Basic Real Analysis
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Series of Numbers –Part II
Lecture No 67

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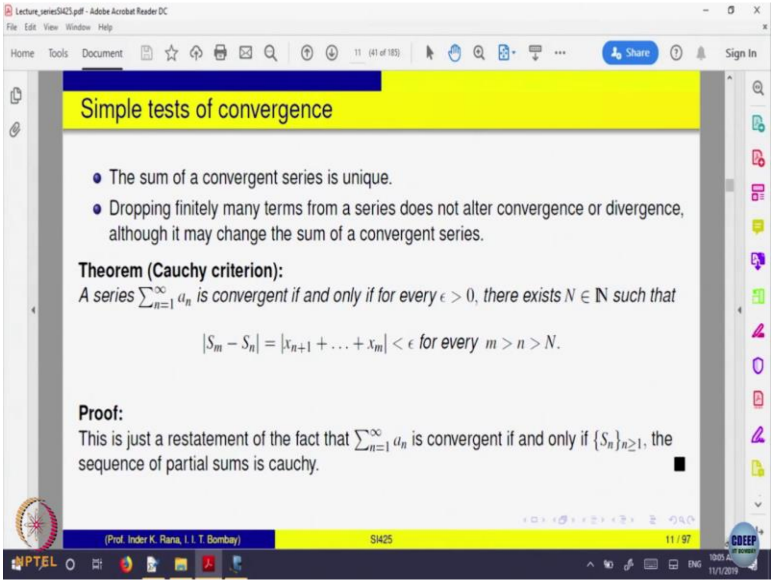

So, here is a, keep in mind one thing, the sum of convergent series is always unique. If a series is convergent its sum is unique. Why, limit of a sequence is unique. What is definition of sum, it is the limit of partial sums, and partial sums can converge only to one limit. So, limit of a convergent sequence is unique that implies, that sum of a convergent series is unique.

Suppose, you drop some of the terms from a, a1, a2, an is a sequence given. I want to know that whether it is summable from 1000 terms onwards or not. That is equivalent to saying whether it is, the series itself is convergent or not, because what is left is the sum of finite terms only. So, if, so you can say the convergence of a series does not depend upon first few terms of the series.

It is the same as the fact that we did for sequences. Convergence of a sequence depends only on what is the tail of the sequence; so, convergence or divergence of series does not depend on the first few terms, whether those are added in the series or not. But the sum may change. If you sum it from 1 to onwards or 1000 onwards, the sum may change. But convergence or divergence will not depend upon, it depends only on the tail of sequence an which is given.

Cauchy's criteria, which is a consequence of, a sequence is convergent if and only if it is Cauchy. A series is convergent when these partial sums converge, and partial sums will converge only when the partial sums is a Cauchy sequence.

So coupled with that fact, the Cauchy criteria, that sequence an is convergent if and only if partial sums is Cauchy and that is same as saying given epsilon. You can write epsilon delta, epsilon n naught definition, given epsilon, the difference between the nth and the mth term should be small and that is that sum from mod of xn plus 1 to xm should be small, for m bigger than n, Sm minus Sn that should be small. So, there is nothing, it is a simple consequence of the fact that every sequence, a sequence is convergent if and only if it is Cauchy. So, apply it to the partial sums.

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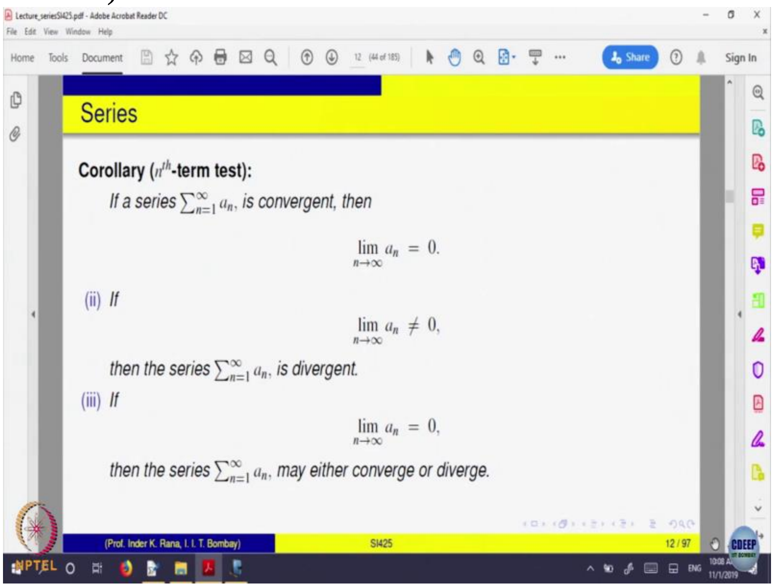

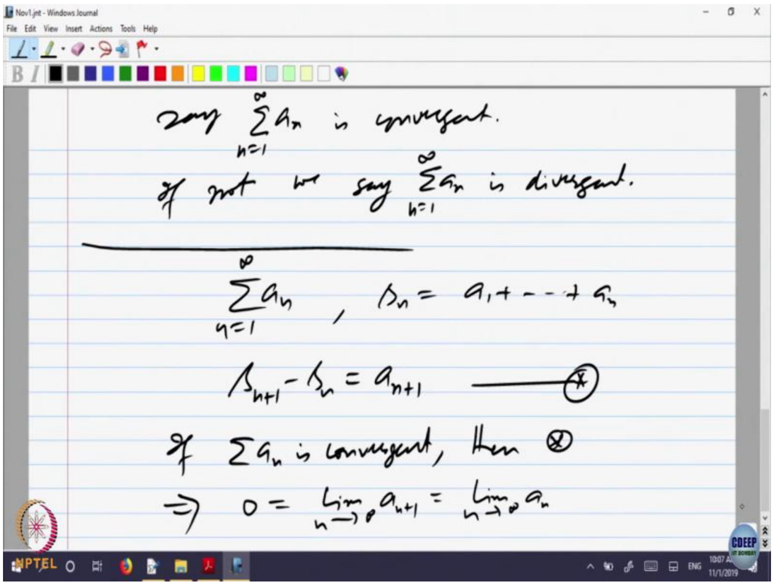

Now here is another simple fact that, suppose a series is convergent, then what is the nth term? If a series is convergent what is the nth term? This is a series, whether convergent or not, let us not bother. Sn is a partial sum, then what is Sn plus 1 minus Sn, that is a n plus 1, simple arithmetic.

Now if, sigma an is convergent then, in this left hand side if I take the limit n going to infinity. Then star implies what? What is the limit of the left hand side? Series is given to be convergent.

Then what is the left hand side, that is 0 is equal to limit n going to infinity of a n plus 1. And if you like, this is same as limit n going to infinity of an, does not matter. Or I could have just written here, Sn minus Sn minus 1, I could have written, that is equal to an, either way.

So, I get a consequence, very simple observation that if a series is convergent then the nth term in that sequence, the sequence of nth term must go to 0. So, an must go to 0. So, this gives me a necessary condition for a series to be convergent.

A series is convergent, then it should necessarily happen that the nth term should become smaller and smaller and go to 0. So, that is a necessary condition and a very useful one because if the nth term does not go to 0, then the series cannot converge. So, not convergence is useful proving.

So that is same as saying, if an is not equal to 0 it is divergent. But if it is, it is only necessary condition, it is not sufficient. That means, if the nth term goes to 0 that does not imply that the series will always converge.

Remember, just now we said 1 over n series is not convergent, nth term is 1 over n that goes to 0. But the alternating series again the nth term goes to 0, but that is convergent. So, this is not the sufficient condition that the nth term should go to 0, it is only necessary. It may be either converge or diverge. So, you can give examples we just now given.

You can apply, if you like you can apply it geometric series, we proved for mod r less than 1 or r bigger than 1, nth term will not go to 0 if r is bigger than, it goes to infinity so that does not converge.

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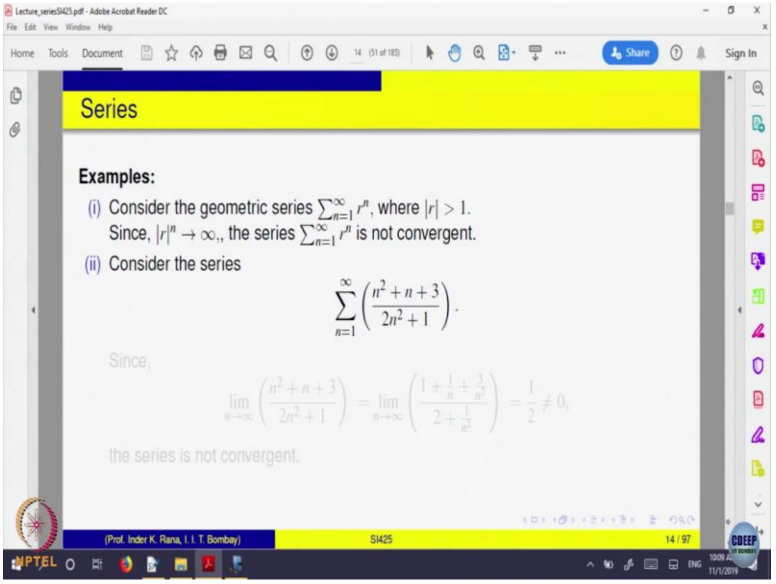

So, let us look at this kind of a series. It looks like n square plus 3n plus 1 divided by 2n square plus 1. It looks like the numerator and denominator both are increasing, at the same rate essentially, n square, power is n square.

So, when n goes to infinity it will stabilize somewhere, because both are increasing at the same rate. How do I analyze that, so look at the nth term, nth term is n square plus n plus 3 divided by 2 n square plus 1; I want to analyze what happens as n goes to infinity. So, the simplest thing is divide numerator and denominator by n square, because I know 1 over power of n goes to 0.

So that gives you 1 plus, numerator will give you 1 plus 1 over n plus, 1 over n plus 3 over n square, and the denominator will give you 2 plus 1 over n square as n goes to infinity the limit will be equal to, numerator will go to 1, denominator goes to 2. So, by the theorems on sequences, if an is convergent, bn is convergent, and bn is not convergent to 0 then an divided by bn is convergent to limit of an divided by limit of bn.

So that theorem says, that the limit of this nth term of this series is converging to 1 by 2, which is not equal to 0. So, this series cannot converge, because the nth term does not go to 0. So, nth term that is how it is used to say, analyze not convergent of a series. So, this does not converge.

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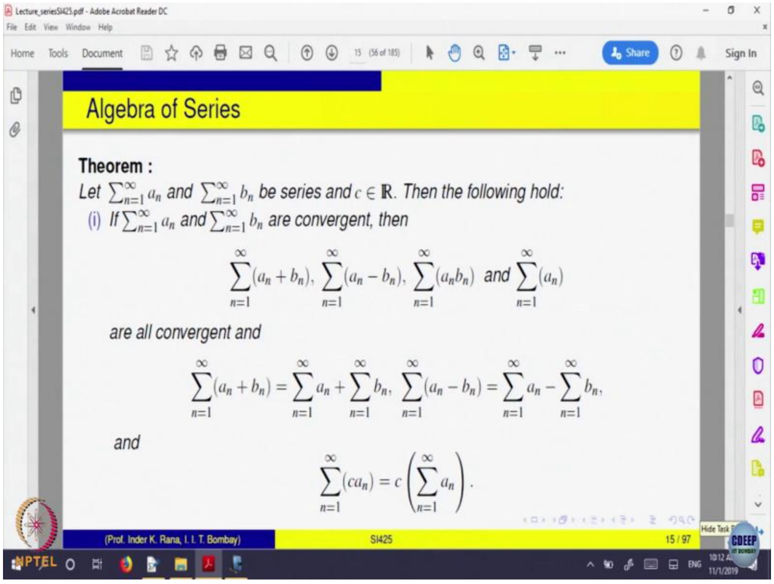

Now, here are the theorems about algebra of limits, giving you algebra of convergent series. If an is a series which is convergent, bn is a series which is convergent you can add nth term of both, get a new series whose nth term is an plus bn.

Then what will be the partial sums of an plus bn, it will be partial sum of an plus partial sums of bn. And if an is convergent then partial sums converges, so partial sums of the sum will converge to sum of the partial sums by limit theorems on sequences.

So, if an is convergent, bn is convergent then, an plus bn is convergent and sum is equal to sum of an plus sum of bn, because of the limit theorems on sequences. Same logic applies to the other, you can have difference; you can have the scalar multiplication. One can wonder what happens if you multiply 2 series.

Can you multiply two series? Does not matter, you can just look at an bn if you want, that is one way of multiplication. But the partial sums will not be multiplication of partial sums, so that will not work out. So, it is only for the additions. One can think what could be way of multiplying series so that the corresponding result for series is valid. You understand what I am, I am throwing a question.

What could be, given a series an, given a series bn, what could be the multiplication of these 2 series so that the limit of that product whatever we define is product of the sums. Think of it, it is a good thing to think. It is possible to do such things, but let us not do into that.

So, this is algebra of the sums of the convergent series, this is for convergent only. If an is convergent, if bn is convergent, then an plus bn, that series is also convergent, using the sequences, theorems on sequences. How is that useful, you can always make examples.

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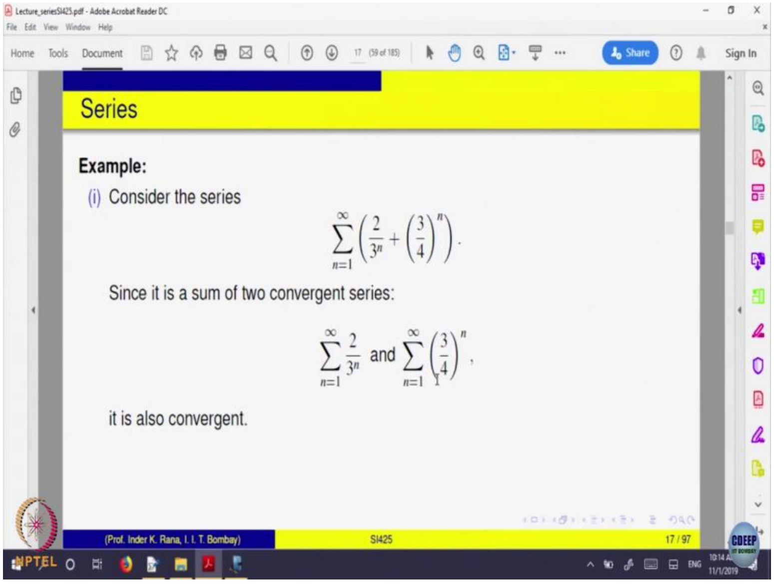

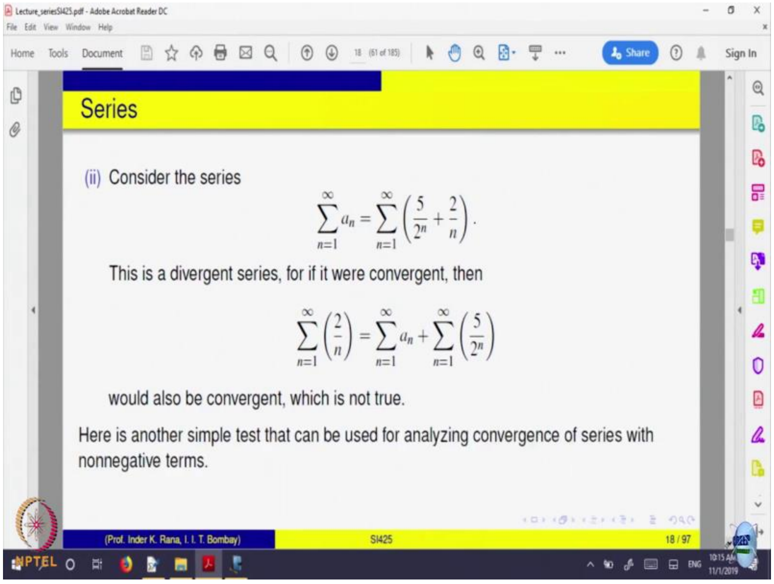

Look at this series, 2 by 3 to the power n plus 3 by 4 whole to the power n, is it convergent? So, we will sum up 2, so 2 over 3 raised to 1 over 2 by 3 raised to power n and you try to show that both of them are convergent and the sum will be the sum of that. So, I am leaving for you to check why both are convergent.

More examples one can give, I think. This is not convergent, why, because if it were, I know 5 by this is convergent when you subtract it should give me 2 by n should be convergent which is not true. So, one can play with this kind of things. The usefulness of saying that sum of convergent series is convergent.

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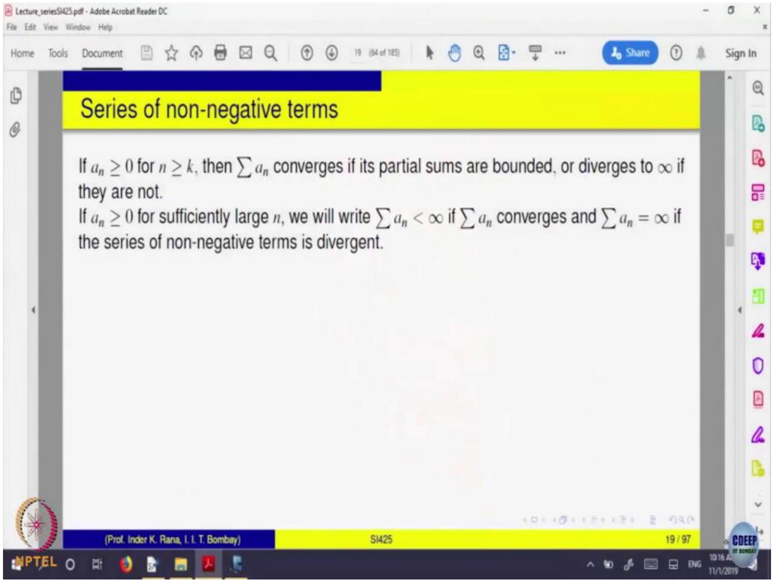

Here is something. So let us, what I am going to do is I am going to specialize for some time on series with non negative terms only, an's are all nonnegative. When an's are non negative partial sums are going to be increasing, partial sums are going to be increasing.

Because we will be adding something all the time non negative. So, either the series will converge if the partial sums are bounded above or what will happen, the partial sums will keep on increasing and go to plus infinity.

So, in some, in such case when sequences of non negative terms are given, series of non negative terms, if convergent we write what is the sum or sigma an less than infinity. Other only other possibility is, it is divergent and, in that case, partial sums converge to plus infinity, so one writes sigma an 1 to infinity equal to plus infinity. That is the notation, nothing more than that. So, it is just a notation saying that they are…

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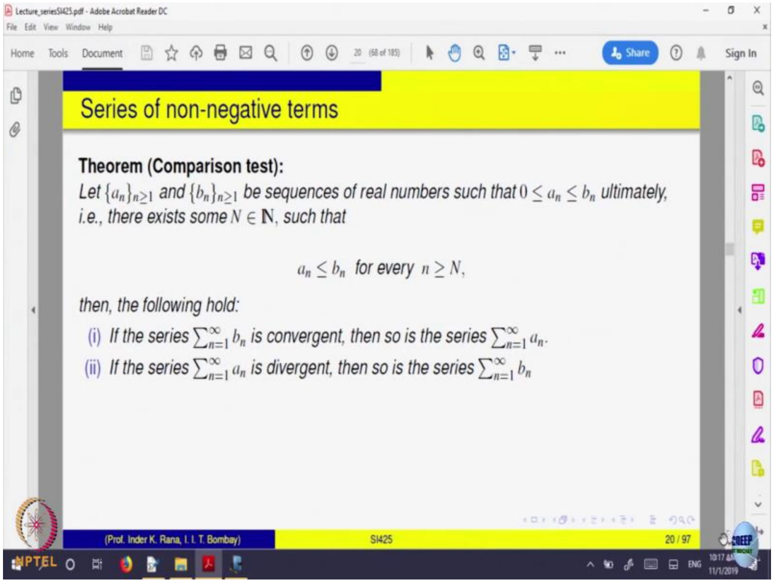

So, here is, one of the simplest tests which can help us to analyze convergence or divergence of a series. We are given 2 sequences of non negative terms an and bn, such that an is less than bn. Ultimately, what does ultimately mean, from same stage onwards, ultimately means for some stage onwards because it is the tail that is going to matter for convergence. So, you can write ultimately, that is for some n bigger than capital N, an is less than or equal to bn. So that means what, each term of the sequence bn is dominating the term an from some stage onwards.

So, what will happen to the partial sums, partial sum of an will be dominated by the partial sums of the series bn. Because an is less than bn. So, if the partial sums of bn's converge, partial sums of an's are dominated by partial sums of bn, so that will converge because it is less than or equal to. And if an's do not converge, then bn's cannot converge, because an's are less than bn.

So, you get 2 ways of writing it. Just writing the partial sums of the corresponding things, that if bn's have converged, if the series bn is convergent then the series an is also convergent. Because the partial sums of an's, will be less than or equal to partial sums of bn and that converges so no problem.

And because an is less than bn, if an is divergent, that means what, the partial sums go to infinity. Partial sums of bn are bigger than partial sum of an, so bn's also will, partial sums of bn also will go to infinity. So, it implies that if an is divergent then bn is divergent, if bn is convergent then an is also convergent. Simple comparison of 2 series, from stage onwards an is less than bn. Very simple proof, so no problem about that. So, let us skip the proof. Try to write the proof yourself. What will be involved?

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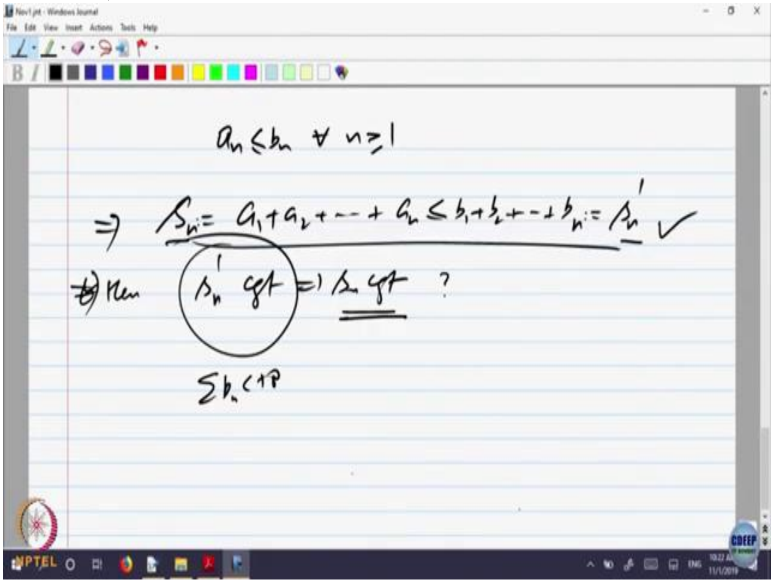

Let me just write probably one, so that you understand why some minor modifications are required. So, we have got an less than equal to bn for every n, for every n or for some stage onwards, that does not matter, from some stage onward.

So, what is Sn equal to, a1 plus a2 plus an will be less than or equal to b1 plus b2 plus bn and that is, that is Sn of this, what should I write for Sn of that, some notation, so let me write Sn dash. We are calling that as Sn dash, we are calling it as Sn. So, this implies this. So, Sn dash convergent implies Sn convergent. Why? How should I justify? How should I justify this statement?

Student: (())(19:18)

Professor: These are partial sums only, Sn is partial sum, this is a partial sum.

Student: (())(19:23) we can write

Professor: We can write this, this is okay. But I am saying this imply, so this statement implies convergent is, or hence Sn dash convergent implies Sn convergent. Why is that?

Student: (())(19:39) bn's itself are convergent.

Professor: bn's are not convergent.

Student: The series Sn, the series bn is convergent.

Professor: Series bn is convergent is same as saying this is convergent. So, this is because series bn is convergent. Why does it imply sigma Sn is convergent, you have to say something more.

Student: Because they are bounded.

Professor: Bounded by what?

Student: Partial sums of (())(20:14)

Professor: Partial sums, this partial sum is less than or equal to partial sum that. So, the claim is Sn is convergent, why? What is the reason, Sn is a sequence of numbers. Why is it convergent?

Student: It is bounded by limit of Sn prime.

Professor: Sn prime

See both are non negative terms. So, Sn dash is increasing. So, limit of Sn dash will dominate all Sn's, limit of Sn dash will dominate all Sn's. And Sn itself is also increasing, is non negative terms, so partial sums are again increasing. So, this is also increasing sequence of non negative terms which is bounded above and hence it must converge, so we are using both things.

Professor: If a sequence of, if a sequence is monotonically increasing and convergent, so what is the limit?

Student: Upper bound

Professor: Limit is upper bound, least upper bound.

So, all Sn's dash for every n is less than or equal to the upper bound which is the limit, which exists. So, Sn's are bounded, monotonically increasing, so they converge. So that is the argument that we have to supply in between.

And similarly, if we want to say that an's are, sigma an is divergent, that means what, that the partial sums Sn's are converging to infinity, and Sn is less than Sn dash. So, Sn dash will also converge to infinity because they are bigger than Sn and Sn is going to infinity. So, if an is divergent then, bn also is divergent; simple observations about sequences only. So that is the comparison test. So, let us skip the proof of that.

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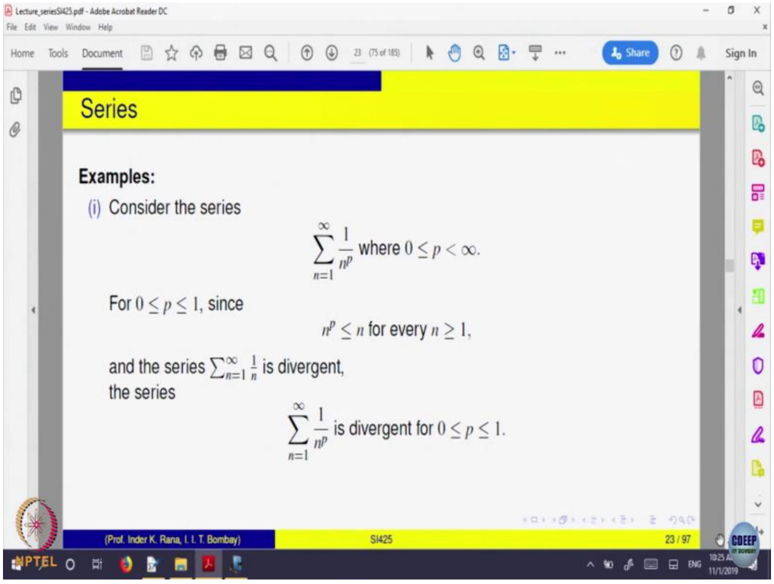

So, let us look at examples of comparison test. So, we are comparing 2, so look at the series. Remember we did 1 over n; that was divergent, 1 over n square was convergent. So now, we are looking at for any p between, bigger than 0, less than infinity, what can you say about that. So, let us assume, so it depends on p of course. Because p is equal to 1, we know it is divergent. So, let us look at when p is between 0 and 1 then n to the power p, is less than or equal to n, because p is between 0 and 1.

So, what happens to 1 over np, that is bigger than, 1 over that will be bigger than 1 over n. And 1 over n is divergent. So, 1 over np will be divergent, the sigma 1 over np is divergent for p between 0 and 1 by comparison test, comparing it with 1 over, the series 1 over n. Simple observation. So, that is divergent, so this is divergent between 0 and 1.

Let us look at when p is bigger than 2, we know 1 over n square was convergent, p was equal to 2. So, let us take p bigger than or equal to 2. What happens to np and n square?

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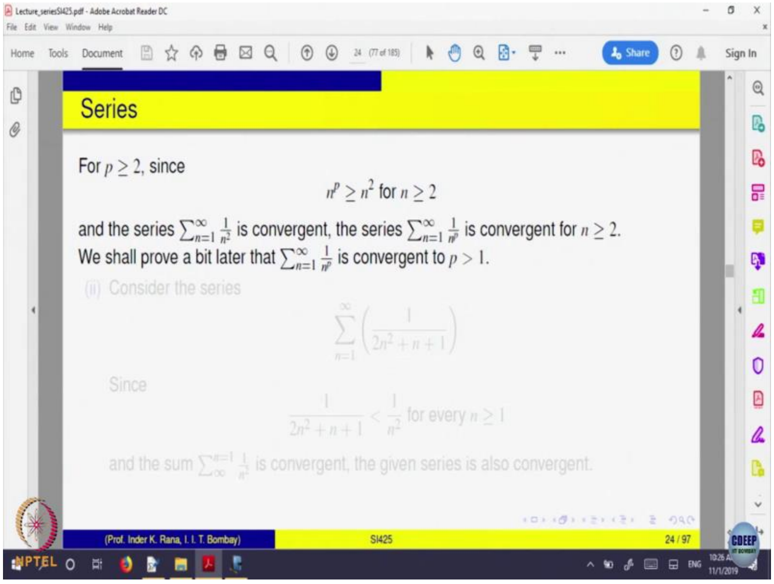

If p is bigger than 2, compare n to the power p and n to the power 2, n square, n to the power p will be bigger than n square. So, convergence of 1 over n square will give you convergence of 1 over n to the power p for p bigger than or equal to 2. Already proved convergence of 1 over n square, already proved divergence of 1 over n, comparison gives you, for p between 1 and 0, np is divergent for p bigger than or equal to 2, 1 over np sigma np is convergent.

Between 1 and 2, we have not done anything yet. Because we are just, known things, we have to compare with something known, an, bn, an less than or equal to bn. If you know something about bn convergent then you can say an convergent, sigma an convergent, we will do that also a bit later.

For example, let us look at this kind of a thing. So how these things help us analyzing, 1 over 2 n square plus n plus 1 kind of thing. It looks like 1 over n square, it looks like 1 over n square. So, can we compare 1 over 2n square plus n plus 1 with 1 over n square? nth term? Already 2n square, we are increasing the denominator, so making it smaller anyway.

So, 1 over 2n square plus n plus 1, that you call as an is less than, 1 over n square which is bn, that is convergent so this is convergent. So, how you think and what you have to compare with, that you have to sort of keep in mind. So, this is convergent. I think there are more examples you can study later on.

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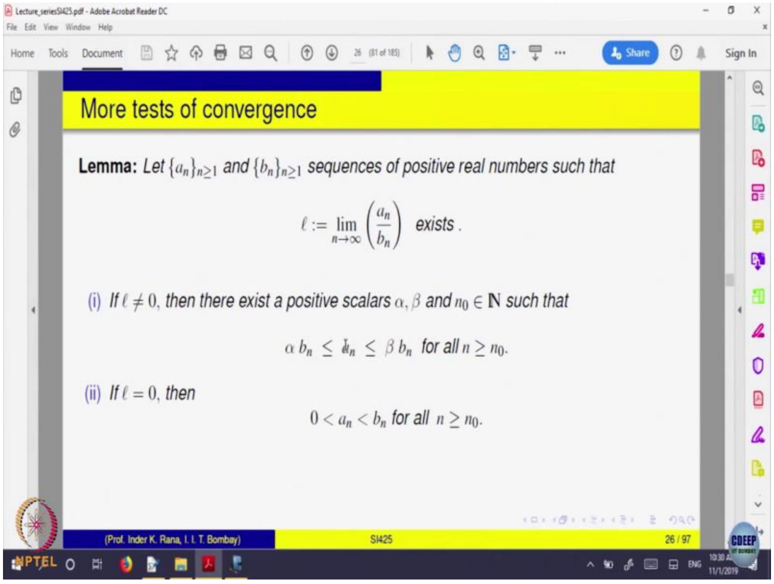

Let us look at, something which requires a bit of thought but not much. Suppose, is again a kind of comparison but we are going to look at, see we looked at that 1 over 2n square and all that, and 1 over n square. So, 2n square and n square sort of compatible kind of a thing, we could compare.

So here, we are looking at an and bn, 2 sequences of positive real numbers, only for the time being positive and look at the limit of that, suppose that limit exists. Suppose, the limit exists and is l. So, what is the meaning of saying this l exists, an over bn, that means eventually an and bn are stabilizing. They are becoming, sort of coming to a common kind of, proportion kind of a thing.

So, the claim is if this limit exists and if this limit is not equal to 0, then you can have inequality which says for some alpha and beta, alpha bn will be less than or equal to an is less than or equal to beta times bn, if this limit is not equal to 0. In what way that is useful, it helps you to compare an and bn. It says from some stage onwards, you can compare an and bn.

So, convergence of one can imply the convergence of other or the divergence of one can imply the divergence of the other. So, this limit becomes important. And if this, so let us look at first why is this thing happening.

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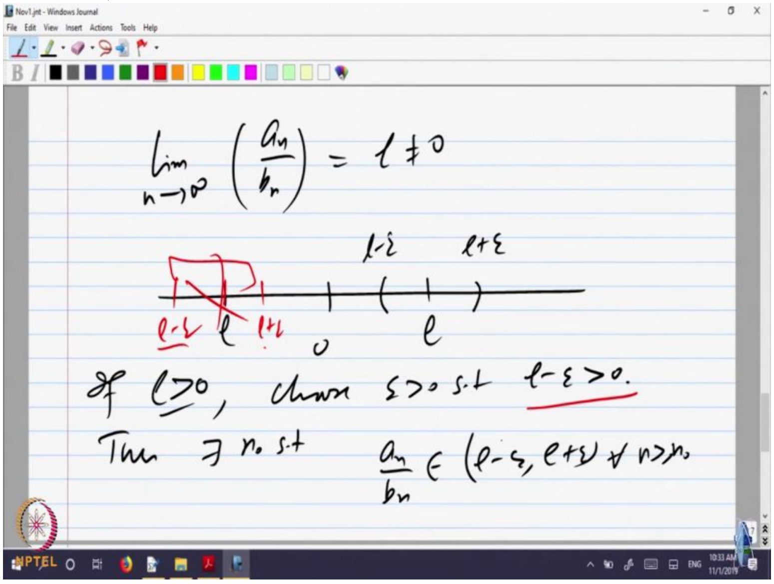

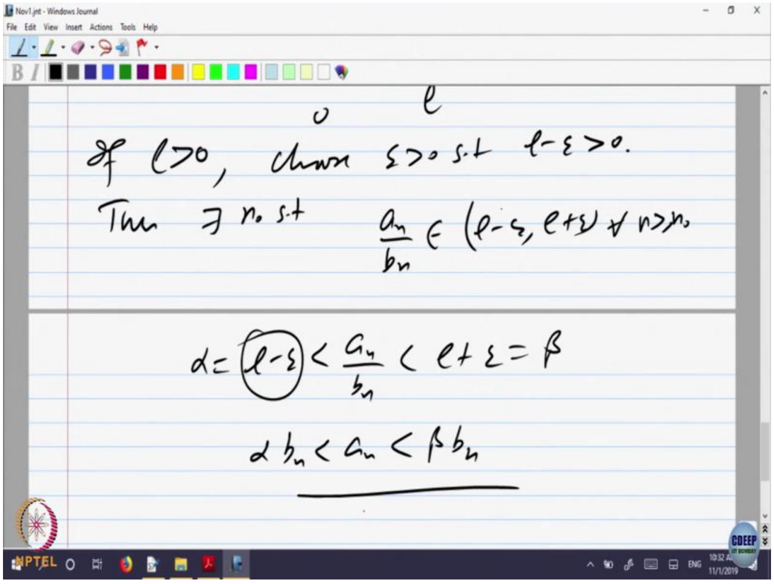

So, I want to look at a simple, you see analysis of real line is coming back to picture. So, an divided by bn limit n going to infinity equal to l and that is not equal to 0. So, here is 0, here is l, either on positive side or on the negative side, does not matter, it is away from 0. So, what is the meaning of saying that its limit exists, limit exists means the terms should come in the neighborhood of that point after some stage onwards.

So, let us define neighborhood like this. Say l minus epsilon and l plus epsilon. So, let us choose epsilon such that, so if l is bigger than 0 choose such that l minus epsilon is also bigger than 0, then convergence there exists some n naught, such that an by bn belongs to l minus epsilon and l plus epsilon for every n bigger than n naught. That is simple convergence.

It comes in a neighborhood. I want to stay away from 0, so I will just take l minus. So, what does that mean, that means l minus epsilon is less than an by bn is less than l plus epsilon. Call this number as alpha, call this number as beta. So, then alpha times bn is less than an is less than beta times bn, you got the inequality. Simple definition of limit, if limit is not 0, an over bn should stay away from 0, that is all.

And if this limit is equal to 0 will mean what? So, this was l bigger than 0. Similarly, l less than 0, does not matter, bigger or, l is bigger than 0, if it is less than 0, what would be your argument.

Say if l is here, if l is here then you will have l plus epsilon and then you will choose epsilon size that l plus epsilon is bigger than 0. Here we had taken this, in this case we will choose epsilon such that l plus epsilon is bigger than 0, so everything will be inside this. Again, this will be your alpha, that will be your beta.

Student: (())(31:27) positive, then how can l be less than 0?

Professor: Pardon?

Student: a and b are positive sequences

Professor: They are positive, I am saying this…

Student: If they are negative.

Professor: So, it is a valid point. I am just giving you general arguments. For any sequence, an and bn, not necessarily positive, in our case, in other case this does not matter. But in general I am saying the limit of a sequence, if it is not 0 then everything should be in the neighborhood of, away from 0.

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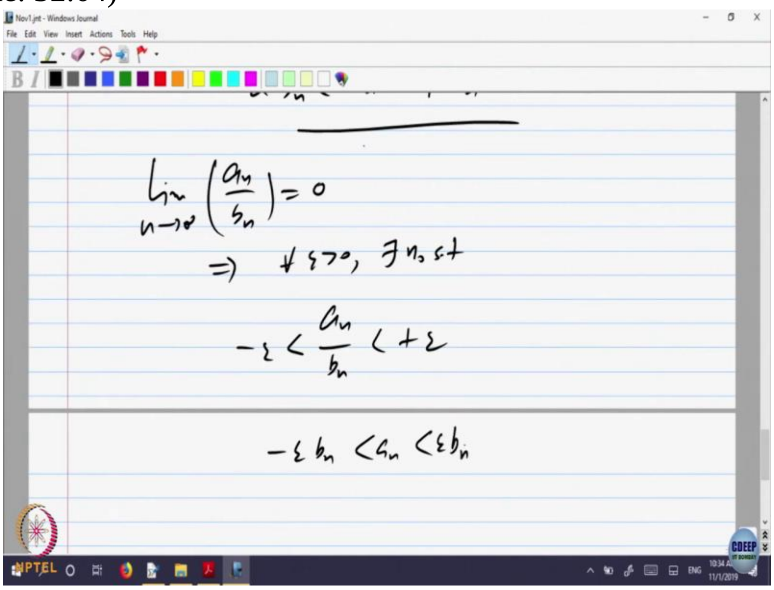

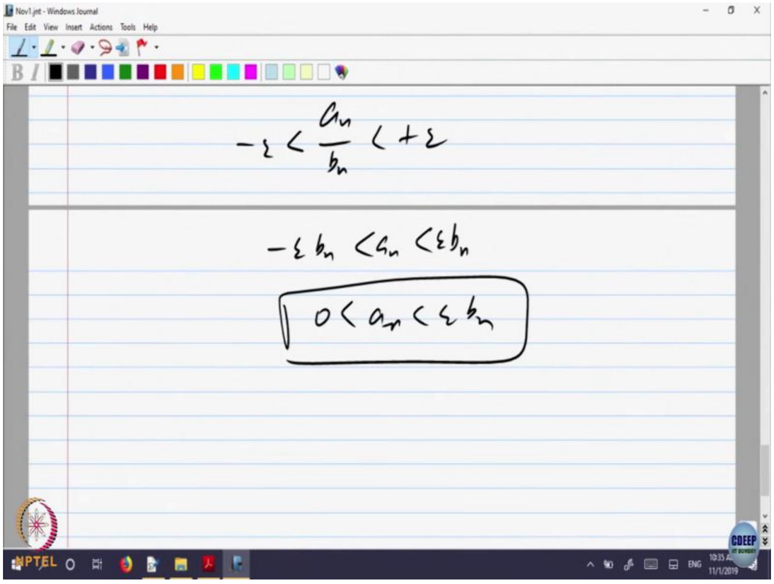

Now, if the limit is equal to 0 that is possible. So, let us say that limit of an by bn is equal to 0, that means what? By the same argument, then there exists, so for every epsilon bigger than 0 there is a n naught, such that an by bn between minus epsilon and plus epsilon, in neighborhood of 0. So, that means what, minus epsilon bn is less than an is less than an plus epsilon, bn times, bn times epsilon.

But an, in our case all the an's are non negative, so 0 less than an less than epsilon bn, because all the an's are non negative. So, if the limit is 0 now you get only one inequality an and less than or equal to bn. But still it gives you a comparison between an and bn, you can compare an with bn. So, that was the lemma.

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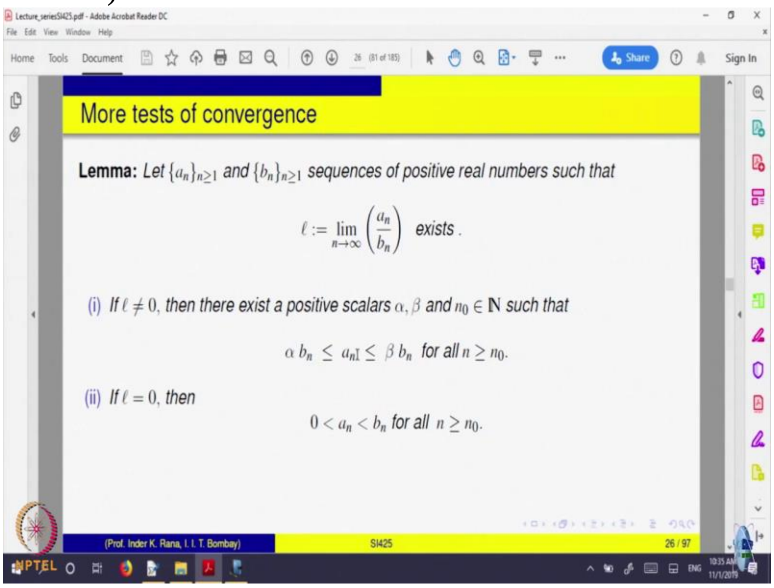

Now, you can use this lemma in view of the comparison test. So, what will that give me, if the limit is not equal to 0, of this thing and supposing bn is convergent then look at this, this part of the inequality, an less than or equal to the beta times bn, bn is convergent so that will imply, an is convergent. And other way round, if an is convergent I can use this part to say bn is convergent.

So, if the limit is not equal to 0 of an by bn then either both an and bn converge or both diverge. If sigma an is convergent, then sigma bn is convergent and other way round. If the limit is equal to 0 then only convergence of bn can apply, convergence of an or divergence of an can apply divergence of bn, it will not give you if and only if.

The idea is that comparison test, when you want to find the sum of a series, it is only some point onwards the things matter. How are we getting this? Saying that l minus epsilon, l plus epsilon, an by bn is important for n bigger than or equal to n naught. So, this comparison is valid only for, and that is good enough for convergence.

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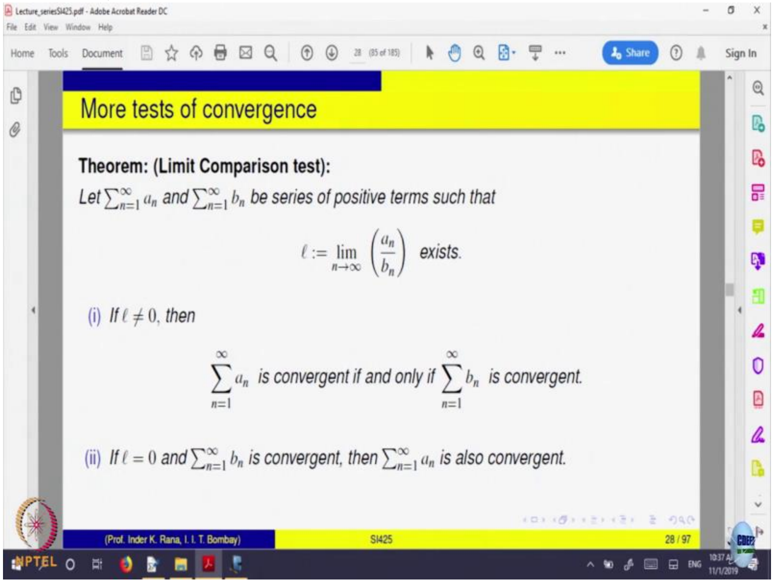

So, that gives me that test which is limit comparison test. So, either sigma an is convergent if and only if sigma bn is convergent. If limit is 0, when bn is convergent implies an is convergent. So, how this test helps in analyzing series of non negative terms that is important.