Basic Real Analysis
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Lecture 68
Series of Numbers - Part III

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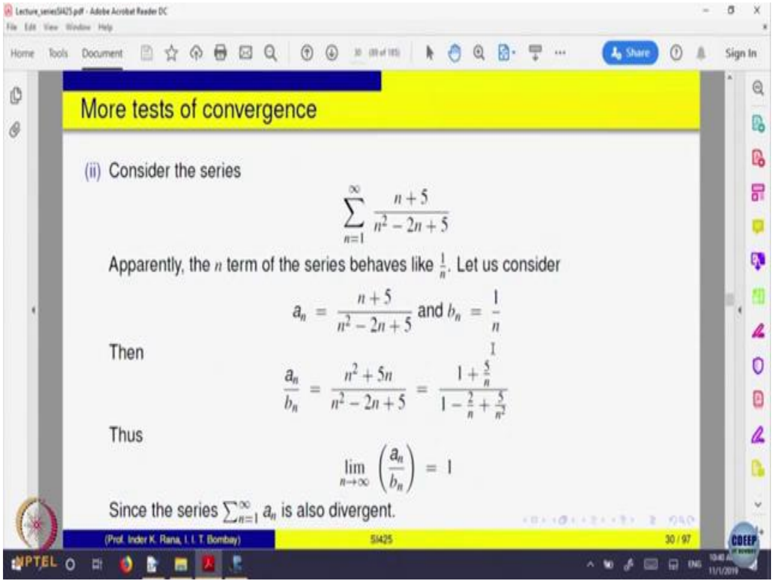

So, let us look at some example examples of that, I think that we have now b bigger than or equal to 1, you can analyze that. Like this one, I said n, n square, so eventually if I numerator, I call as an, denominator I call as bn. Then eventually, it looks like n over n square 1 over n, the limit looks like to be equal to 0. So, I can try to apply a limit comparison test here. So, find out the limit of, I it is something wrong here. I should not compare n with n square, let us, if you directly you try to compare numerator and denominator, then you will end up into problem because you do not know either of they may convergent or not.

You understood what I am saying, you should not take an to be equal to n plus 5. I do not know that series is divergent, so I cannot help it. So, this is my series, it looks like one over n, n over n square, it looks like 1 over n. So, let us try to compare this with, 1 over n ratio an by bn. So, an is equal to this, bn is 1 over n. What is an by bn, so an by bn that will be n square plus 5n, 1 over n, bn was 1 over n. So, and that limit is 1 over n squared by an square that limit will be equal to 1. So, limit of an by bn is equal to 1 which is not 0, 1 over n is divergent. So, this also will be divergent.

So, that is what I said to you, if I look at the n general term, it looks like 1 over n, it looks like 1 over n and 1 over n is not convergent. So, this series should not be convergent and that we are formalizing by comparing with an by bn limit of that, eventually it looks like 1 over n. That is how your thinking should go. So, that is divergent

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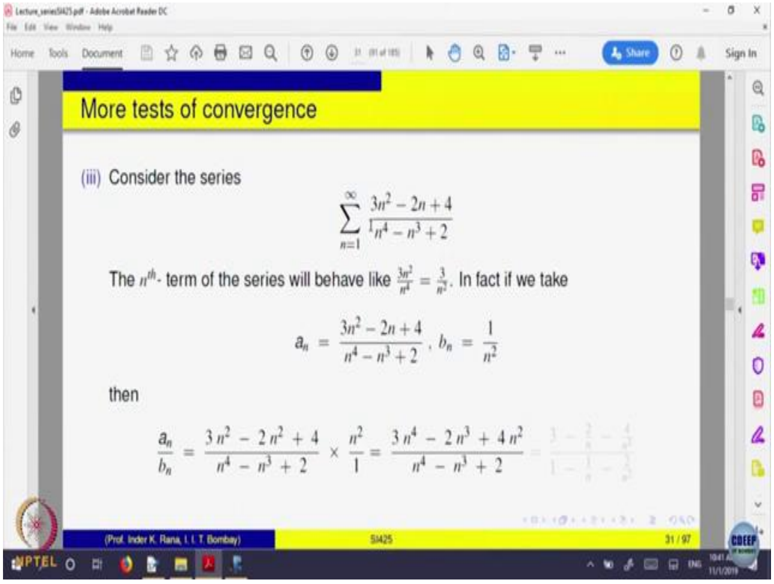

n square n to the power 4, what does it look like, it looks like 1 over n square. So, I should compare it with 1 over n square, same technique, an equal to this, bn equal to 1 over n square, when you divide an by bn, the limit will be equal to 3n to the power 4 and so on. So, what will be the limit, that will be, what is the limit of that, it will be 3 by 1, so it will equal to 3. It will be equal to 3 by 1 here, so that will be equal to 3. So, limit is not equal to 0. So, convergence of 1 over n square will imply convergence of this series which is 3 n square minus 2n plus 4 divided by that. Eventually it looks like 1 over n square.

Professor: Yes?

Student: Sir, what is the problem in saying directly that it will be something like 1 over n square, so at point state is convergent, so this will also be convergent.

Professor: But how do you formalize that? That is the end that is how you should think that the sequence general time looks like 1 over n square. So, eventually it should look like 1 over n square and that is what is the limit comparison test says that, justification for that is limit comparison test. So, look at an divided by bn, an is this, bn is equal to 1 over n square, compute the ratio and take the limit that tells you eventually how does this an compare with bn. So, that is a rigorous way of saying the same thing. So, that is convergent.

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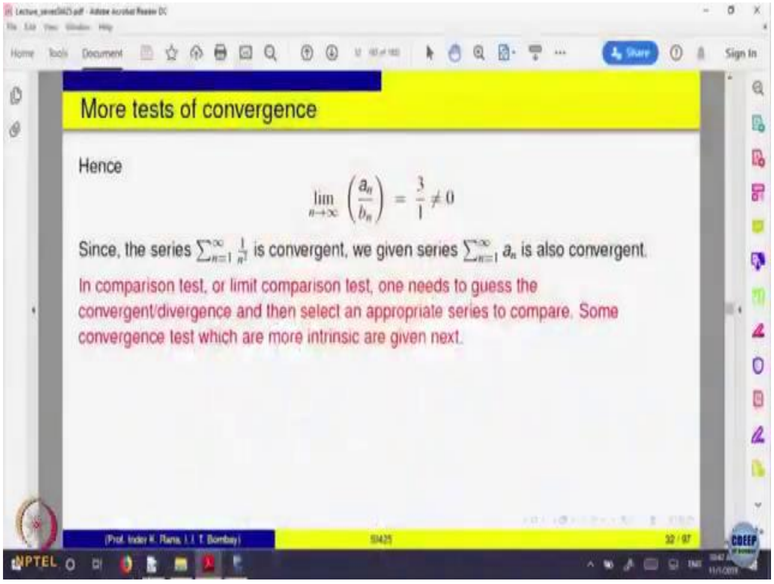

So, here is a question saying that, in all these things you have to think of with what you should compare. You have to make a sort of a guess, that I should compare it with this by looking at power up and down and so on, and then only compare. So, these test are not intrinsic test. Once again, I am bringing that word intrinsic. Remember, we use the word intrinsic when we looked at convergence of sequences, finding a sequence of a limit exists, a sequence is convergent, you require the limit, which is not given to you.

But saying is cushy. I do not require anything outside, I only have to analyze whether the terms are coming closer to each other or not. So, cushiness is an intrinsic property. And the beauty is it is equivalent to saying the limit exists. Now here root test, comparison test, and all these things, asked me to guess something outside the given knowledge that I had to find someone like 1 over n square, 1 over n and then compare them. But can I have a test, which does not require me to do that kind of a thing. By looking at the series itself I can say something. So, there are tests possible like that.

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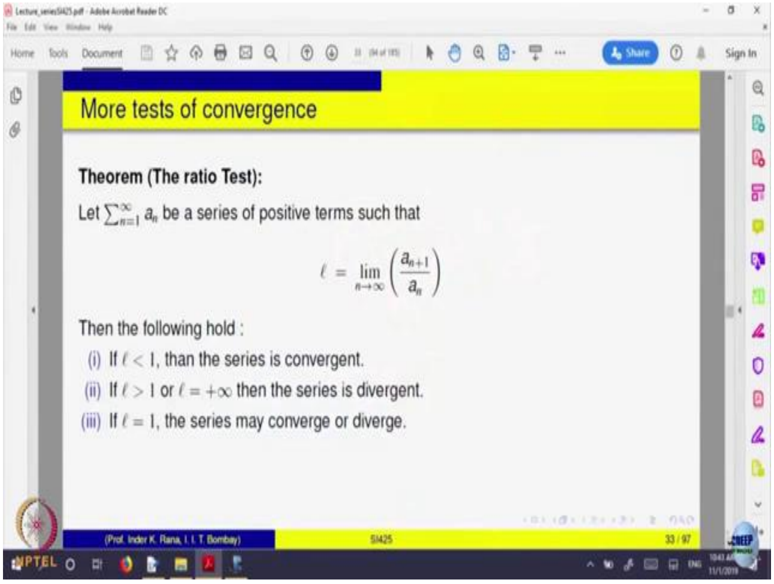

So, let us look at that is called the ratio test. So, here given the sequence an, look at the series an. So, look at the limit of an plus 1 divided by an, the next term it is ratio with the previous 1, look at that ratio. Why limit eventually, so take as a limit of this that is l, if l is less than 1 then the series is convergent, bigger than 1 it is divergent and equal to 1, is a same kind of problem that I can convert it can diverge. So, what could we prove of this? Till now what are the techniques we know, we only know the geometric series is convergent, the compare and then we knew that 1 over n square is convergent.

And then the comparison tests gave me something for bigger than 2 and less than 1 and 1 over np. So, those are known facts already. And here l is equal to this limit.

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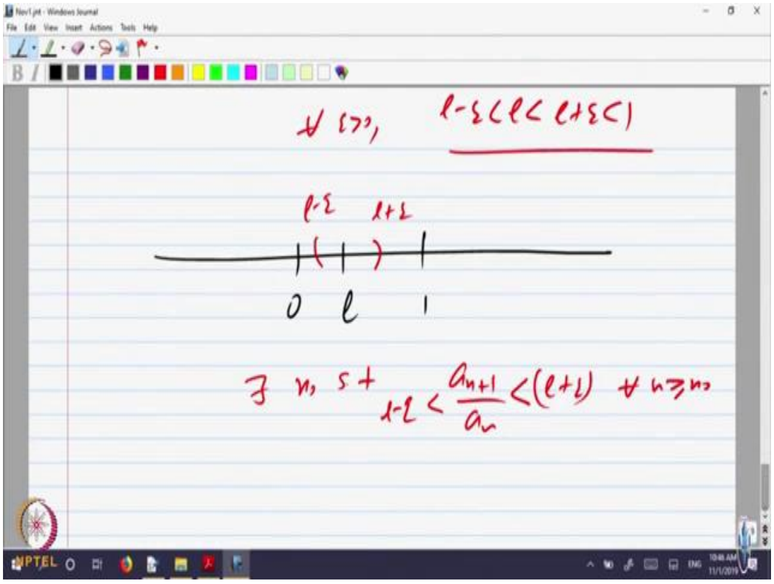

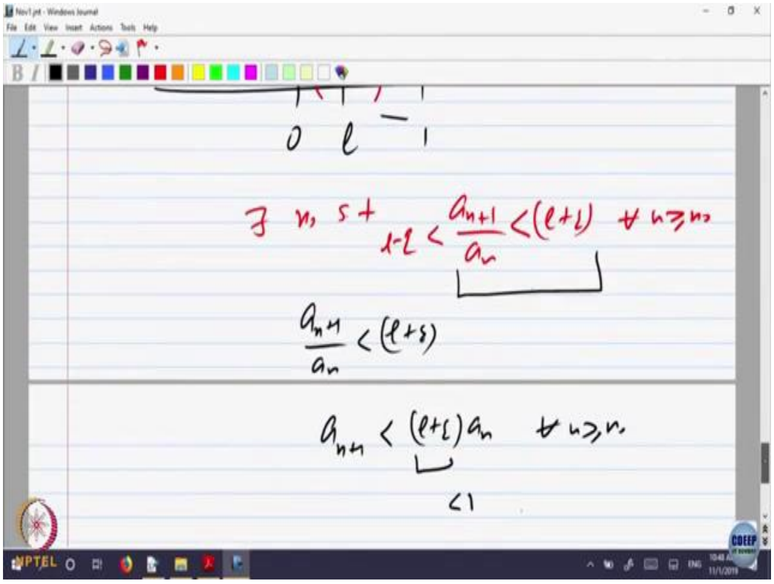

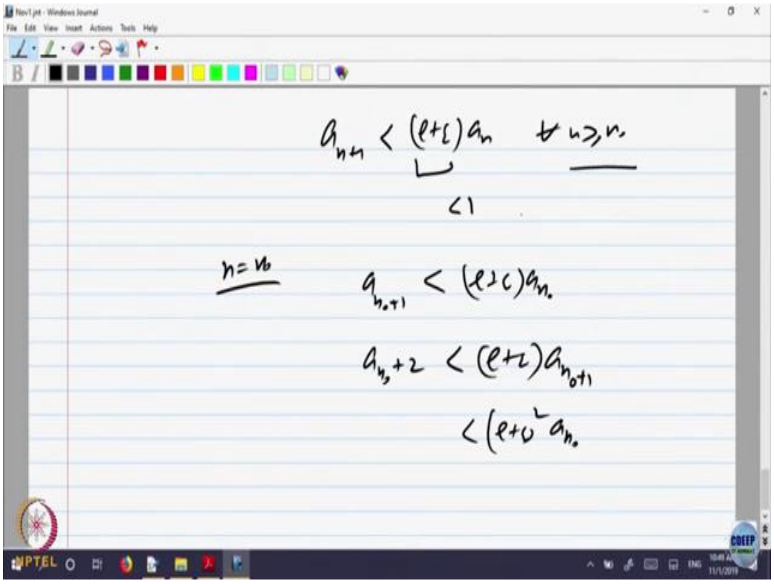

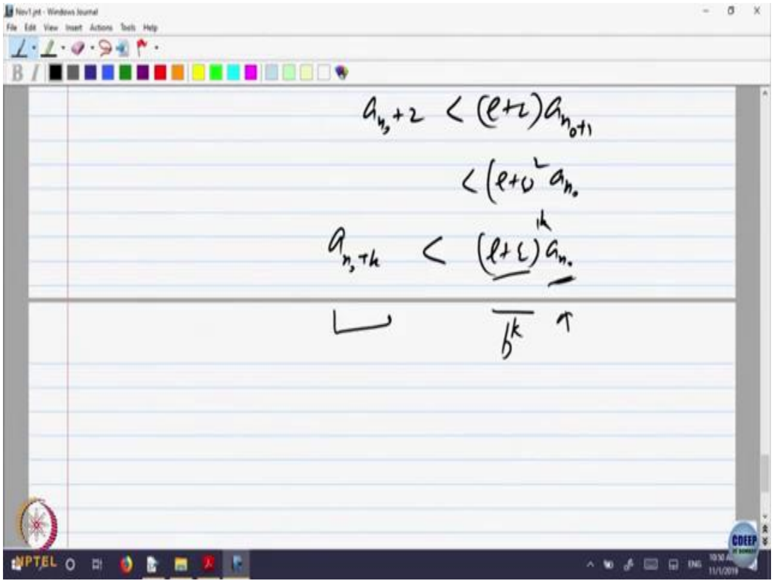

So let us analyze, what is the meaning of limit means here, if l is the limit of this ratio, that means what? And l is less than 1, l is less than 1. So, what does it mean? Here is 0, here is 1 and here somewhere l, I should, one should point out, it is strictly less than 1. So, it is not equal to 1, it is something in between, that means what, that means after some stage all the terms of the sequence must be close to l, so let say they are here.

So, let us l minus epsilon and l plus epsilon. So, for every epsilon bigger than 0, such that l minus epsilon less than l less than l plus epsilon still less than 1, let us keep it less than 1. Is it possible? Where l is between 0 and 1, so there exists some n naught such that an plus 1 by an is less than l plus epsilon and bigger than l minus epsilon for every n bigger than n naught. Is it okay definition of a limit? Now, see what is happening is, it says look at this part, it says an plus 1 by n is less than l plus epsilon. That means an plus 1 is less than l plus epsilon times for every n bigger than n naught.

So, this is telling me, how much an plus 1 is in comparison with an, and this thing is less than 1, this is less than 1. So, now let us try to use it inductively, this is for any n, n naught onwards. So, what will happen? an, so let us take equal to an, n equal to n naught. So, I got an naught plus 1 is less than l plus epsilon of an naught, it is bigger than n naught. Let us, go to the next step, an naught plus 2 will be less than l plus epsilon an naught plus 1. But an that is already less than, so it is l plus epsilon to the power 2 an naught.

So, what is it happening? So, what will happen to the n plus k, say if you look at an naught plus k, that will be less than l plus epsilon to the power k of an naught inductively. So, what we are saying is, the terms of the given sequence an naught from some stage onwards are bounded by this time is constant that is fixing thing, an naught. And what kind of, so if I call this as b, b to the power k, what is b, it is less than 1. So, that is geometric series. So, that is convergent. So, we are saying an's are less than bn's and bn's are geometric series from some stage onwards. So, that is convergent, so that implies and will be convergent. So, that is how this is useful.

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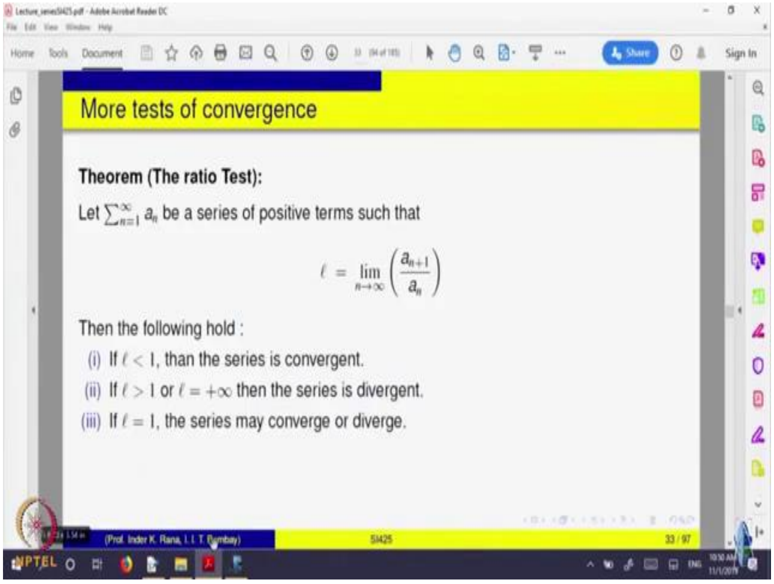

It says if l is less than 1, the series is convergent. If l is bigger than 1 what will happen, an by bn will stay away from 1, on the right hand side. So, an by bn will be bigger than l minus epsilon which is bigger than, so when inductively again powers. So, it will be a geometric series with common ratio bigger than 1. Again, comparison test will give me that they should be divergent. So, that will say the series is divergent. When it is equal to 1 either thing is possible, one can give examples.

So, definition of the limit of the ratio an plus 1 and n is falling back upon comparison tests and geometric series, the proof involves writing the limit and using the fact that the geometric series is convergent if the common ratio is less than 1. So, that gives us the result. So, you have already seen that, so you can write it out the proof, an plus 1 is less than that, that is less than 1 and so on.

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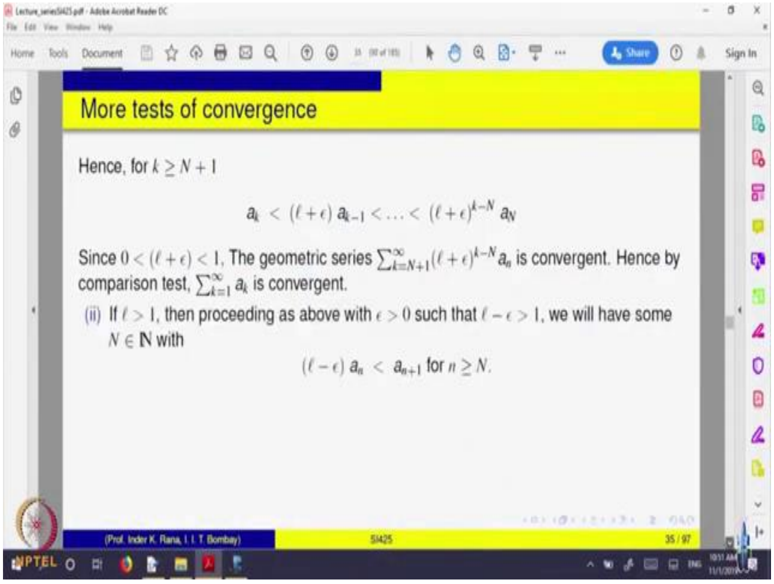

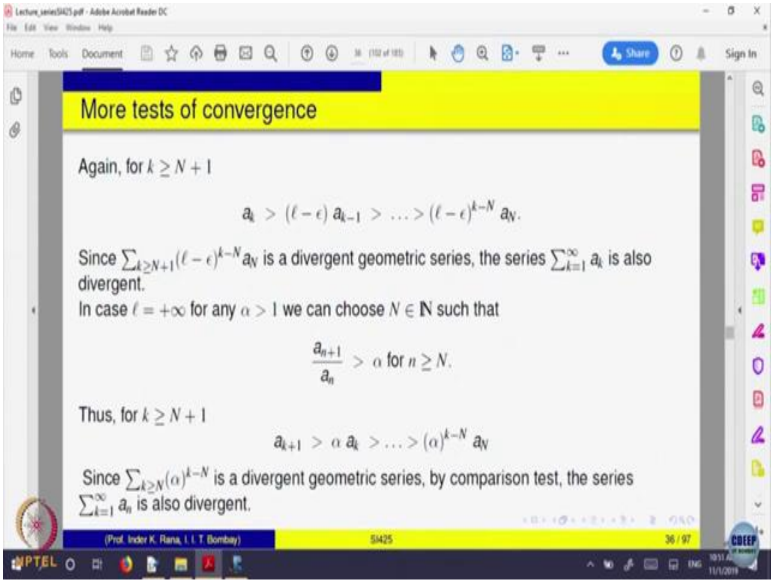

l bigger than 1, so I it will keep on increasing comparison, so that it is bigger than 1 you can...

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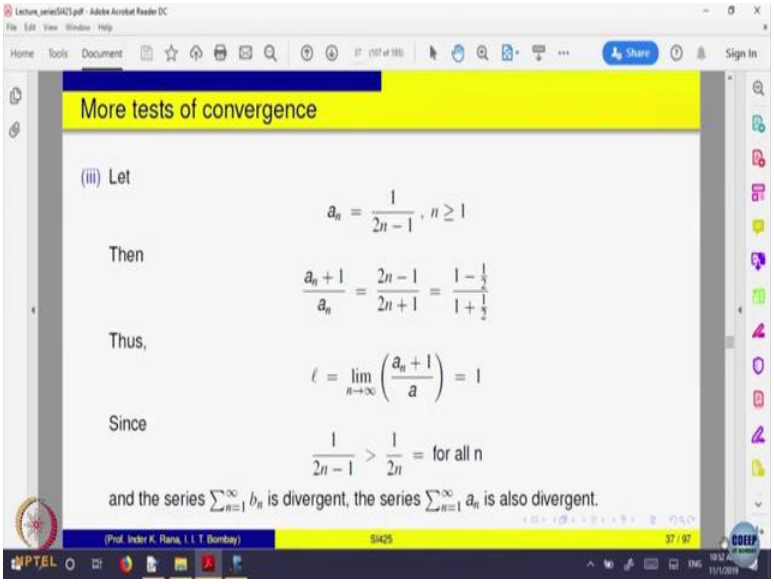

So, let us look at, some applications of this simple applications an is 1 over 2n minus 1. It looks like one1 over n anyway, so I do not have to really do anything just compare it with that. But let us take the ratio, the ratio is equal to, with 1 over n or ratio of this itself. that limit is equal to 1, so divergent.

As we guessed, you can compare it with that if you like. Either way, so limit is equal to 1, so divergent, an plus 1 divided by an and the limit is equal to 1 then the series is divergent. That is what way, less than 1, it was convergent.

Student: (())(5:18) l is equals to 1.

Professor: l strictly less than one convergent, because in the geometric series common ratios strictly less than 1 only will give you convergence. And combination is equal to 1 that gives you divergence.

Student: But here, limit value is equals to 1.

Professor: 1, so there is not strictly less than 1.

Student: Sir, we just say that if l is equals to 1, it may converge or diverge.

Professor: Which one?

Student: The upper part where theorem was taken.

Professor: So, here it is diverging, that is all. Limit is 1, but it is divergent compared to that.

Student: Sir, ultimately they are using that theorem (())()16:03)

Professor: Which theorem? So, now I giving examples that when it is equal to 1, anything can happen. For the third case, I said l equal to 1 anything can happen, strictly less than 1 it is convergent, say q bigger than 1, it is divergent. So, in this case, it is strictly it is equal to 1 but if you apply the comparison test, it is divergent, equal to 1 but divergent.

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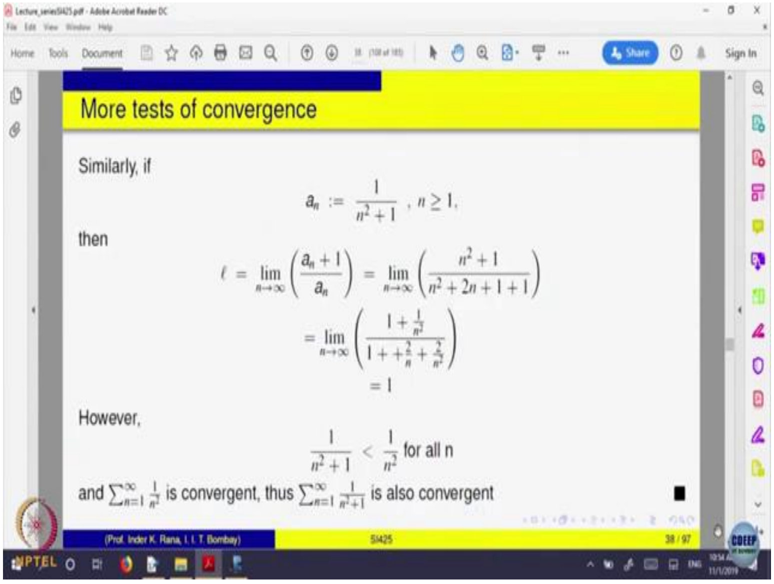

You can look at n square similarly, the ratio will be equal to 1, but it is convergent. If you look at n square, the ratio will be equal to 1, it is convergent. So, either is possible in that case, we are not applying the theorem but we are saying that counter examples for the third case.

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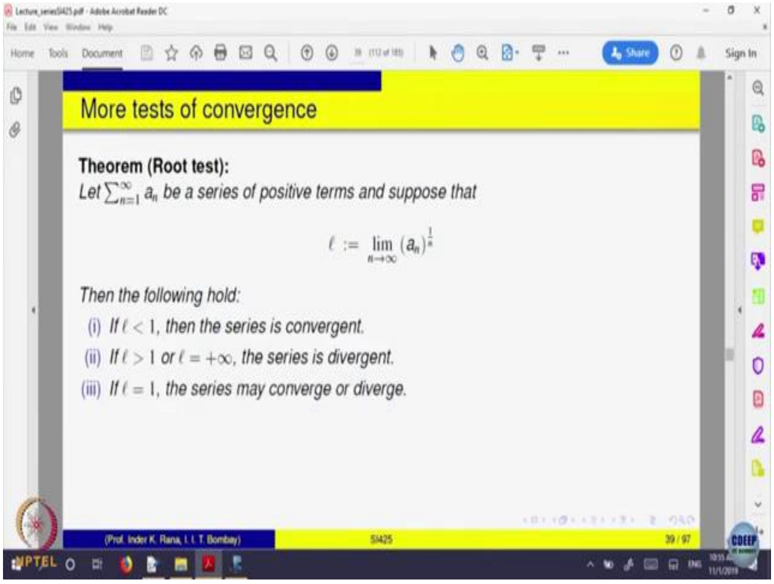

So, there is something called the root test is something similar to the ratio test. It says an's are non negative, look at the nth root look at the limit, if that exists less than one is convergent, bigger than 1 or infinity it is divergent, equal to 1 anything can happen. Basically, eventually analyzing the limit and bring it back to something that is already known, that is the idea of the proof. So, if limit of an raise to be 1 over n, that is l is less than one then what happens, it should say in a neighborhood of 1. It should say in a neighborhood of, 1 limit is 1, if the limit l is less than 1, it should say on the left side of one.

That means what, an raise to power 1 over n will be less than something which is less than one. So, when you raise the power, so an will be less than that small quantity which is less than 1 raise is to power n. Again, dramatic series will convergence. So, basically definition and geometric these are giving these tests. Similarly, for l bigger than 1, again geometric series the common ratio bigger than 1 will be divergent, equal to 1. We have to give examples to illustrate that.

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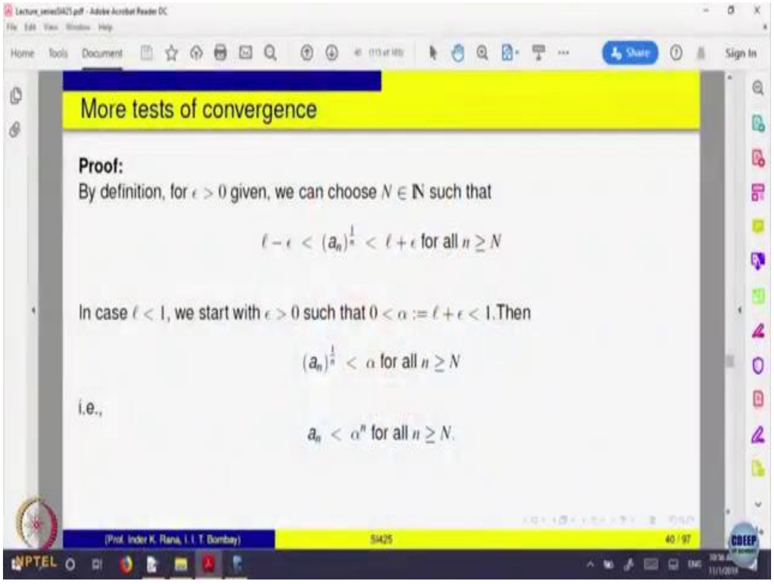

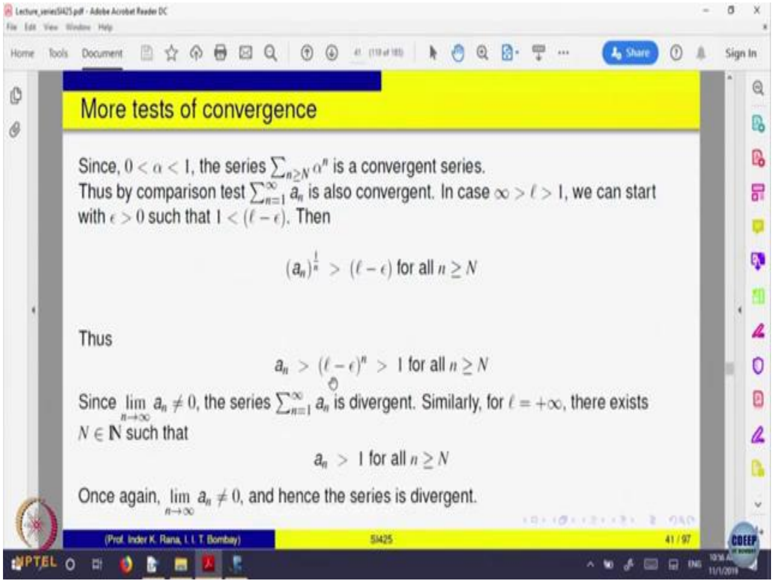

So, an will be less than alpha to the power n when alpha is less than 1 and said, though the will be geometry series and convergent and.

Similarly, by the comparison tests and comparison with geometric series. When bigger than 1, an will be bigger than on the right hand side of 1. So, l minus epsilon will be bigger than 1. So, power an will be bigger than l minus epsilon raise to power n that is bigger than 1. So, again, comparison will give you it is divergent. So basically, you can give examples.

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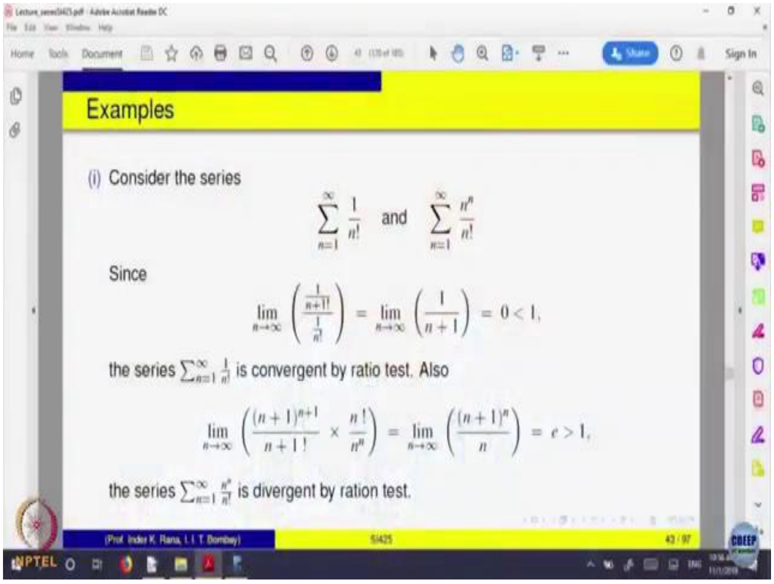

So, I think these examples you should study and try to do it, because I can explain examples and it will be okay, you will nod your head, yes, it is okay and all that, but you should understand why. So, look at the examples, when the limit is less than 1, bigger than 1 and so on. I do not know whether, the limit of this quantity is equal to e and we prove that in the tutorial classes or something, limit of 1 over 1 plus n or 1 plus 1 raise to power n that limit is equal to the number e, Euler’s number. So, that is being used here actually.

That it is an interesting thing, I do not know was it part of tutorial 1 plus 1 over n raise to power n that limit exists is equal to e, anyway. So, that is being used here, because that is actually the definition of a number e, e is a number, which is called Euler’s number. And the same which comes in the exponential function also, e raise to power 1, exponential of 1 is the same number as this. So, there are connections I think, let me not to go into that.

Those who are interested read, see the slides and try to figure out why it is same number as that, if you are interested in mathematics. As such you unity here coming, here coming they are both are same or not. So, I think that is a root test.

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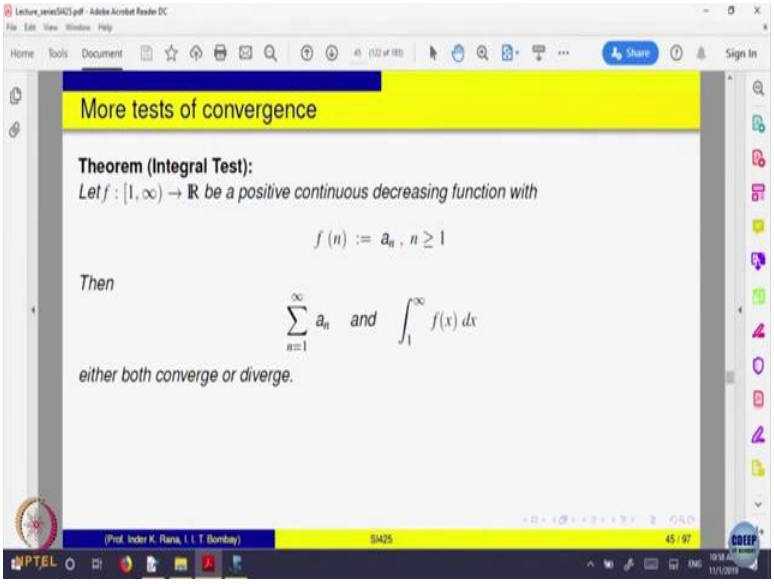

There is integral test, I will not discuss much about this. Because this is something is not difficult, but anyway it is a, it 1 to infinity that is a improper Riemann integral kind of a thing. So, let f be a continuous function from 1 to infinity, evaluate the value of f at the point n and if that is an, then the series and this integral either both converge or both diverge. Remember we had defined what is called the convergence of improper integral, this relates with improper integral. I think let me not go into the proof of this, it is easy, but let me not go into the proof and let me not, just statement of this.

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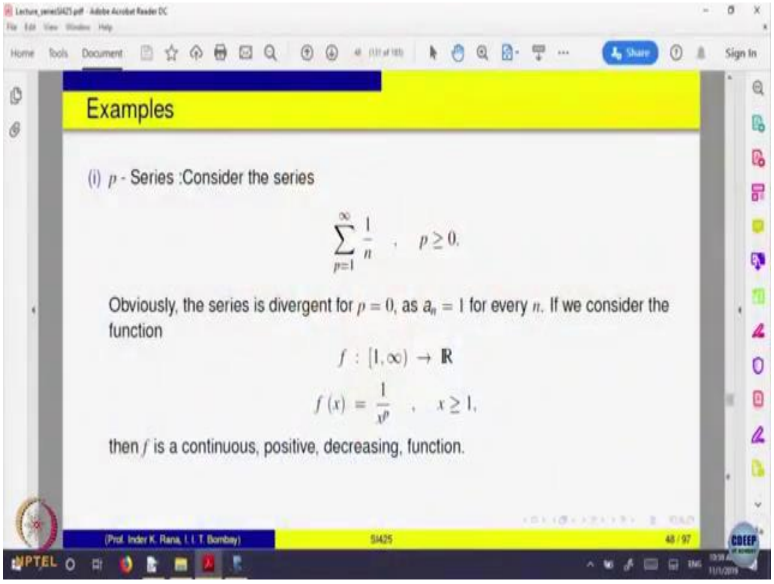

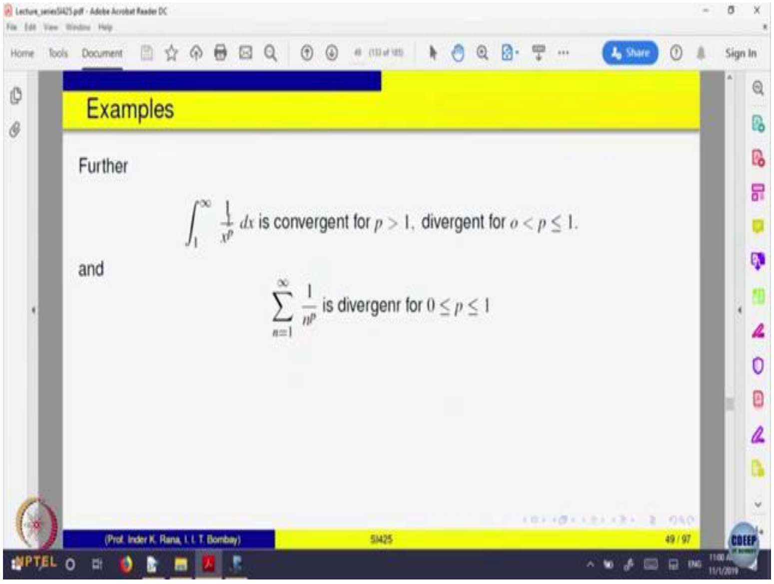

This you can look at, to apply with something like function being 1 over x to the power p and p between 1 and 0 and then you show it is improper integral is convergent. And then, you see how, remember I said between 0 and 1, 1 over n to the power p, we did not analyze, we analyze only when equal to 1, 1 over n or bigger than that. So, this is the integral test gives you, but that requires a fact that this is a convergent improper integral, so one requires that.

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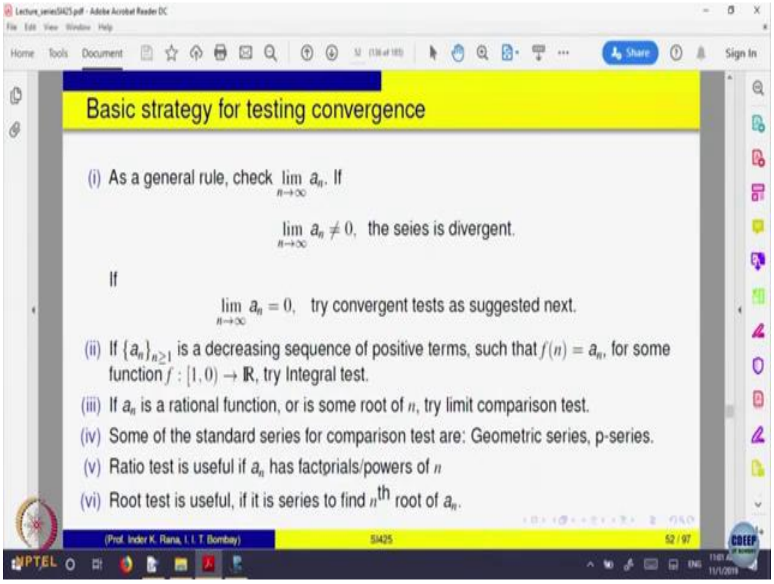

So, let me probably sum it up what we have done today. How do to analyze testing and convergence of a series. The general rule is, check first of all whether the nth term goes to 0 or not. If it does not go to 0, it is not convergent. So, proceed only when it goes to 0 analyze conversions, you can apply integral test. If it looks like a rational function, something divided by something, that n square divided by something then the limit comparison test may work.

And for the standard series, you can try to compare it with geometric series, p series and so on.

Ratio tests works when there are factorials and powers coming, where because when you divide powers will try to cancel it out. So, you should try that. Root test is useful when nth root of, somewhere it is coming. So, a general rules is kind of, not rules general hints of how to analyze convergence of series. So, we have looked at only today for non negative terms, series. But we saw one series, alternative series was convergent.

So, and if series is not convergent, you can always take the absolute values each term and see whether that is convergent or not. So, there is something called absolute convergence of the series and series for alternating terms. So, we will look at it in next time.