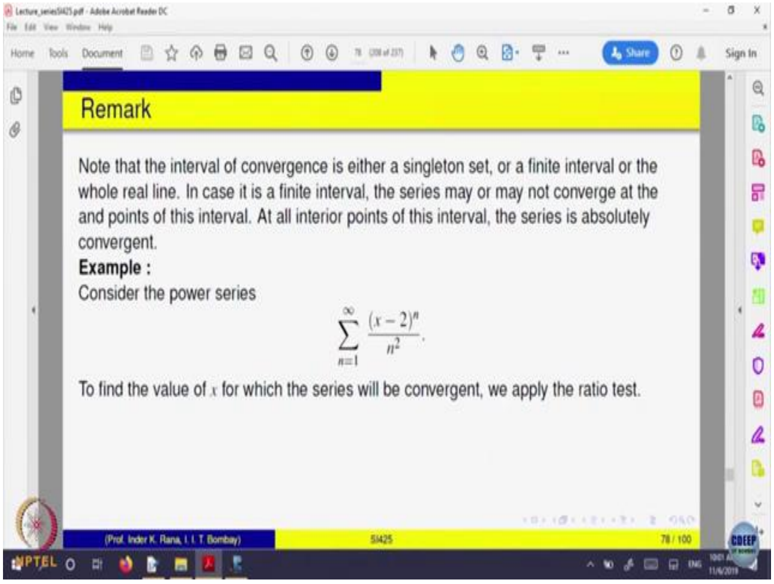

So, take the ratio and limit comes out to the mod of x minus 2 so that is simple, powers cancel out limit mod x minus 2 n over n plus 1 and that gives this, remember that series goes to that sequence goes to 1 limit of that, if you want to see y, n plus 1 minus 1, so it is 1 minus 1 over n plus 1, so that goes to 1. So, this is mod x minus 2. So, it says that this series will converge absolutely when mod x minus 2 is less than ratio test, less than 1, so ratio test less than 1. So, it will converge absolutely and that means between these limits, and it will be divergent outside.

When it is equal to 1, see what is the interval, minus 1 to 3 that is an interval of convergence. When x is equal to 1, what is at series n point, x is equal to one. So, minus 1 to the power n divided by n square is that convergent, there is alternating series. You can apply the alternating series test, mod of there is 1 over n square that goes to 0, so that is convergent for at the endpoint.

So, there is absolutely convergent for that also, when x is equal to 3, what happens x is equal to 3, that was other endpoint. So, x is equal to 3, 3 minus 2 that 1 over n square that also is convergent. So, not only the radius of convergence is equal to that 1, 2, 3 at open interval, but even the endpoints it is converging for this series, particular series. So, endpoints you have to analyze separately. So, that is convergent. So, interval of convergence is 1, 2, 3. That may not happen always.

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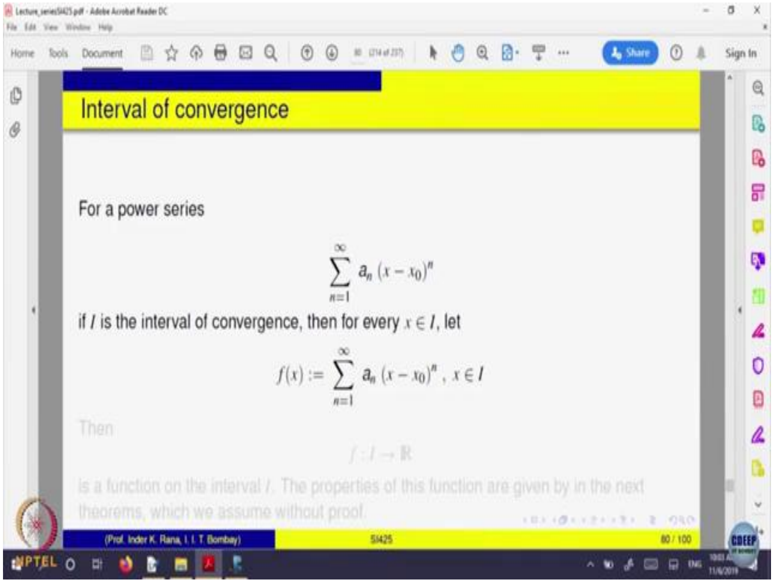

So let us, why we are bothered about this interval of convergence, because in the interval of convergence, the power series converges absolutely and when it converges absolutely, the limit you can call it as f of x. So, if the series converges absolutely in the interval of convergence, we can define the function f of x to be equal to this. Now, you can ask a question. So, what is the question, you would like to ask, each term on the right hand side seems to be differentiable, x minus x raise to the power n that is differentiable function.

So, we are adding up differential powers of a differentiable function is the sum differential, is the sum integrable, if because they are integrable also x minus a to the power n is integrable. So, the question comes that in a power series, is the function defined by the sum in the domain of convergence, differentiable or integrable.

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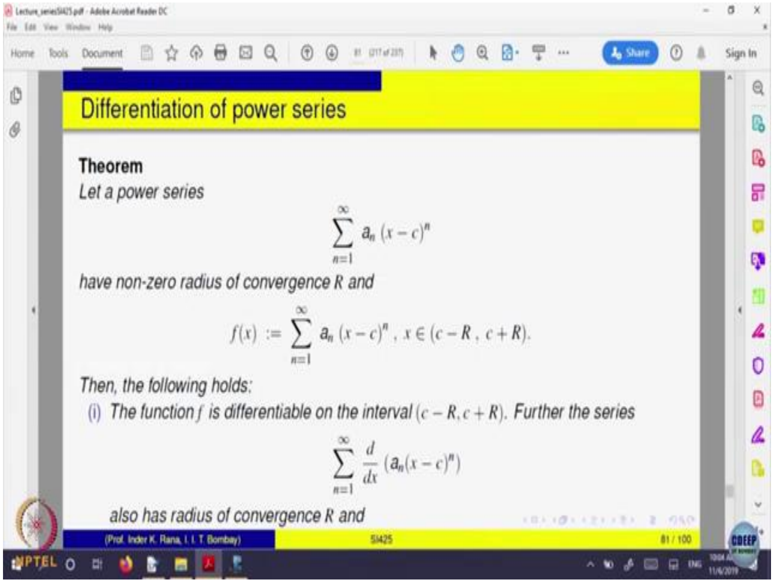

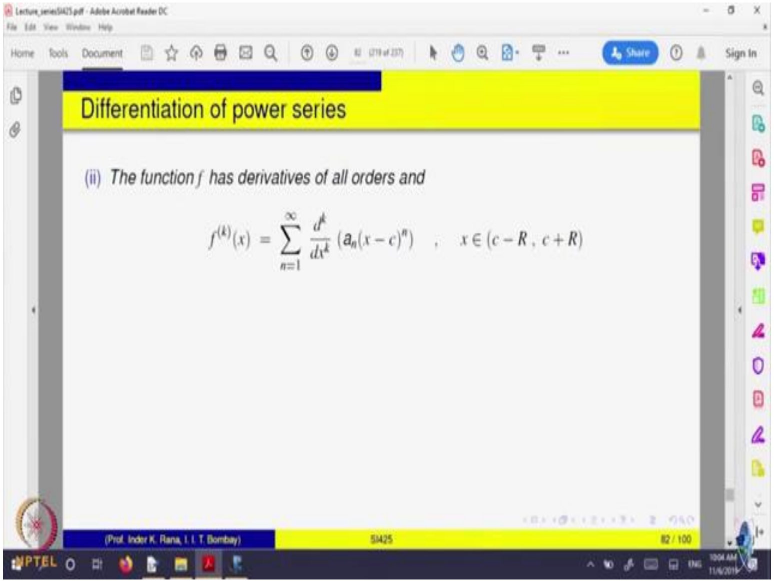

So, these are 2 theorems, we will not again go to the proof because so it says consider the power series this and as a non zero radius of convergence R, so this is the interval of convergence, then it says then the derivative f is a function defined by the series. So, what is the derivative meant, f of x plus h minus f of x divided by h limit h going to 0. So, we are asking whether you can compute the derivative of that function, it says yes, you can not only compute, you can take the derivative inside the series. So, it again interchange, derivative of the sum is equal to sum of the derivatives.

So, it says a function is differentiable and the series also, has the same radius of convergence which and the sum is equal to f dash. So, that is a very nice thing, if a function is defined by a power series in a radius of convergence, the function is differentiable and the derivative is the derivative of the each term added up together.

So, that is what it says, because x is power n is any number of times differential. So, when you add up a series, power series so a power series in the radius of convergence is infinitely differentiable, every k the derivative exists, every k the derivative exists and that is given by this. So, that is about derivative. So, the function defined by power series in the interval of convergence is differentiable and derivative first, second, third anything is equal to the sum of those corresponding derivatives of x minus a raise to power n.

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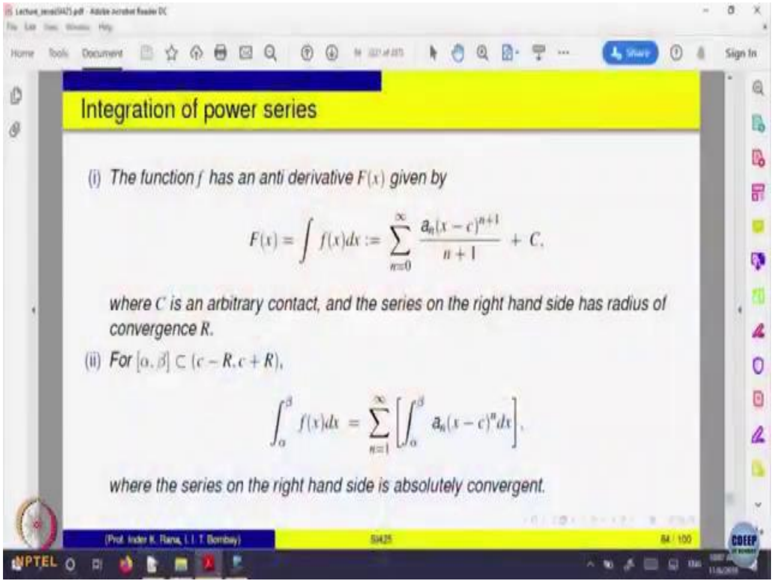

What about integration? Is something similar happens that f of x is the function defined by the series, then this function f as a anti derivative and what is, and how do you get the antiderivative of a function, Fundamental Theorem of Calculus by integrating a to x. So, it says anti derivative is given by integrating each term one at a time. To get the anti derivative f, you have to just take the anti derivatives of x minus a raise to power n and what is the anti derivatives of that, we know, so that is anti derivative, so f of x. So, as a consequence of this f of x is equal to sigma of the integrals.

So, integration which is again a limit process, integral a to b, f x dx is a limiting process, limit of partition sums and so on, upper sums, lower sums. It says, you can take that limit also inside the summation sign. So, in power series is a very useful results that you can take differentiation inside derivative of the sum is equal to sum of the derivatives, integral of the sum is equal to sum of the integrals, a very useful results.

And if you will, proof is there in the slides, you can read if you like but from examination point of view, we will not ask you the proofs. But those who are keen to know, how the proof goes, you can have a look at it and essentially you will have to look at the upper sums and lower sums, because integrals are coming into picture.

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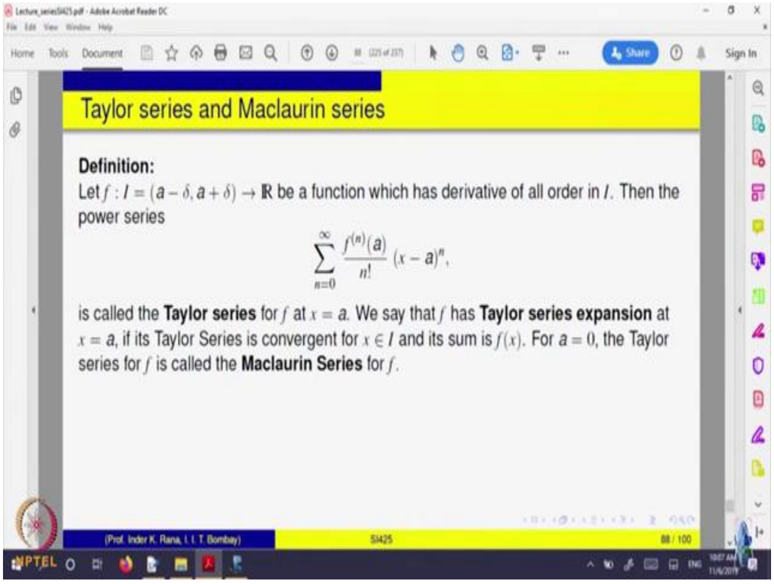

In the power series, there is very special case of power series. So, let me just say that what is the special case, this is a special case of power series. Remember, power series at an x minus a raise to power n, where an’s are the coefficient of the nth term. If the coefficient of the nth term is the nth derivative at the point a where it is centered divided by n factorial, then this series is called the Taylor series for the function f. f is a function given to you and suppose the function is any number of times differential, nth derivative exists.

So, if you can write f of x, so this is a function with the some as f, so f of x is equal to this thing and this is called Taylor's series. So, you can ask that question for what functions have Taylor’s series expansions. The series is called Taylor series expansion, because it involves derivatives of the function f. But it may or may not converge, it is a power series. So, what is the domain of convergence of this series, in that domain of convergence will it represent the function f itself or not? That is a question.

So, this, most of you must have done this in your undergraduate course in Taylor series and when the a is 0, that is called Maclaurin series and all those series, sin x, cos x, e raise power x, they have that series expansions.

What are those series, they are precisely Maclaurin series, x is equal to 0, a is equal to 0. So, those are, you asked, I hope you have done these things if not, anyway probably, you will do it sometime or you should revise. So, you want to know that when will this converge to the function f. So, what are you interested in, you are interested in knowing f minus the sum, partial sum when the series converge and the partial sum goes to 0.

So, the sum from some stage k to infinity that is the remainder term, in this series that remaining thing is called the remainder term. So, you have many ways of representing remainder terms and analyzing when does remainder term goes to 0, in what domain it goes to 0. So, for the trigonometric functions, exponential functions you show that the remainder terms go to 0 for all values of x. So, that is a particular form of power series. So, with that we end the course.